Chapter 1
Geometric Measurement

Big Idea 1
Behind every measurement formula lies a geometric result.

> ### Essential Understanding 1*a*
> Decomposing and rearranging provide a geometric way of both seeing that a measurement formula is the right one and seeing why it is the right one.

Big Idea 4
Classifying, naming, defining, posing, conjecturing, and justifying are codependent activities in geometric investigation.

The big ideas that teachers need to understand for teaching geometry in the middle grades are explored in *Developing Essential Understanding of Geometry for Teaching Mathematics in Grades 6–8* (Sinclair, Pimm, and Skelin 2012). With each big idea, the authors delineate two or three associated, subordinate ideas—the essential understandings that support the broader concept. Altogether, Sinclair, Pimm, and Skelin identify four big ideas and ten essential understandings that flesh them out (Appendix 1 provides a complete list). Clearly, middle-grades teachers need to have more than a superficial grasp of these big ideas and their related essential understandings before approaching the teaching of geometry in the classroom. A thorough, robust understanding of each one is necessary to prepare teachers to offer rich, effective instruction to students in grades 6–8.

Because teachers' own understanding must extend beyond the concepts and skills that teachers expect their students to learn, key questions arise for classroom instruction:

- Which of the big ideas and essential understandings outlined by Sinclair, Pimm, and Skelin for teachers are critical for middle-grades students to develop while they explore geometry in a more focused and sophisticated way than they did in the elementary grades?

- At what level should students understand these ideas in the middle grades to be ready for the more advanced work with geometry that they will undertake in high school and beyond?

- How can teachers nurture these ideas and understandings most appropriately and effectively in the middle-grades classroom?

These are the questions that this book sets out to answer.

Research and experience provide invaluable assistance in this endeavor, pointing to a number of pedagogical practices that are particularly effective for developing students' mathematical understanding. Useful practices include the following:

- Challenging misconceptions that students commonly form

- Having students analyze one another's work

- Inviting students to describe and critique the thinking of other, sometimes fictitious, students

- Thoughtfully creating cognitive dissonance in students' thinking

- Purposefully selecting counterexamples for students to consider

These are among the practices that this book emphasizes in sample tasks that illustrate ways in which you can develop your middle-grades students' understanding of the core ideas of geometry.

Although this book is not intended to be a comprehensive treatment of middle-grades geometry, it presents tasks that illustrate how you can help your students develop their own conceptual understanding. Each task is designed to provide a rich learning opportunity for students while offering you a window on their thinking. The insights into their misconceptions that these tasks will give you are intended to show you the ideas—large and small—that they do understand, as well as those that they do not understand, thereby supplying you with the information that you need to shape subsequent instruction and tailor it to your students' needs.

While your students complete and discuss these tasks, you will be able to ensure that they receive the kind of feedback that Hattie (2009) claims is one of the most

important for the development of student learning. He is not describing teacher feedback regarding student performance but rather students' own feedback to one another on what they know and can apply. This feedback not only provides observant teachers with valuable information about students' understanding of concepts and skills but also prompts students to think about what they have done, thus helping them to self-assess their work.

Not surprisingly, designing such tasks and preparing to present them involves one of the key pedagogical practices for supporting the productive use of students' thinking. This is the instructional practice of anticipating students' solutions—successful and unsuccessful—to set up opportunities to assess and advance that thinking (Stein et al. 2008; Smith and Stein 2011).

Chapter 1 explores topics in geometric measurement. It highlights ways to capture student thinking by using critical pedagogical components. These components include, but are not limited to, the following:

- Determining instructional pathways through the selection and sequencing of activities

- Connecting work in ways that illuminate big ideas of geometry

- Generating a series of questions that have the potential to prompt and probe students' thinking

The first topics that we examine are the concepts of area and perimeter, and the core ideas that come into play in this work are Big Idea 1, Essential Understanding 1*a*, and Big Idea 4.

Working toward Big Idea 1, Essential Understanding 1*a*, and Big Idea 4

Geometric measurement typically lies at the heart of middle-grades investigations of geometry as students expand their understanding of geometric shapes and their attributes and measurement. When students arrive in grades 6–8, they have reached a point in their mathematical development where they can use their growing understanding, together with their emerging skill with algebra, to make generalizations that lead to measurement formulas. Big Idea 1, the notion that behind every measurement formula lies a geometric result, is the central idea that supports and guides this work.

In grades 6–8, students no longer routinely make measurements by counting or working directly with numbers as they did in the elementary grades. To understand geometric measurement more abstractly—to make the connection between a

measurement formula and a geometric result—middle-grades students need a powerful tool of geometric comparison: decomposition and composition. As Sinclair, Pimm, and Skelin (2012) express it, "One very powerful means through which such a comparison [without numbers] might occur involves decomposition and re-arrangement" (p. 9). Thus, Essential Understanding 1*a* is a critical component of Big Idea 1 that middle-grades students must grasp to investigate geometric measurement: "Decomposing and rearranging provide a geometric way of both seeing that a measurement formula is the right one and seeing why it is the right one" (p. 9).

Furthermore, Big Idea 4 naturally underpins students' work with geometric measurement in the middle grades. Big Idea 4 is the notion that classifying, naming, defining, conjecturing, and justifying all play roles in any geometric investigation and are inextricably interrelated. As Sinclair, Pimm, and Skelin (2012) explain, "Geometry arises as the endpoint of geometric investigation" (p. 55), and the actions in Big Idea 4 are "central along the way" because "geometric investigation has a continuity that invokes each of these … intellectual actions at some point" (p. 55). Chapter 1 focuses particularly on the importance of justifying conjectures to support middle-grades investigations of geometric measurement.

Geometric measurement is a significant area of study for the middle grades, a fact that is underscored by its place in the geometry domain for grades 6–8 in the Common Core State Standards for Mathematics (CCSSM; National Governors Association Center for Best Practices and Council of Chief State School Officers [NGA Center and CCSSO] 2010). Throughout grades 6–8, students are expected to solve real-world and mathematical problems involving the measurement of core attributes of shapes and solids:

> Grade 6 (Geometry, 6.G)
> **Solve real–world and mathematical problems involving area, surface area, and volume.**
>
> 1. Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems. (p. 44)
>
> Grade 7 (Geometry, 7.G)
> **Solve real–life and mathematical problems involving angle measure, area, surface area, and volume.**
>
> 6. Solve real-world and mathematical problems involving area, volume, and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms. (p. 49)

Grade 8 (Geometry, 8.G)

Solve real–world and mathematical problems involving volume of cylinders, cones, and spheres.

> 9. Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems. (p. 56)

The work of quantifying the attributes of objects in everyday life to make sense of the physical world offers many opportunities for students to meet the expectations of these standards in the CCSSM geometry domain while recognizing their relevance to career options. However, this rich opportunity for application is often ignored. Instead, the typical instructional approach has often been formula-centered, with emphasis on definitions, rules, variables, and dimensions, without adequate attention to students' understanding of these ideas. As Young (1911, pp. 4–5) observed a little more than a century ago,

> The mere memorizing of a demonstration in geometry has about the same education value as the memorizing of a page from the city directory. And yet it must be admitted that a very large number of our pupils do study mathematics in just this way.

Sadly, not much has changed in slightly more than a hundred years, as measured by student performance on major assessments such as the Trends in International Mathematics and Science Study (TIMSS) and National Assessment for Educational Progress (NAEP). Geometric measurement is regularly found to be an area of weakness (Sowder et al. 2004).

Steele (2006) reports that students' misconceptions often lie at the intersection of geometry and measurement. A geometric measurement topic that is especially problematic at the middle-grades level involves the relationships among the measurable quantities of geometric figures, such as area and perimeter.

Interpreting units of length and units of area

The unit of measure is a core concept that underlies both perimeter and area. This is so, in part, because either of these measurements can be found by iterating a unit. For example, the perimeter of a triangle can be determined by identifying a length unit and iterating it around the boundary of the shape. Similarly, the area of a triangle can be found by defining a square unit and iterating it in the interior of the shape. Although the iteration process may not be exact, it can provide at least a good estimate of the measurement in either case. In a broader sense, all measurement requires the integration of spatial and numerical concepts into the unifying idea of an iterated unit, thereby making an emphasis on unitizing critical (Hiebert 1981).

What is more significant for the teaching of area, which usually comes after the teaching of perimeter, is that students tend to confound units of measure for area with units of measure for perimeter. Perimeter, as the measure of the boundary of a shape, is a length, with only one dimension. Unlike perimeter, area, as the measure of the region inside the shape, has two dimensions. Measuring area involves coordinating the measures of the two dimensions of the shape (Outhred and Mitchelmore 2000). In finding the area of a parallelogram, for example, students must coordinate and differentiate two different types of units—linear units when they are finding the length of either of the parallelogram's dimensions or its perimeter, and square units when they are reporting the area enclosed by the parallelogram. In light of students' frequent confusion of linear and square units in the measurement, a critical question for instruction emerges: What would happen if the nature of the unit itself became the focus of consideration in instruction and teachers highlighted the contrast between linear measure and area measure? Linear measurement is an iteration of a linear unit, and area measure is an iteration of an area unit. Reflect 1.1 invites your thinking about this approach in relation to multiplication.

Reflect 1.1

Consider the area of the shape below:
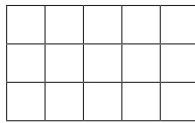
How would thinking of this area of 15 squares as an iteration of a square unit 5 times to make one row unit and then an iteration of the row unit 3 times to make three row units be different from thinking of the same area as length times base?

Are both approaches multiplicative in nature? Why or why not?

An iteration of units suggests an additive approach to measurement because the joining together of the units is indicative of addition. Placing length units on the boundary of a shape fits naturally with perimeter and is closely tied to the additive nature of perimeter formulas. However, covering a shape with square units and counting the number of units to find area does not intuitively align with the multiplicative structure of area formulas.

Multiple researchers—for instance, Battista (2006); Kamii (2006); and Mulligan, Mitchelmore, and Prescott (2005)—have studied students' understanding of unit in relation to perimeter and area. Noticeably fewer researchers have explored the coordination of units across multiple measurement types, such as perimeter and area.

Barrett and colleagues (2011) found that shifting from an additive perspective on units to a multiplicative one requires a purposeful sequencing of tasks that move from qualitative comparisons to quantitative ones, coupled with specific questions focused on the multiplicative relationships.

Outhred and Mitchelmore (2000) proposed a trajectory that moves students from additive thinking to multiplicative structures to develop a relational, rather than an instrumental, understanding of the area formula (Skemp 1978). These researchers found that, for students to progress along this trajectory, they should be able to do the following in sequence (adapted from Outhred and Mitchelmore [2000, p. 161]):

- Completely cover a rectangle by fixed units, without gaps or overlaps

- Spatially structure the units in the array with the same number of units in each row

- Relate both the number of units in each row and the number of rows to the lengths of the sides of the rectangle

- Identify the multiplicative structure of the units in a rectangular array as associated with the number of units in each row and in each column

The last point in Outhred and Mitchelmore's sequence raises an issue: How can students see that this is a multiplicative situation, beyond recognizing that you multiply the dimensions (height × base) in the formula? One way to consider the area of a 3-unit-by-5-unit rectangle is to start with a 1-unit-by-1-unit square, then iterate the square unit 5 times to make one row unit, and then iterate 3 row units to create an area of 15 square units. The area of the 3-unit-by-5-unit rectangle is 15 times larger than the area of the 1-unit-by-1-unit square. The structure of this thinking moves beyond merely substituting the linear dimensions in a formula and instead focuses on the relationships of the unit of measure and the quantity being measured.

Finding perimeter and area, given dimensions

One of the most important responsibilities that teachers have is to create or locate appropriate tasks to advance their students' understanding. Rich tasks can tap into students' thinking and assist them in addressing—and teachers in identifying—their challenges with concepts and procedures that students sometimes understand only in a very fragile way. Such tasks can show both teachers and students what students understand or do not yet understand about the fundamental geometric concepts of perimeter and area. For example, giving students a figure and labeling it in a purposeful way can induce students to demonstrate any weaknesses in their

conceptual understanding. Consider the tasks in figure 1.1 as directed by the questions in Reflect 1.2.

Reflect 1.2

Figure 1.1 shows problems 1 and 2, which seek the area and the perimeter, respectively, of a 5–unit by 12–unit rectangle. The rectangle's dimensions are intentionally labeled differently in each case.

How do you suppose the labeling of the rectangle in these problems might affect a student's approach to the problems?

What misconceptions might the student's solutions indicate?

Problem 1: Find the area	Problem 2: Find the perimeter

```
Problem 1: Find the area          Problem 2: Find the perimeter

            12                                12
      ┌───────────┐                     ┌───────────┐
      │           │                     │           │
   5  │           │  5               5  │           │
      │           │                     │           │
      └───────────┘                     └───────────┘
            12
```
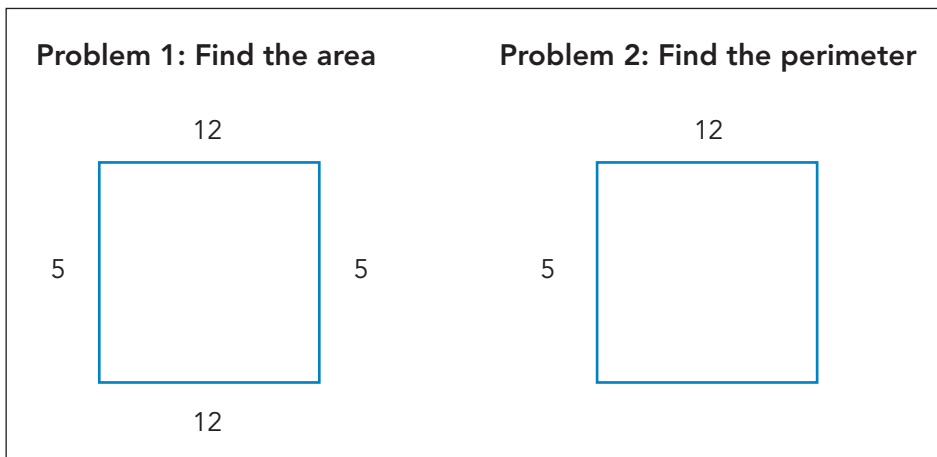
Fig. 1.1. Sample problems with possible labeling of the same rectangle

Samples of student work indicate that the labeling does indeed affect students' solutions. Figure 1.2 shows the work of Anthony, a sixth grader, who attempted to find the area in problem 1 by multiplying all the numbers given by the labels. When asked to explain his work, he said, "I know that when you find area, you have to multiply length and width, so I just multiplied." Clearly, Anthony had difficulty discerning the measures of length and width that he needed. In further discussion, he said that because all the sides were labeled, "That must mean that you have to use all of the measurements."
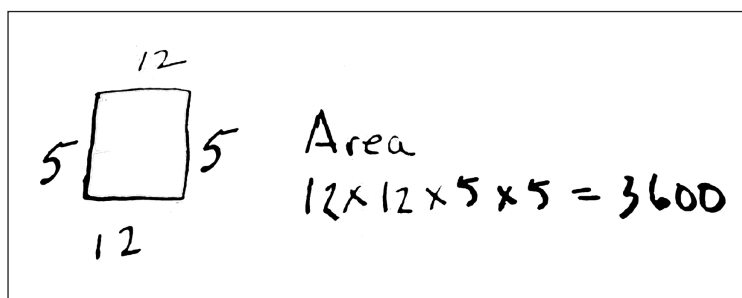
Fig. 1.2. Anthony's (grade 6) solution to problem 1 about the area of the rectangle

Zoie's (a sixth grader) work on problem 2, which seeks the perimeter of the rectangle, is shown in figure 1.3. Using a process similar to that demonstrated by Anthony with respect to the area of the rectangle, Zoie tried to find the rectangle's perimeter by multiplying the two dimensions given by the labels, as shown in the sample of her work in the figure. In oral discussions, she too described the need to use the measurements that were given rather than interpret the meaning of the information in the figure. Neither Zoie nor Anthony considered the reasonableness of their answers. Instead, they focused on selecting and using a formula that seemed to fit the labeling of the shapes.
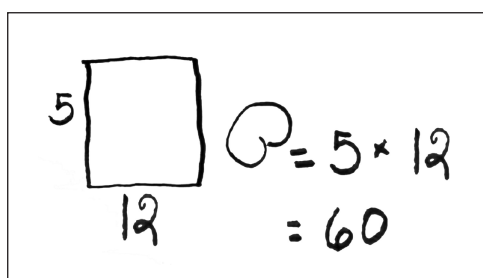

Fig. 1.3. Zoie's (grade 6) solution to problem 2 about the perimeter of the rectangle

The solutions that Zoie and Anthony produced are representative of the various ways in which many students interpret the labeling of figures with respect to a formula. If students are asked to calculate the area of a rectangle that is shown with all four sides labeled, they may opt to use the perimeter formula and find the rectangle's perimeter instead of its area. Or they may seek to apply the area formula but unreflectively. Knowing that this formula involves multiplying a rectangle's dimensions, they may simply multiply all four numbers given. In contrast, if students are asked to find the perimeter of a rectangle that is shown with two adjoining sides labeled, they may resort to using the area formula and find the area instead of the perimeter.

Providing tasks that motivate students to interpret the given information in the context of the problem is important. One of the best ways to do this is to make students stop and think by varying the information given, ranging from giving students the length of one side of a square to giving them rectangles with all four sides labeled in both area and perimeter problems.

When students do not apply a formula or algorithm thoughtfully, deliberately, or prudently, their approach to the formula and their understanding of the relationships expressed in it are more procedural (or instrumental) than relational (Skemp 1978). Stein and colleagues (2008), much like Thompson and colleagues (1994) before them, noted that when the focus of instruction is on the computational aspects of geometric shapes, significant connections between the formulas and the attributes of the actual shapes are missing from students' understanding. By contrast, instruction that varies the information given and focuses the discussion on the relationships between the information given and the formula components can help students develop a stronger conceptual understanding that extends beyond calculations.

Determining the relationship between perimeter and area

While students explore the concepts of perimeter and area, they begin to generalize the relationships between the two. To assist them in this work, they may make comparisons of the perimeters of two shapes if they are given only the areas of the shapes, or vice versa. Consider how students might respond to the task presented in Reflect 1.3.

Reflect 1.3

How do you think your students would respond to the following task:

> **Carmen said, "As the perimeter of a rectangle increases, the area also increases."**
>
> **Do you agree with Carmen? Explain your reasoning. (Adapted from Dougherty [2006, p. 56])**

It is not uncommon for students to think that a direct relationship exists between perimeter and area. That is, they often generalize that if the perimeter of a given shape increases, the area will also increase. This misconception can be addressed by having students engage in tasks that explicitly direct their attention to the relationship. Consider the two tasks, A and B, shown in figure 1.4, guided by the questions in Reflect 1.4.

Reflect 1.4

Figure 1.4 presents tasks A and B, which are designed to elicit and address students' common misconception that the perimeter and the area of a shape are directly related to each other.

What concepts or big ideas might emerge from discussions about the solutions that students found?

How could those concepts or big ideas be used to address students' misconceptions about the relationship between area and perimeter?

Task A

Use square tiles to create as many different rectangles as possible, each with an area of 36 square units. Draw your rectangles on grid paper.

Task B

Use square tiles to create as many different rectangles as possible, each with a perimeter of 24 units. Draw your rectangles on grid paper.

Fig. 1.4. Two tasks that focus on the relationship between perimeter and area

One way of using tasks A and B in the classroom is to ask half your students to work on task A and the other half to work on task B. When students complete the tasks, ask those who worked on task A to find the perimeter of each of the rectangles that they created with an area of 36 square units. At the same time, ask those students who worked on task B to find the area of each of the rectangles that they created with a perimeter of 24 units. When you monitor their work, you can ask students to describe their process of determining the perimeters of their shapes in task A. Their responses indicate the level of their thinking in addition to offering you opportunities to link the physical, material representation with a corresponding symbolic representation and develop the relationship in further class discussion. After your

students have found all the possible rectangles for their task and have determined the perimeter (task A) or the area (task B), you can ask the groups to describe what they notice. Figures 1.5–1.9 show sample sixth-grade students' observations, paired with the students' drawings of their rectangles.
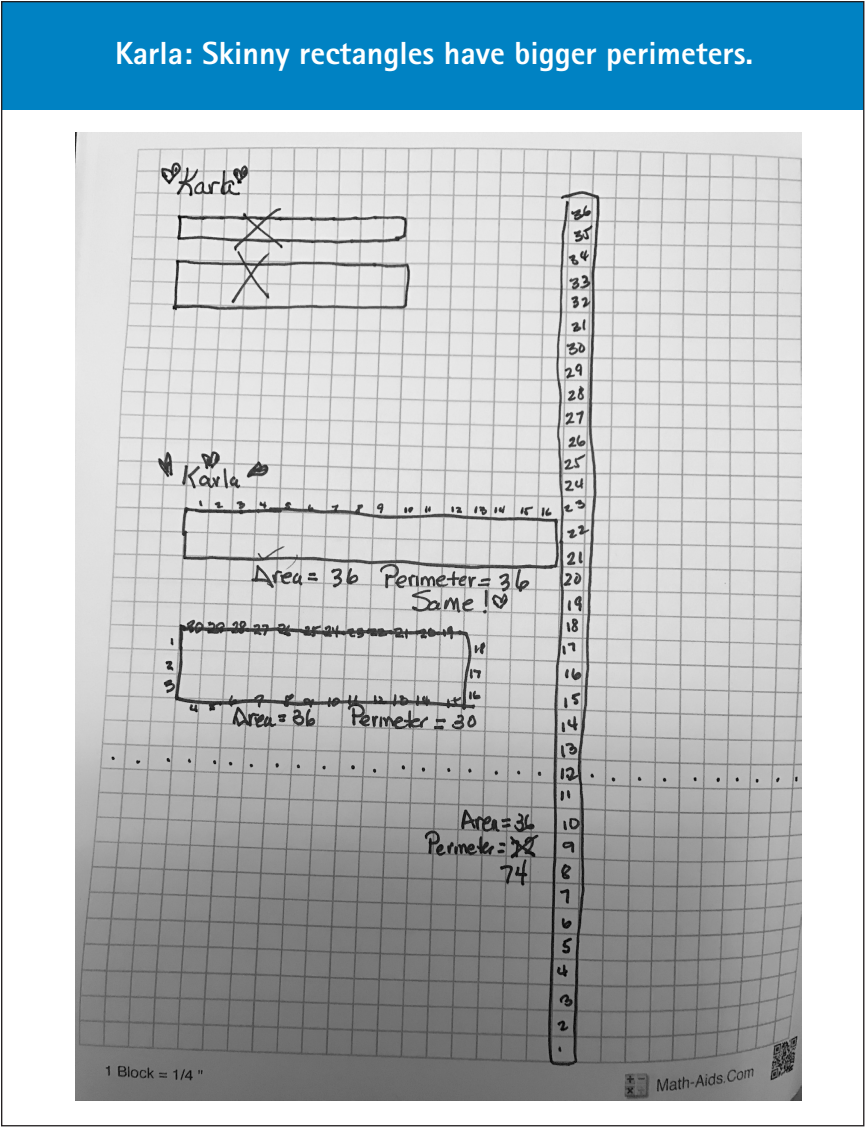

Fig. 1.5. Karla's (grade 6) work and observation on task A

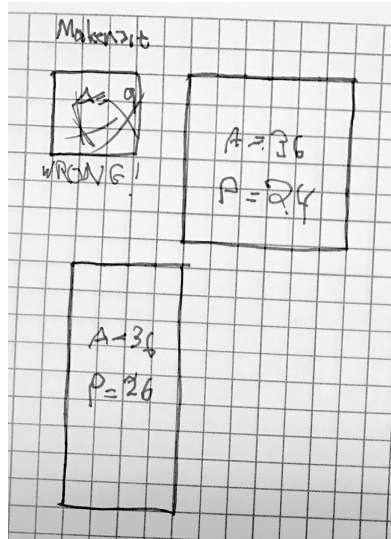**Mackenzie: Fat rectangles have less perimeter but more area.**

Fig. 1.6. Mackenzie's (grade 6) work and observation on task A

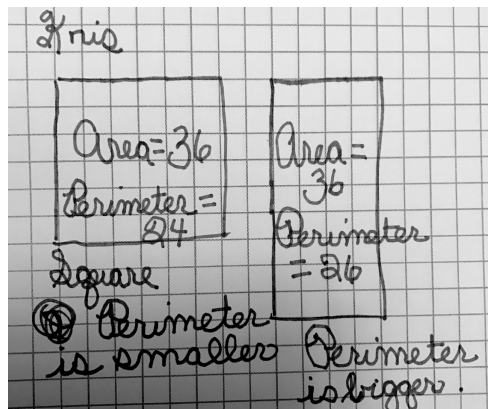**Kris: Squares have smaller perimeters but more area than regular rectangles.**

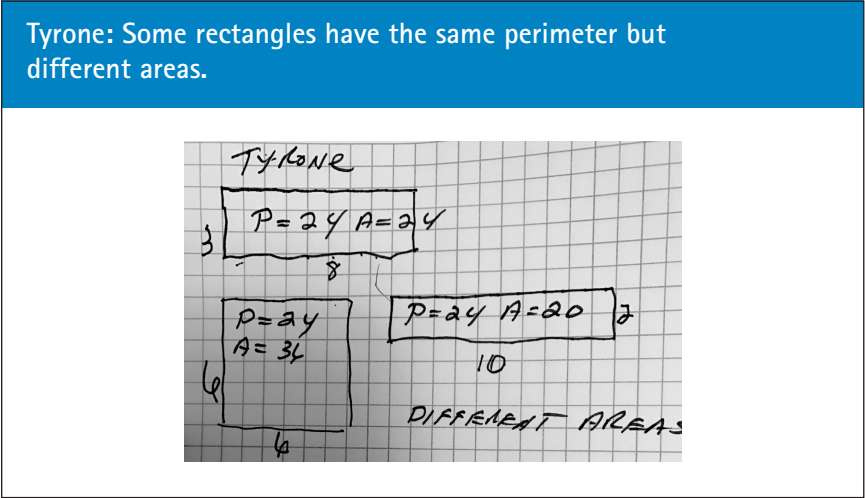Fig. 1.7. Kris's (grade 6) work and observation on task A

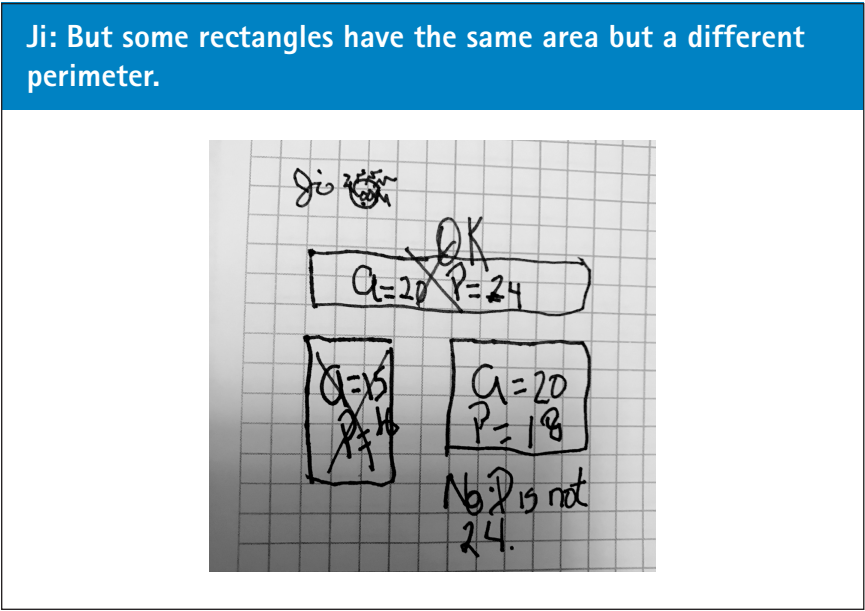Fig. 1.8. Tyrone's (grade 6) work and observation on task B

Fig. 1.9. Ji's (grade 6) work and observation on task B

Conjecturing and justifying conjectures are two key activities that Big Idea 4 identifies in geometric investigation, and the class discussion that ensues as students share their observations should include their justifications for the statements that they make. The rationale or support that the students offer for their observations

may provide a new perspective that will help other students who have not noticed a particular relationship to see it for the first time or gain further insight into it. The class discussion should also incorporate opportunities for students to ask clarifying questions or furnish further support for the pattern that a student is sharing.

Additionally, the discussion should give students an opportunity to share and examine the processes by which they determined the perimeter and area of the rectangles. To consider the potential variety in these processes, examine the two rectangles in figure 1.10, using the questions in Reflect 1.5 to help you think about the multiple ways in which students could have found the perimeter and area of these rectangles.

Reflect 1.5

Figure 1.10 shows two rectangles on a grid of unit squares. Assuming that students are given rectangles that are presented in this manner—

(a) determine at least three ways in which students might find perimeter;

(b) determine three ways in which students might find area.

How do these methods relate to formulas that can be used to find perimeter or area of rectangles?
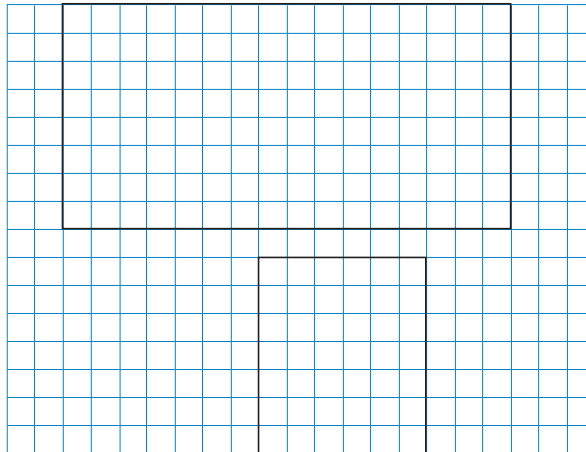

Fig. 1.10. Two rectangles on a grid of unit squares

Students might proceed in multiple ways to find the perimeter of a rectangle like either of those in figure 1.10, composed of unit squares on a square grid. For any of these methods, they would first need to understand that perimeter is a length—the total length of the boundary around a closed figure—and in the case at hand, it is the length of the boundary around the rectangle. These methods include the following:

1. Counting the sides of the unit squares in the grid that adjoin one another to form the boundary of the rectangle.

2. Counting the sides of the unit squares in the grid that adjoin one another to form two adjacent sides of the rectangle and then doubling the count.

3. Counting the sides of the unit squares in the grid that adjoin one another to make up the height of the rectangle, then counting the sides that adjoin one another to make up its base, then adding the two counts, and finally multiplying this sum by 2.

4. Counting the sides of the unit squares in the grid that adjoin one another to form each of the four sides of the rectangle and then adding these four side lengths.

5. Counting the sides of the unit squares in the grid that adjoin one another to make up the height of the rectangle and multiplying this count by 2, then counting the sides that adjoin one another to make up the base of the rectangle and multiplying that count by 2, and finally adding the two products.

Each of these methods can be linked to a symbolic representation, as shown in figure 1.11.

Method	Symbolic representation
Counting the sides of the unit squares in the grid that adjoin one another to form the boundary of the rectangle.	$P = \underbrace{1+1+\cdots+1}_{n}$, where n is the number of sides of the squares around the boundary of the rectangle.
Counting the sides of the unit squares in the grid that adjoin one another to form two adjacent sides of the rectangle and then doubling the count.	$P = (h + b) + (h + b)$, where h is the number of sides of squares on one side of the rectangle (its height), and b is the number of sides of squares on the adjacent side of the rectangle (its base).
Counting the sides of the unit squares in the grid that adjoin one another to make up the height of the rectangle, then counting the sides that adjoin one another to make up its base, then adding the two counts, and finally multiplying this sum by 2.	$P = 2(h + b)$, where h is the height and b is the base of the rectangle.
Counting the sides of the unit squares in the grid that adjoin one another to form each of the four sides of the rectangle and then adding these four side lengths.	$P = s_1 + s_2 + s_3 + s_4$, where s_x is the length of each side of the rectangle.
Counting the sides of the unit squares in the grid that adjoin one another to make up the height of the rectangle and multiplying this count by 2, then counting the sides that adjoin one another to make up the base of the rectangle and multiplying that count by 2, and finally adding the two products.	$P = 2h + 2b$, where h is the height, and b is the base.

Fig. 1.11. Counting methods and associated symbolic representations for determining the perimeter of a rectangle drawn on a square grid of unit squares

Linking the symbolic representation in this way with the physical act of counting to determine perimeter can help students understand why a specific formula or generalization "works." It can give them a deeper understanding of the relationships expressed in a formula, positioning them to apply the formula more appropriately. It helps students interpret their results and determine the reasonableness of their solutions. Additionally, explicitly discussing the multiple techniques for finding perimeter and linking them to the associated symbolic representation gives students multiple strategies to use, helping ensure that if they forget one strategy, they have another one available.

Attending to justification (Big Idea 4)

As emphasized in Big Idea 4, conjecturing and justifying conjectures are inextricably bound up with classifying, naming, defining, and posing—all the activities that intertwine in geometric investigation (Sinclair, Pimm, and Skelin 2012). Reasoning and proof serve several critical purposes in the geometry classroom. At the middle school level, instruction that nurtures students' ability to make conjectures, justify, verify, and explain is essential. You should encourage your students to discuss conjectures—their own and those made by others—and offer justifications for them. Ensuring that they have opportunities to critique the reasoning behind conjectures is part of building a classroom climate that establishes both respect and rigor. Actively and regularly inviting students to make conjectures and justify them are pedagogical practices that are tightly aligned with NCTM's Geometry Standard, which expects instruction in the middle grades to enable students to "create and critique inductive and deductive arguments" (NCTM 2000, p. 232), while it engages them in "making and validating conjectures, and classifying and defining geometric objects" (NCTM 2000, p. 233).

To help students to provide clear justifications for their mathematical conjectures and to support their efforts to shape those justifications into mathematical arguments (including foundational components of theory and proof that they will use in subsequent grades), instruction must move the students away from seeing mathematics as a set of irrefutable rules. Instead, students need to evaluate conjectures and statements and apply the generalizations that they make to other contexts so that they can probe them more deeply.

This pedagogical goal can be accomplished in multiple ways, but a method that you can use to engage your students actively and effectively in the work of examining, explaining, and justifying—or refuting—conjectures is to introduce the work of fictitious, "other" students for their consideration. To do this, you should generate statements to present to your students as the work of "a student in another class" or "other groups of students," and have your students evaluate their accuracy. The statements could include a variety of assertions; below are just two examples:

- "As the perimeter of a rectangle increases, the area also increases."

- "The units used to measure area must be identical in shape to the shape measured—squares must be used for squares, circles for circles, and so on."

Then you should pose such questions as the following:

- "Why did the group make that statement?"

- "How would you respond to their solution?"

- "Can you explain what they are thinking?"

Consistent use of the practice of having students explain the ideas of others can reinforce their awareness of the need for self-examination of their own reasoning and logic.

Use of this teaching practice gives students an opportunity to apply their own conjectures in the process of critiquing the reasoning of other students. It can extend and challenge students' thinking about the relationship between perimeter and area in a different context while giving you meaningful opportunities to assess their understanding.

Extending and assessing students' understanding of perimeter and area relationships

The students whose work on tasks A and B was discussed previously made substantive observations and conjectures regarding the relationships between perimeter and area (see figs. 1.4–1.9). Nevertheless, to internalize these ideas and make them part of an enduring, growing mathematical understanding, students must encounter them and apply them in more than one task. The major points about perimeter and area relationships seem relatively straightforward in tasks A and B. However, calling these points to mind and putting them to work in a new setting are challenging, especially when students are asked to measure shapes that do not offer the uniformity of rectangles or other regular figures. Figure 1.12 shows the Leaf task (Ronau and Gilbert [1988]; adapted by Wilson and Chavarria [1993]), designed to extend and challenge students' thinking about perimeter and area. Examine this task, guided by the questions in Reflect 1.6.
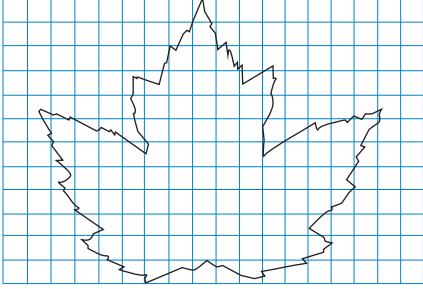
Reflect 1.6

Figure 1.12 presents the Leaf task (adapted by Wilson and Chavarria [1993] from Ronau and Gilbert [1988]).

How does the structure of this task compare with that of the paired tasks, A and B, in figure 1.4?

Suppose that you gave this task to your students. What misconceptions do you think their work on the task might elicit, bringing those misunderstandings to the surface for the students' own examination as well as your assessment?

Russell's group and Jordan's group were asked to find the area of the leaf shown below. Decide whether each group used accurate methods, and explain your thinking.
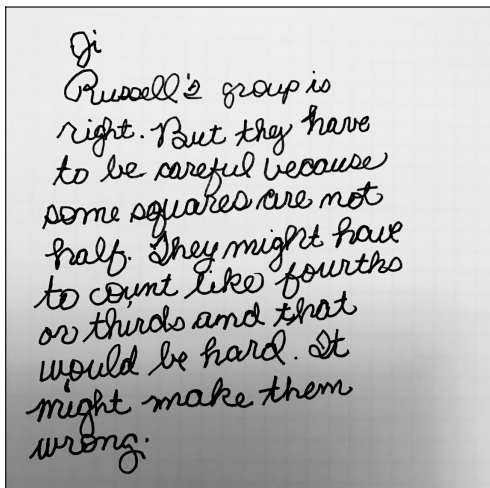

- Russell's group found the area of the leaf by counting all the squares inside the leaf's boundary. His group paired half squares to create full square units and totaled all the square units.

- Jordan's group took a string and placed it closely around the perimeter of the leaf. Then they created a rectangle with the measured string and counted the length and width of the shape in sides of squares and multiplied them to find the area.

Fig. 1.12. The Leaf task, a perimeter and area problem using a nonuniform shape. Adapted by Wilson and Chavarria (1993) from Ronau and Gilbert (1988).

The Leaf task presents an irregular shape drawn on a grid, along with solutions purportedly from two groups of students—Russell's group and Jordan's group. Students completing the task are asked first to decide whether they would expect the methods that the groups used to produce an accurate result and then justify their determination. The methods that Russell's and Jordan's groups used are likely to be similar to those used by students to solve tasks A and B, but the groups are now applying them to a shape that does not have an associated formula. In evaluating the solution strategies offered by Russell's and Jordan's groups, students must draw on their own understanding of the concepts of perimeter and area and the relationships between them.
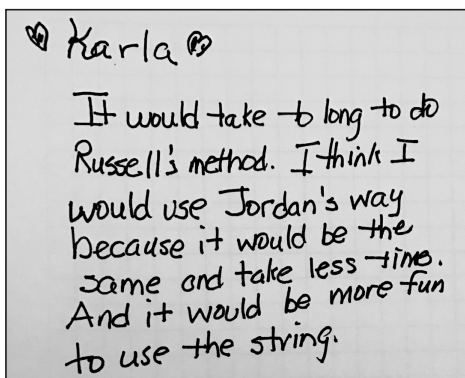
We gave the Leaf task to the same group of sixth-grade students that we had previously asked to examine tasks A and B (see fig. 1.4)—the group that produced the five samples of student work shown in figures 1.5–1.9. Figures 1.13–1.15 show responses to the Leaf task from three of the five sixth graders whose successful work on the more straightforward task is included in the previous group of figures. Ji's work on the Leaf task appears in figure 1.13 (fig 1.9 shows Ji's earlier work),

Karla's work appears in figure 1.14 (fig. 1.5 shows her earlier work), and Kris's work appears in figure 1.15 (fig. 1.7 shows Kris's earlier work). Their conclusions about the accuracy of Russell's and Jordan's groups' solution methods and their justifications of their determinations indicate their varying levels of understanding of the relationships between perimeter and area.

Fig. 1.13. Ji's (grade 6) response to the Leaf task

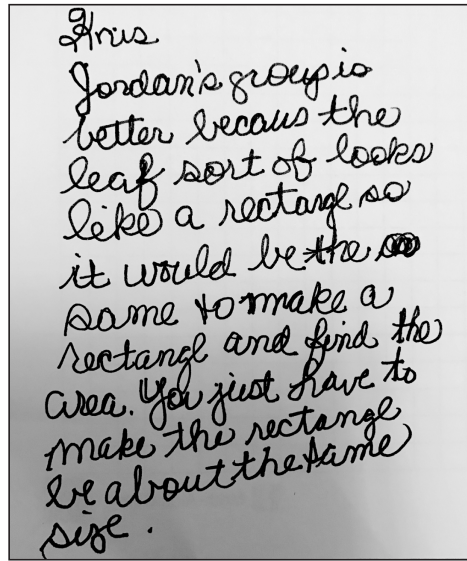Fig. 1.14. Karla's (grade 6) response to the Leaf task

Fig. 1.15. Kris's (grade 6) response to the Leaf task

Note the high value that Karla placed on efficiency relative to accuracy in her assessment of the groups' methods. Her response seems to suggest that she believed that both methods could yield an accurate result, but "it would take to [sic] long to do Russell's method." Students may suggest that the string approach used by Jordan's group is superior to the counting method used by Russell's group for a variety of reasons that either ignore accuracy or assume it for both methods. Their reasons in favor of Jordan's group's method may include more efficient use of time or greater ease of computing or counting. According to Kris, by applying Jordan's "better" method, "You just have to make the rectangl [sic] be about the same size [as the leaf]." Ji was the only student of the three who correctly stated that the counting method used by Russell's group was "right." Nevertheless, Ji pointed to the inefficiency and potential for error in the method:

> But they have to be careful because some squares are not half. They might have to count like fourths or thirds and that would be hard. It might make them wrong.

Students' assessment of this drawback in Russell's group's method may go beyond Ji's. Instead of simply urging care with Russell's counting method, they may conclude that making complete squares as the method calls for from such "messy" pieces along the edge of the leaf is so difficult as to make the method impractical to use or even unlikely to yield, or incapable of yielding, an accurate result. By contrast, because the string method used by Jordan's group beguilingly offers a way to create a "nice" rectangle and facilitates performing a deceptively easy calculation of the area through multiplication, students may identify it as the "better" method.

Clearly, in spite of their observations about perimeter and area relationships in tasks A and B, students tend to be drawn to a method that contradicts the very observations that they made earlier. If, after responding to tasks A and B, your students evaluated Russell's and Jordan's groups' methods and decided that Jordan's group's method was "better," how could you challenge this conclusion and motivate them to reflect on and reassess their thinking? One possibility would be to project the leaf on the whiteboard and use string to measure around its edge. You could then cut the string to give a length equal to the perimeter and tape the ends together to form a loop, which you could show to your students and ask, "If this string is the same length as the perimeter of the leaf, then should it enclose the same area as the leaf?" From those who believe that it should, you could select a student to create a rectangle whose area he or she thought was similar to that of the leaf. Then you could take the loop of string and, by pulling it into a narrow rectangle with very little area, show a counterexample. Some students would almost certainly be surprised, at which point you could ask them to reflect on the observations that they previously made about tasks A and B. This process may motivate them to question why they did not use their thinking from those tasks—specifically, their conclusion that the area cannot be known from the perimeter—to make their decision in the Leaf task.

The fact is that the Leaf task is very challenging—particularly because it offers a measuring approach that students find very inviting, lending itself as it does to efficient use of a familiar procedure and formula. Even though students may have had previous experiences in which they produced multiple figures with fixed perimeters and areas and developed generalizations about the relationship between perimeter and area, they are often unable to transfer that knowledge to an unusual application such as the Leaf task, which calls on them to move beyond a procedural approach. However, confronting students with a task such as this one after they have investigated the previous, more straightforward tasks A and B gives them an important opportunity in a different, more complex context to reinforce their previous observations and reflect that procedural knowledge regarding the use of formulas may not be enough and that building on and applying earlier discoveries and understandings is critical.

Extending understanding to three-dimensional measures

CCSSM first addresses three-dimensional geometric measurement in grade 5, where students are expected to explore "and understand concepts of volume and relate volume to multiplication and addition" (NGA Center and CCSSO 2010, p. 37 [5.MD]). CCSSM expects fifth graders to use unit cubes to pack a rectangular prism without gaps or overlaps to obtain a volume measurement, then progress to the use of standard and improvised cubic units to measure volume, and ultimately develop

the formulas $V = l \times w \times h$ and $V = b \times h$ for the volume of a rectangular prism (5MD.3–5).

This initial development of students' understanding of volume at the end of the upper-elementary grades provides the foundation for the gradual expansion of concepts and formulas for other, more complex three-dimensional shapes, such as cones and spheres, in grades 6–8. This middle-grades expansion is elaborated in NCTM's Geometry Standard and CCSSM, both of which expect students to move away from hands-on experiences in packing rectangular prisms with cubes and counting them to more abstract visualizations of three-dimensional shapes in two-dimensional space, with increasing attention to relationships between and general-izations about surface area and volume. As NCTM's Geometry Standard expresses it, students in grades 6–8 should be able to "use two-dimensional representations of three-dimensional objects to visualize and solve problems such as those involv-ing surface area and volume" and should "use visual tools such as networks to represent and solve problems" (NCTM 2000, p. 232). CCSSM traces the arc of this work from grade 6 to grade 8. In grade 6, CCSSM expects students to build on their initial work in grade 5 by using unit cubes of the appropriate unit fraction edge lengths to pack a right rectangular prism with fractional edge lengths (6.G.2). Then they should show that the volume that they measure in this manner is the same as that obtained by multiplying the prism's edge lengths (6.G.2). This work leads, in grade 7, to solving problems "involving area, volume, and surface area of two-and three-dimensional objects" (7.G.6), and ultimately, in grade 8, to students' develop-ment of the formulas for the volumes of cones, cylinders, and spheres and the use of these formulas to solve real-world and mathematical problems (8.G.9).

Surface-area concepts and skills follow directly from students' understanding of the area of two-dimensional shapes as well as from their understanding of nets. Nets—two-dimensional representations of three-dimensional shapes—provide opportunities for substantive spatial explorations as well as opportunities to link a physical model to surface-area formulas. Consider, for example, the task in figure 1.16 while responding to the questions in Reflect 1.7.

Reflect 1.7

Figure 1.16 shows a task that asks students to find all the unique nets of a cube. Suppose that you gave your students this task.

What ideas about a cube do you think their exploration of all the different nets might support?

What ideas about three–dimensional shapes in general might their work help to reinforce?

Below is a net for a cube:
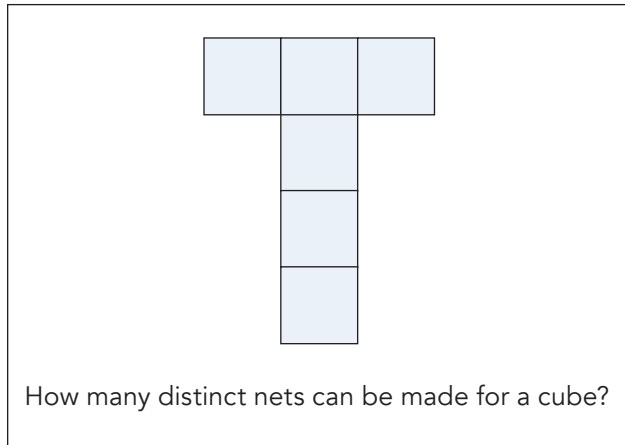

How many distinct nets can be made for a cube?

Fig. 1.16. A task asking students to find all the nets for a cube

Eleven nets can be made for a cube, as shown in figure 1.17. In each of these eleven nets, students can easily see that a cube's faces are six squares. Students' work in finding the nets and examining them can help them see that the surface area of the cube is the sum of the areas of the squares, or six times the area of one of the squares. Similar explorations with nets of other three-dimensional shapes can result in the development of a better understanding of surface area.
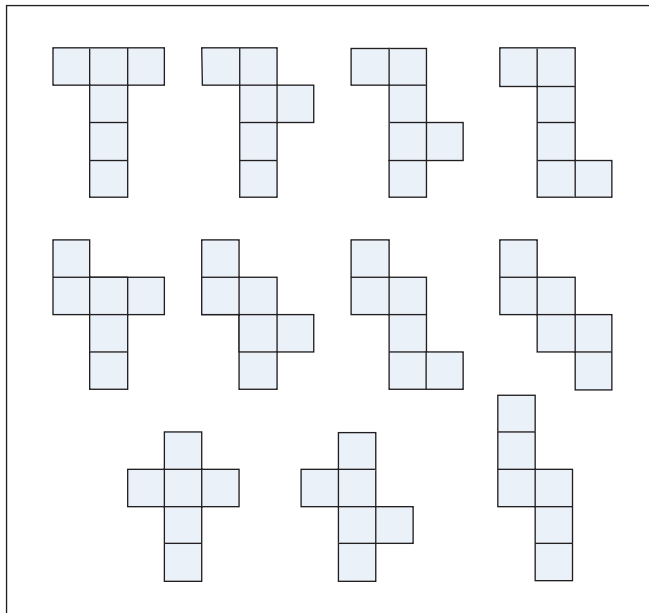

Figure 1.17. The eleven unique nets of a cube

Grasping the concept of the surface area of a three-dimensional shape requires the application of ideas about the area of a two-dimensional shape and recognizing the direct link between the two. A net physically models area, showing all the faces of a three-dimensional shape and making the calculation of the shape's surface area straightforward and obvious.

However, finding volume is not quite as obvious. Just as computing area requires students to coordinate two dimensions—and two length measures—to produce a two-dimensional measurement, applying the volume formula requires students to coordinate three length measures to produce a three-dimensional measurement. For some students, this process is not intuitive, and their difficulty in conceptualizing it often results in their applying formulas for volume in a procedural fashion rather than with understanding.

Battista (2003) describes four processes that students must use to become competent in working with volume, as detailed in figure 1.18. These include (1) forming and using mental models, (2) attending to spatial structuring, (3) engaging in what Battista calls "units locating," and (4) organizing by composites. Students' internalization and mastery of these four processes rely on their having the many and varied experiences with physical models that they need to develop them fully.

Process	Example
Forming and using mental models	Visualizing and reasoning about situations
Spatial structuring	Abstractly identifying, interrelating, and organizing objects' components
Units locating	Mentally locating a cube, for example, within a three-dimensional object
Organizing by composites	Creating composite units from a cube to form a new unit

Figure 1.18. Battista's four mental processes for competence in working with volume (2003)

Battista's four processes are a foundation for subsequent, more sophisticated thinking and visualizing related to volume. Having multiple experiences in measuring volume enables students to develop and use these processes fluently, and this competency supports them in achieving the larger, more general, more abstract insights that Sinclair, Pimm, and Skelin (2012, p. 8) identify as Big Ideas 2 and 3, which address work with imagery and mental objects:

Big Idea 2
Geometric thinking involves developing, attending to, and learning how to work with imagery.

Big Idea 3
A geometric object is a mental object that, when constructed, carries with it traces of the tool or tools by which was constructed.

Essential Understanding 2*c,* one of the three subordinate ideas that Sinclair, Pimm, and Skelin (2012) associate with Big Idea 2, resonates particularly with the four processes that Battista asserts that students must use with competence with volume: "Geometric awareness develops through practice in visualizing, diagramming, and constructing" (p. 8).

Many tasks paralleling the perimeter and area tasks presented in this chapter offer opportunities to expand students' thinking and understanding of measures related to 3-dimensional shapes (surface area and volume). Higher-order questions can push students' thinking, motivating them to create conjectures, test them, and use them to justify other conjectures or observations. Some tasks to consider might include the following:

1. Ben said, "As the volume of a prism increases, the surface area of the prism also increases." Do you agree with Ben? Why or why not? Support your answer.

2. The edge of a cube measures 5 cm. Another cube has an edge length of 10 cm. Without performing any calculations, make a conjecture about the relationship between the volumes of the two cubes. Also make a conjecture about the relationship between the surface areas of the two cubes. Justify your answers.

3. Below are two-dimensional representations of an open-top box:
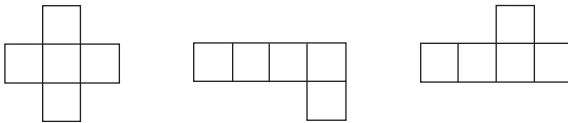
    a. How many ways can you add a square to create a net for a cube?

    b. Find three other two-dimensional representations that will fold into an open-top box.

    c. How many ways can you add a square to your other representations to create a net for a cube?

    d. What observations can you make?

Additional tasks are identified in Appendix 2; see especially the descriptions of articles by Garimella and Robinson (2015); Haberern (2016); Jeon (2009); and Prummer, Amador, and Wallin (2016). Implementing tasks beyond traditional computation problems, which require only an application of a formula, provides opportunities to extend students' thinking. As students apply conjectures and generalizations that originated in their own experiences, they gain a deeper understanding of relationships within these measurement contexts.

Summarizing Pedagogical Content Knowledge to Support Big Idea 1, Essential Understanding 1*a*, and Big Idea 4

Teaching the mathematical ideas in this chapter requires specialized knowledge related to the four components presented in the Introduction: learners, curriculum, instructional strategies, and assessment. The four sections that follow summarize some examples of these specialized knowledge bases in relation to the geometric measurement in connection with Big Idea 1, Essential Understanding 1*a*, and Big Idea 4.

Knowledge of learners

Studies have shown that students' geometric understanding correlates with their spatial visualization and achievement in other areas of mathematics that may not appear to be geometric (de Hevia, Vallar, and Girelli 2008; Stavridou and Kakana 2008). In fact, strong spatial visualization skills have been shown to predict student success in the areas of science, technology, engineering, and mathematics (for example, see Shea, Lubinski, and Benbow [2001] and Wai, Lubinski, and Benbow [2009]).

However, students' experiences from previous grades in geometry may have been less than optimal in developing their understanding, for a variety of reasons. For example, geometric concepts and skills are often taught at the end of the academic year to allow more time to teach number topics. CCSSM includes fewer standards for geometry than for number in the elementary grades, and consequently teachers may perceive geometry as an area that merits less emphasis than the number domains.

Therefore, students entering middle school may be functioning at lower levels of thinking on the van Hiele continuum (Battista 2002; Senk 1989). The van Hiele model includes five levels of thinking, usually numbered 0–4:

- Level 0, Visualization

- Level 1, Analysis

- Level 2, Informal deduction

- Level 3, Formal deduction

- Level 4, Rigor

Thinking at each level is progressively more sophisticated than at the previous level, culminating in thinking at level 4, rigor, at which point students are capable of operating within an axiomatic system and making relatively abstract deductions. The goal in grades 6–8 is to move students at least to level 2, informal deduction, at which point they can identify and work with relationships among shapes and their properties.

However, students arriving at the middle grades may be functioning only at level 0, visualization, where they are limited to naming shapes. They may be able to name some properties of the shapes but lack confidence about the characteristics that are necessary or sufficient for those shapes. If they have not developed beyond visualization, the way in which they approach a task is likely to be to focus on its visual aspects. This can result in immature or overgeneralized observations or rules. For example, they may leap to the generalization that as the perimeter of a rectangle increases, the area of the rectangle also increases.

It is thus important to recognize that students may bring to the classroom a diverse set of experiences. More instructional time may be needed so that students have significant experiences that can move their thinking from a rudimentary level to a more sophisticated level that will allow them to consider relationships among geometric shapes and their properties thoughtfully and analytically.

Knowledge of curriculum

Knowing that students come into middle grades with varied experiences in geometry can affect and potentially strengthen your selection of tasks for use in your instructional sequence. Tasks of types that you might otherwise have chosen may seem less appropriate than others in light of this realization. The tasks presented in this chapter offer students opportunities not only to explore perimeter and area but also to confront misconceptions that they may have formed about these foundational concepts. These new encounters enable students to revisit notions that they have imagined to be true in the context of tasks that challenge their thinking. The resulting cognitive dissonance motivates them to resolve the contradictions between previously held ideas and the mathematics that they are seeing, using, and thinking about in the task that they are currently doing.

Tasks that ask students to create multiple solutions function as generalization tasks. As students share their solutions, they have opportunities to observe common characteristics that then lead them to generalizations. For example, task A (fig. 1.4), which asks students to use square units to create as many rectangles as possible with an area of 36 square units, provides them with an opportunity to observe that a rectangle with a fixed area can have different dimensions and thus does not, unless otherwise specified, have a fixed perimeter. In fact, when students use unit squares to create rectangles with an area of 36 square units, they discover that they can create five distinct rectangles with very different dimensions and perimeters: 74 units (1 unit by 36 units), 40 units (2 units by 18 units), 30 units (3 units by 12 units), 26 units (4 units by 9 units), and 24 units (6 units by 6 units). Likewise, task B (fig. 1.4), which asks students to use square units to create as many rectangles as possible with a perimeter of 24 units gives them a chance to observe that a rectangle with a fixed perimeter can also have different dimensions and thus does not have a fixed area. The students' work on the task allows them to discover that, by using square tiles, they can create six distinct rectangles with perimeters of 24 units but with very different areas: 11 square units (1 unit by 11 units), 20 square units (2 units by 10 units), 27 square units (3 units by 9 units), 32 square units (4 units by 8 units), 35 square units (5 units by 7 units), and 36 square units (6 units by 6 units).

Both the richness of the tasks and the sequencing of them in the classroom are critical to the success of these kinds of learning opportunities for students. One fairly typical characteristic of a rich task is a call for students to create multiple cases, as in "create as many different rectangles as possible, each with an area of 36 square units" (task A) or "create as many different rectangles as possible, each with a perimeter of 24 units" (task B). Creating these multiple cases allows students to take note of and internalize relationships between perimeter and area. Specifically, they should note that knowing the perimeter of a rectangle does not mean knowing its area, and, conversely, knowing the area of a rectangle does not mean knowing its perimeter. These insights then lead to a realization that an increase in the area of a rectangle does not necessitate an increase in its perimeter. Notice that these tasks are neither complex nor difficult; they are accessible to all students, but the solutions to the tasks provide opportunities to move the simple task to a much deeper level.

The sequencing of the tasks also is linked to their richness. In using rich tasks, teachers do not begin with skill development. Initial tasks do not focus on the formulas for geometric measures. Instead, early tasks begin with the conceptual aspects related to perimeter and area, allowing students to explore geometric measurement in accessible tasks. Later lessons, beyond the scope of this chapter, can focus on the skill of using a formula for finding area and perimeter.

Knowledge of instructional strategies

As Young (1911) remarked more than a century ago, geometry instruction has often included tasks that focus on memorization of definitions and procedures. This chapter, however, has presented tasks that emphasize student exploration and sharing of both solutions and solution strategies. Tasks such as the Leaf task encourage students to share their thinking in a very natural way and critique the thinking or reasoning of others. Incorporating tasks such as these requires that you use instructional strategies effectively to optimize the learning outcomes.

Lamberg (2012) provides strategies that can be used to facilitate class discussions effectively. She notes that small-group and whole-class discussions can lead to (1) a common or shared understanding of a problem and its solution, (2) the development of students' metacognition through teacher questioning, and (3) opportunities for students to evaluate and analyze their own and their peers' reasoning. According to Lamberg, the facilitation of the discussions should then be thought of as having three phases:

> Phase 1: Making thinking explicit
>
> Phase 2: Analyzing solutions
>
> Phase 3: Developing new mathematical insights

In each phase, teacher questioning provides the impetus for the discussion. However, as students become accustomed to a discourse-based classroom, their questions also move the discussion forward because they begin to emulate the teacher's questioning. Those questions will then lead to their own questions.

Smith and Stein (2011) identify five practices that teachers can use to promote productive discussions. These practices are as follows:

- Anticipating: Solve the problem yourself beforehand, thinking about how your students are likely to solve it and determining the mathematics that is embedded in it.

- Monitoring: Listen and observe students while they solve the problem, identifying strategies that they are using and refocusing them through questions if they are not on track.

- Selecting: Decide what solution approaches or solutions you want to highlight, identifying ones that will further the mathematical thinking.

- Sequencing: Identify the order in which you want students to share their thinking or solutions, determining the method by which students will share.

- Connecting: Ask questions that connect the solutions and the mathematical ideas to highlight, comparing multiple students' solutions and methods.

While students are sharing, they will be thinking critically about the solutions and the solution methods. You will also be analyzing the students' work, using their solutions as a way to assess the level of their thinking and the depth of their understanding.

Knowledge of assessment

The tasks presented in Chapter 1 provide opportunities for you to assess student learning through their sharing of solutions and discussions. According to Wiliam (2007), tasks that provide such opportunities can function as types of formative assessments, enabling teachers to make instructional decisions that are based on the understandings and misunderstandings that come to the surface in the students' work and their discussions of it. You can gather information about their thinking, both while you circulate as they work on the task and while you attend to their approaches and solutions in whole-class discussions after they have completed the task. The use of reversibility, flexibility, and generalization tasks, as described by Dougherty (2001) and illustrated in the Introduction, enhance the information gained from student solutions.

By presenting tasks that prompt students to solve in multiple ways or find multiple solutions, you can determine the depth of their understanding with greater precision. The questions that you can then create to extend the problem or explore or probe student thinking can provide you with further evidence of the way in which students have constructed their knowledge about geometric measurement. These windows on their understanding can also show you limitations of their thinking that may hinder their ability to use particular approaches to solving the tasks.

It is worth emphasizing that you need to determine what your students *don't know,* as well as what they do know. In addition to giving you insight into the depth of their understanding, tasks of the kind presented in Chapter 1 allow you to discern pervasive student misconceptions that need further explorations. Explicitly addressing misconceptions by providing tasks that create cognitive dissonance or that make a misconception evident is critically important for advancing students' understanding.

Conclusion

Chapter 1 has demonstrated possibilities for designing instruction that can provide optimal opportunities for students to develop an understanding of geometric measurement that goes beyond rote memorization. Effective instruction requires that you understand your students as learners, identify and implement rich tasks, and use student responses to the tasks to make instructional decisions. The tasks presented in this chapter have provided some examples of tasks that can support your teaching of geometric measurement.

Chapter 2 continues to highlight big ideas and essential understandings from *Developing Essential Understanding of Geometry for Teaching Mathematics in Grades 6–8* (Sinclair, Pimm, and Skelin 2012). Another major strand of middle-grades geometry becomes the focus in that chapter: transformational geometry.