About This Book

Navigating through Number and Operations in Grades 9–12 provides materials for you to use selectively in your classroom to deepen your students' understanding of number and operations. At the same time, working with these materials can consolidate your own awareness of the challenges of the Number and Operations Standard.

Principles and Standards for School Mathematics (National Council of Teachers of Mathematics [NCTM] 2000) places number and its nearly inseparable counterpart, operations, first among the Content Standards. This primacy of place recognizes the fundamental role that these entwined elements play in the development of all the other content strands of school mathematics. Number and operations certainly deserve this priority in grades 9–12. A large part of high school algebra is the study of operations on variables that stand for numbers, for example.

Principles and Standards is clear about the primary goal of instruction for number and operations in grades 9–12: "Whereas middle-grades students should have been introduced to irrational numbers, high school students should develop an understanding of the system of real numbers" (NCTM 2000, p. 291). Three words hold the keys to what high school students must attain over and above what they achieved in middle school: *understanding*, *system*, and *real*. An introduction to irrational numbers suffices for middle-grades students, but high school students must develop an understanding of the system of real numbers. A *system* is a unified set rather than a simple collection, and the real numbers are the fundamental numbers that students will need for measuring and counting in the world around them. In addition, as *Principles and Standards* emphasizes, students will need the system of real numbers to develop understandings of such new systems as vectors, matrices, and complex numbers.

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Overview of the Chapters

The book's four chapters present activities that show how to achieve goals of the Number and Operations Standard for grades 9–12. A consideration of the real numbers in chapter 1 prepares the way for an examination of number theory and algebra in chapter 2. Chapter 3 explores some applications of number and operations in the real world, and chapter 4 concludes the book by investigating some extensions of basic concepts in the world of mathematics.

Chapter 1, "The Real Numbers," focuses on the relationship between the rational numbers and the irrational numbers, the two systems that make up the real number system. From one perspective, these classes of numbers are extremely different. From another perspective, however, they are not so very unlike. Students encounter both perspectives in the chapter's three activities—When Fractions Are Whole, Designing a Line, and Trigonometric Target Practice:

"[Students] should understand that given an origin and a unit of measure, every point on a line corresponds to a real number and vice versa." (NCTM 2000, pp. 291–92)

See "So That's Why 22/7 Is Used for π!" (Burke and Taggart 2002; available on the CD-ROM) for a graphical method of building students' understanding of rational approximations for irrational numbers.

- When Fractions Are Whole helps students build and deepen their sense of rational and irrational numbers as fundamentally different. Students use what they know about simplifying fractions to discover—and prove—that they cannot express several common classes of numbers as rational numbers. They see, in other words, that these numbers are irrational.
- Designing a Line helps students take the opposite perspective. Now they consider rational and irrational numbers as not so very different from one another. *Principles and Standards* recommends that students understand the real number line—the geometric interpretation of real numbers that treats rational and irrational numbers in the same way—as points on a line. This activity focuses on geometric representations of both real numbers and the operations on real numbers.
- Trigonometric Target Practice pushes the students' understanding of rational and irrational numbers a step further, helping them realize that irrational numbers are more pervasive than they might have thought. The activity provides a graphical way of "seeing" when a number is irrational.

Principles and Standards also asserts that high school students "should understand that irrational numbers can only be approximated by fractions or by terminating or repeating decimals" (NCTM 2000, p. 292). Our inability to represent irrational numbers exactly by these means reflects an important element of the relationship between rational and irrational numbers. Although the book does not treat this aspect explicitly, Burke and Taggart (2002) have investigated how teachers might approach this issue with high school students. Their work appears as supplemental reading on the CD-ROM that accompanies the book.

Chapter 2, "Number Theory and Algebra," illustrates ways to help students understand the properties of number and operation systems, as *Principles and Standards* recommends. Students investigate properties of systems and apply them in exploring new systems in the chapter's three activities—Counting Primes, Complex Numbers and Matrices, and Solve That Number:

- Counting Primes engages students in exploring the divisibility of whole numbers and proving results that are easy to believe from considerations of the number line. The students then use these results to prove that the set of prime numbers is infinite.
- Complex Numbers and Matrices shows students how they can represent the complex number system by systems of vectors and matrices based on the real numbers. The activity engages the students in a cycle of exploring and justifying as they compare the complex number system with the system of matrices that they encounter in the activity. Their investigation of properties shows them that this system of matrices is isomorphic to the complex number system.
- Solve That Number sheds light on the distinction between algebraic and transcendental numbers. An excursion into the theory of equations adds a new twist to the algebra that students have previously encountered. Instead of receiving equations and finding numbers

"In grades 9–12 all students should …

• compare and contrast the properties of numbers and number systems … ;

• understand vectors and matrices as systems that have some of the properties of the real-number system;

• use number-theory arguments to justify relationships involving whole numbers."

(NCTM 2000, p. 290)

that solve them, the students now receive numbers and find polynomial equations, with integer coefficients, that each number solves. The activity refers to this process informally as "solving the number." In the course of the activity, the students encounter numbers that they cannot "solve," such as π. Through such experiences, they gain an appreciation of the difference between algebraic and transcendental numbers. This new insight, in turn, helps them recognize how prevalent transcendental numbers are and what mathematicians mean when they refer to the trigonometric functions as transcendental functions.

Chapter 3, "Numbers and Operations in the World," shows how to help students find and apply ideas about number and operations in the real world. The everyday importance of these concepts is clear in the chapter's two activities—Frequency, Scales, and Guitars, and Rock Around the Clock:

- Frequency, Scales, and Guitars illustrates the important role that irrational numbers play in the world of music. The activity demonstrates the significance of the twelfth root of two in the design and tuning of guitars and other musical instruments. High school students learn that real numbers in their world are often extremely large or extremely small, and they are often irrational. An applet activity, Sound Wave, adapted from the Illuminations Web site, appears on the accompanying CD-ROM as a supplement to this exploration.
- Rock Around the Clock helps students learn another important fact as they discover that the operations they need to work with numbers in the world around them are not limited to the "big four"—addition, subtraction, multiplication, and division. The students use multiplication of integers modulo *n* to make sense of a system of message-coding schemes known as *position ciphers*. A firm understanding of the properties of the basic operations (+, –, ×, ÷) on integers, rational numbers, and real numbers allows students to extend their insights to new operations, and these discoveries, in turn, empower them to grasp mathematics in the world around them.

Chapter 4, "Extending Number and Operation Activities," the book's final chapter, shows how seemingly simple activities focusing on number and operation can lead students to mathematical investigations of increasing depth and sophistication. Such rewards are characteristic of any curriculum that emphasizes the five major aspects of "doing" mathematics—problem solving, reasoning and proof, communication, connections, and representation. These are the facets of mathematical study that *Principles and Standards* highlights in the Process Standards. These processes play central roles in this final chapter's three activities —Number Triangles, Perfect Squares, and Flooding a Water World:

- Number Triangles begins simply, with a problem-solving activity similar to ones that students encounter frequently at the elementary level. The students examine the pattern of numbers in a "number triangle" to supply the numbers that are missing in a second triangle. However, by reflecting on their solutions and solution processes,

Houser (2002; available on the CD-ROM) presents additional information on the use of the twelfth root of two in the design and tuning of some musical instruments.

Hall (2003) and St. John (1998) show how students can use matrices to extend their newly acquired ideas about position ciphers to other encryption methods. Both authors' pieces are available as supplemental reading on the accompanying CD-ROM.

Principles and Standards urges teachers to develop students' abilities to reason with and prove mathematical propositions in whole number contexts such as that in the activity Perfect Squares. Bosangue and Gannon (2003), Francis (1993), Miller (1991), and Shiflett and Shultz (2002) all discuss similar investigations and ideas for helping students delve deeper. Their work appears on the CD-ROM.

the students move to a consideration of such higher-level issues as solvability and the effectiveness of algorithms.

- Perfect Squares opens with an exploration of the properties of the class of natural numbers known as the perfect squares. The investigation quickly directs students' attention to the last digits in very large perfect squares—numbers that conventional calculators typically cannot handle. This consideration in turn leads to an investigation of a number system consisting of the last digits of integers under multiplication—a system that is equivalent to the integers modulo 10.
- Flooding a Water World allows students to explore simple networks of towers and dikes erected to create habitable regions in an imaginary world covered by water. The students establish relationships among the numbers of towers, dikes, and regions in a "legal" network. By working in this accessible, concrete context, the students easily discover and then apply several powerful ideas from graph theory and geometry. This activity is one of many counting tasks that can lead high school students to rich mathematical experiences without using permutations or combinations.

Using the Book

Needless to say, the Standards-based activities in this book are merely snapshots of possible activities, and they cannot do justice to all the recommendations in the Number and Operations Standard. For example, the book does not include an activity that delves into counting techniques for permutations and combinations. This significant topic and others of equal importance, including vectors and matrices, receive only the briefest of nods here. To provide some ideas of possible approaches to vectors, the accompanying CD-ROM includes two applet activities, Driving with One Vector and Flying with Two Vectors, adapted from electronic examples at NCTM's Web site.

"Counting Pizzas: A Discovery Lesson Using Combinatorics" (Nord et al. 2002; available on the CD-ROM) is a useful article on teaching combinatoric counting techniques.

In addition, it is very important to emphasize that for nearly every major recommendation in *Principles and Standards*, the journals of the National Council of Teachers of Mathematics include many useful articles. The topic of permutations and combinations, for example, offers a wealth of material from which to choose. The CD-ROM includes one of these pieces, and the activity Flooding a Water World in chapter 4 shows that rich and meaningful counting activities at the high school level need not involve combinations and permutations.

Navigating through the book and the CD-ROM

The appendix of *Navigating through Number and Operations in Grade 9–12* presents activity sheets for students as reproducible blackline masters. An icon in the margin (see the key on p. xi) signals all the blackline pages. You can make copies of these pages from the book or print them directly from the accompanying CD-ROM. Solutions to the problems posed in the blackline masters appear in the appendix and on the CD.

As this preface has already indicated, the CD-ROM also includes a number of pieces for teachers' professional development. A second icon in the text identifies all supplemental materials on the CD-ROM, including the book's three applet activities—Sound Wave, Driving with One Vector, and Flying with Two Vectors.

Throughout the book, margin notes supply teaching tips as well as pertinent statements from *Principles and Standards for School Mathematics*. A third icon flags these quotations, which highlight the fundamental notion that students should master the processes of mathematics and see mathematics as an integrated whole.

It is the highest hope of the authors and editors that the ideas and activities in this book are clear and useful. If the book encourages reflection on the NCTM Standards and the support that they can offer in the classroom, it will have served its primary purpose.

Key to Icons


Principles and Standards


CD-ROM


Blackline Master

Three different icons appear in the book, as shown in the key. One alerts readers to material quoted from *Principles and Standards for School Mathematics,* another points them to supplementary materials on the CD-ROM that accompanies the book, and a third signals the blackline masters and indicates their locations in the appendix.