Preface

> Recommendation 1: Problem solving must be the focus of school mathematics in the 1980s.
>
> —*An Agenda for Action: Recommendations for School Mathematics of the 1980s*

> Problem solving . . . can serve as a vehicle for learning new mathematical ideas and skills. . . . A problem-centered approach to teaching mathematics uses interesting and well-selected problems to launch mathematical lessons and engage students. In this way, new ideas, techniques, and mathematical relationships emerge and become the focus of discussion. Good problems can inspire the exploration of important mathematical ideas, nurture persistence, and reinforce the need to understand and use various strategies, mathematical properties, and relationships.
>
> —*Principles and Standards for School Mathematics*

THE TWO statements above, made twenty years apart by the National Council of Teachers of Mathematics (1980, p. 2; 2000, p. 182), serve as evidence of a long-term commitment of the Council to making problem solving a central theme of school mathematics instruction. The first statement was made at a time when the NCTM was just beginning to assert itself as a leader in efforts to change the nature of mathematics teaching in our schools. The second statement demonstrates that after two decades of curriculum development, research, and considerable reflection, the Council has developed a mature position about the role that problem solving should play in mathematics instruction.

The second statement also captures the essence of what this volume and its companion for prekindergarten through grade 6 are about, namely, that the role of problem solving in mathematics instruction should change from being an activity that children engage in after they have studied various concepts and skills to being a means for acquiring new mathematical knowledge. But to suggest, as do the authors of *Principles and Standards,* that problem solving "can serve as a vehicle for learning new mathematical ideas and skills" (NCTM 2000, p. 182) is one thing; to furnish the sort of coherence and clear direction that teachers need is another matter. These volumes represent a serious attempt to provide teachers with that coherence and direction.

In conceptualizing these volumes, the Editorial Panel was guided by what it saw as a central message of all four NCTM *Standards* documents (1989, 1991, 1995, 2000), namely, their emphasis on the importance of viewing classroom mathematics teaching as a system. According to Hiebert and his colleagues (1997), the five dimensions of this system are (1) the nature of classroom tasks, (2) the role of the teacher, (3) the social culture of the classroom, (4) mathematical tools as learning supports, and (5) equity and accessibility. Changing any of the elements of this system requires parallel changes in each of the other dimensions.

The system of mathematics classroom instruction that has dominated U.S. schools for at least the entire past century can be characterized in terms of the foregoing dimensions roughly as follows. Classroom tasks come mainly from the worked examples and homework exercises in the textbook. These tasks are predominantly short, out of context, and symbolic, with emphasis on mastering and maintaining procedural skills. The teacher's role is to work examples for the students using direct teaching, with the expectation that students will listen to and learn to apply the same procedures that the teacher demonstrates. Students then practice those procedures through individual classwork and homework, in which they try many more exercises that are very similar to those the teacher just demonstrated. If any applications of these procedures to real-world problems are included, they are briefly stated, straightforward "word problems" presented immediately after the procedures that students are expected to use to solve the problems.

The social culture of the traditional classroom includes the agreement that the teacher and the answer key in the textbook are the sole mathematical authorities. Students who develop proficiency in using the procedural strategies given in the textbook and demonstrated by the teacher are rewarded with praise and high grades. The nature of the students' thinking and the strategies, both mathematically valid and invalid, that they may have tried for solving problems are generally of much less interest than getting the right answer using the method shown in the textbook.

The most unfortunate consequence of instruction of the sort just described is that too often students leave school with at best a command of a set of facts, procedures, and formulas that they understand in a superficial or disconnected way. Even worse perhaps, they have little or no notion of how they might use what they have learned as they pursue their lives outside of school.

The chapters of this book together describe in some detail the characteristics of a classroom system called "teaching mathematics through problem solving," in which the main goal is for students to develop a deep understanding of mathematical concepts and methods. The key to fostering students' understanding is engaging them in trying to make sense of problematic tasks in which the mathematics to be learned is embedded. In addition to the mathematics that is the residue of work on the tasks, the kind of sense making and problem solving in which students engage involves doing mathematics. As students attempt to solve rich problem tasks, they come to understand the mathematical concepts and methods involved, become more adept at mathematical problem solving, and develop mathematical habits of mind that are useful ways to think about any mathematical situation.

This approach to classroom instruction involves much more than finding and using a collection of "fun" problems. First and foremost, the problematic tasks that are chosen must have embedded in them the mathematics that is to be learned. Second, the tasks must be accessible and engaging to the students, building on what they know and can do. Third, the teacher's role is very important in ensuring that the classroom norms are supportive of students' learning in this way and in pressing students to think deeply both about their solution methods and those of their classmates and, more important, about the mathematics they are learning. Teachers also have a role in ensuring that students have access to appropriate technological and intellectual tools for learning, including facility with important paper-and-pencil procedures. A final challenge for teachers and curriculum developers is to find ways to ensure that the understanding that comes from learning mathematics through problem solving is accessible to all students.

This volume focuses on grades 6 through 12 mathematics, and its companion volume deals with the elementary grades. The organization of, and the issues discussed in, the two volumes are similar, reflecting the overlap of teaching issues across all grade levels.

This volume consists of three main sections—Issues and Perspectives, Tasks and Tools for Teaching and Learning, and In the Classroom—and a final chapter that presents a research perspective on teaching mathematics through problem solving. No single section addresses the entire set of issues concerning teaching mathematics through problem solving, but the volume as a whole presents much of what we in the mathematics education profession know about, and have experienced with, the topic.

Section 1 (chapters 1–4) deals with the conceptual and historical background of teaching mathematics through problem solving. In chapter 1, James Hiebert and Diana Wearne discuss understanding mathematics, why understanding is such an important goal, how engaging in problem solving can lead to understanding, and what some of the signposts are of classrooms designed to promote understanding. The authors close their chapter with a discussion of the fundamental change that is required to move to teaching through problem solving from direct instruction and a traditional curriculum that focuses on procedural skill.

According to Jeremy Kahan and Terrence Wyberg in chapter 2, mathematics helps students solve problems, and in the process of making sense of problems, they come to an understanding of the related mathematics. Following a task that illustrates this sort of sense making, the authors discuss two main benefits of teaching mathematics through problem solving: (1) students take part, at their level of sophistication, in doing mathematics, and (2) they develop understanding of, and interest in, mathematics. The authors also strongly caution that the intended mathematics is the most important focus of planning for, and teaching through, problem solving.

Kenneth Levasseur and Al Cuoco, in chapter 3, argue that through problem solving, students can and should not only understand the underlying mathematics but also learn habits of mind that transcend topic knowledge and that are essential aspects of doing mathematics. The authors discuss eight habits of mind that are especially relevant in grades 6–12 and offer concrete examples and suggestions for developing these habits when teaching mathematics through problem solving.

In the next chapter, Beatriz D'Ambrosio takes us on a trip from ancient to modern times to take a look at how conceptions of problem solving and the role it plays in the mathematics curriculum have changed over time. She notes that problem solving has been an important component of the school mathematics curriculum for at least 150 years and argues that teaching mathematics through problem solving began to emerge rather slowly and has recently begun to appear in some school mathematics textbooks.

Section 2 (chapters 5–8) provides different perspectives on how to select and use appropriate tasks and learning tools so that the intended mathematical understanding will result, an important dimension of the system of teaching through problem solving described above. Robin Marcus and James Fey begin chapter 5 by posing four questions to consider when selecting quality tasks. These questions bring out some ideas discussed briefly in previous chapters, namely, that the important mathematical ideas and methods must be embedded in the tasks; the tasks need to be engaging and problematic, yet accessible to the target students; and work on the tasks needs to help students develop their mathematical thinking and habits of mind. Through several examples, the authors also make the very important point that the collection of tasks in a curriculum must build coherent understanding and connections among important mathematical topics.

In chapter 6, Paul Goldenberg and Marion Walter discuss problem posing as a tool for both teachers and students in problem-based classrooms. For teachers who want to enrich a standard, procedural exercise, the authors suggest using problem-posing techniques for asking useful questions about the exercise (e.g., questions about existence and uniqueness and "What if not?" questions concerning various attributes of the given exercise). Similarly, the authors argue, students should be encouraged to ask these same kinds of questions on their own because doing so is a fundamental part of doing mathematics.

Technological tools for teaching mathematics are Rose Mary Zbiek's focus in chapter 7. She argues that many benefits can be derived from using a variety of technological tools in teaching mathematics. Through examples, she shows how these benefits can be realized, emphasizing, in particular, that the use of technology helps make students' thinking more visible to an observant teacher while also allowing students to reflect on their own thinking more conveniently.

Arthur Bakker and Koeno Gravemeijer discuss another use of technology (statistical minitools available for downloading from the Internet) in chapter 8, although the authors' focus is on planning for and teaching the fundamental statistical idea of "distribution" through problem solving. They use Simon's (1995) idea of a hypothetical learning trajectory that involves considering in advance the learning goal, the learning activities, and the thinking and learning in which students might engage. The authors illustrate their points about planning, learning, and technology with excerpts from students' solutions and conversations.

Section 3 (chapters 9–15) focuses on how teaching mathematics through problem solving might play out in the classroom, assuming appropriate use of the tasks and tools described in the previous section. Together, sections 2 and 3 serve to describe how the five dimensions of the classroom teaching system discussed previously in this preface might be thought of when problem solving becomes the means through which students attain understanding of important mathematics. Nearly every author in this volume refers to the importance of the teacher's role, and many chapters touch on aspects of that role in some depth, but the opening chapter of this section by Douglas Grouws provides the most complete description of that complex role. In fact, he "examine[s] the teacher's role in instruction before, during, and after lessons designed to teach mathematics through a problem-solving approach" (p. 129). As a result, the reader sees a fairly complete picture of what a teacher needs to do to teach mathematics through problem solving well.

In chapter 10, Chris Rasmussen, Erna Yackel, and Karen King describe ways for teachers and students to develop and sustain a classroom environment that fosters and promotes teaching and learning mathematics through problem solving. On the way, they focus on two aspects of the classroom environment, (1) supporting and promoting students' explanations and justifications of their activities, and (2) the distinguishing features of mathematical explanations and justifications that can and should become the classroom norm.

An essential ingredient of teaching mathematics through problem solving is "listening" to students as they do mathematics. For Mark Driscoll, the author of chapter 11, listening carefully to students as they attempt to make sense of rich problems can be a powerful tool for teachers. "The impact of consistent, purposeful listening, especially in a problem-based classroom, can be a powerful way to elicit and understand students' deeper thinking and, perhaps, to propel them toward a more generalized way of thinking" (p. 175).

An important part of a system of instruction based on teaching mathematics through problem solving is classroom assessment, the topic of chapter 12, by Steven Ziebarth. He illustrates some ways to align classroom assessment with teaching through problem solving, mainly focusing on assessing students' understanding. The author also draws on comments about assessment, both from secondary school teachers who teach through problem solving and from their students, in a discussion of several recurring assessment issues, including how to assign grades.

Larry Copes and Kay Shager, the authors of chapter 13, address the important question of how to phase teaching through problem solving into a traditional educational environment. They give practical and gentle advice for the traditional teacher who wants to move, perhaps slowly, toward teaching mathematics through problem solving. The areas that these authors address are articulating the mathematics, structuring class sessions, sources of rich problems, working in groups, and changes in the teacher's role.

In chapter 14, Sarah Lubienski and Jean Stilwell use a researcher-teacher team approach as the basis for their discussion of tough issues and promising strategies for teaching mathematics through problem solving to low-SES students. They describe strategies for structuring problem-based classes and providing extra support outside class that have potential for helping low-SES students learn in a problem-based environment, but they caution that questions remain in this area that teachers and researchers must continue to address.

The final chapter in this section switches the focus completely, to gifted and talented high school students in Russia who were taught through problem solving. Nina Shteingold, who was a student in this system, and Nannette Feurzeig describe an interesting approach that differs somewhat from the U.S. version of teaching mathematics through problem solving. For example, sample problem sets look more traditional than the rich tasks in many of the other chapters of this volume. However, a deeper look suggests that the differences are at least partially superficial. The Russian students were new to the content of the problem sets when they worked on them, and the problems were carefully sequenced to build toward the mathematical learning goals.

In chapter 16, Mary Kay Stein, Jo Boaler, and Edward Silver address the question of what research tells us about the feasibility and efficacy of teaching mathematics through problem solving. They summarize research in two areas—(1) the impact on student outcomes of recently designed curricular programs that support teaching mathematics through problem solving and (2) particular ways that problem-solving approaches are enacted in classrooms. The authors conclude that the evidence from research supports the feasibility and efficacy of teaching in this manner, while acknowledging that implementers face many challenges, including how to support teachers through appropriate curriculum materials and professional development.

A special feature of this volume is the inclusion of a collection of Teacher Stories that amplify the perspectives and suggestions offered by the chapter authors. These stories, written by secondary school teachers who teach mathematics through problem solving, serve to illustrate many of the ideas discussed in the chapters. The teachers were asked to write about their experiences with their students in their own classrooms. These stories, therefore, were not created to illustrate exemplary practice, to explain what works, or to tell other teachers how to carry out a particular activity or implement a teaching technique. Rather, they are attempts to capture stories of mathematical activity in real classrooms, accompanied by the teacher's own thoughts. In a sense, the stories bring to life many of the ideas about teaching mathematics through problem solving presented in the other chapters. The stories represent a wide range of classroom settings—large cities, small urban centers, and suburban towns. We are grateful to these teachers for sharing their practice with us. We hope that this collection of teachers' stories, together with the perspectives offered by chapter authors, will provide both the coherence and the clear direction concerning how to teach mathematics through problem solving that teachers have been seeking.

Finally, the conceptualization and preparation of this volume were undertaken by a small team of mathematics educators who thought long and hard about what it might mean to use problem solving "as a vehicle for learning new mathematical ideas and skills" (NCTM 2000, p. 182). Without their very able assistance, this volume would never have been completed. Not only did several of them write chapters, but each of them reviewed drafts of chapters and gave us invaluable feedback whenever we asked for it. We wish to extend our sincerest thanks to these dedicated individuals, the members of the Editorial Panel:

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