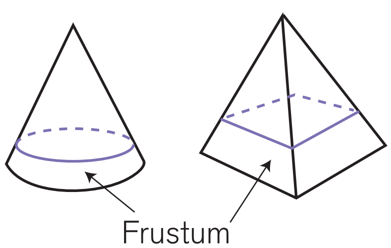
NAVIGATING *through* MEASUREMENT

Chapter 4

Classic Examples of Measurement

The lower part of a cone or pyramid cut by a plane parallel to the base is the frustum.



"In addition to reading measurements directly from instruments, students should have calculated distances indirectly and used derived measures."
(NCTM 2000, p. 321)

Throughout history, scientists and mathematicians have derived formulas and procedures that have built on simpler, easier measures to achieve measures that are more difficult or complex. Historians still wonder how the ancient Egyptians developed a formula for the volume of the frustum of a pyramid—a formula whose rigorous proof requires limit arguments akin to those at the heart of calculus. Whether the stories about Thales (ca. 640–540 B.C.) are myth or fact, they celebrate many brilliant measurement feats, including the use of shadows to measure the heights of pyramids and the application of proportional reasoning associated with similar triangles to measure the distance from shore to ships at sea.

The list of inspired uses of relatively simple mathematics to make other, more complicated measurements—sometimes with a surprising degree of accuracy—is impressive. This chapter selects examples from the list for you to share with your students, in keeping with the recommendation of *Principles and Standards for School Mathematics* (NCTM 2000) that teachers help their students experience the power of mathematics through indirect measurement.

The activities in this chapter explore intriguing settings from the history of mathematics and show how these contexts can engage high school students in the process of measuring and the analysis of measurement error. Three activities highlight some ancient methods and mathematical models for measuring sizes and distances pertaining to the earth, moon, and sun. These methods and models are early, truly remarkable examples of the use of simple observations and basic mathematics to make challenging measurements.

Standard textbook problems pertaining to measurement can often lead to productive hands-on investigations, which in turn can deepen students' understanding of the mathematics in the problems.

The first activity, If the Earth Is Round, How Big Is It? takes students through the steps of measuring the circumference and diameter of the earth by a method that Eratosthenes invented in the third century B.C. Many high school geometry textbooks illustrate Eratosthenes' method in notes or exercises. Students who have studied basic right triangle trigonometry can use their knowledge to facilitate their work, but the exploration does not depend on trigonometry.

The second activity, Moon Ratios, follows up this investigation of ancient measurements of the earth with an exploration of methods that early astronomers developed for finding the visual angle of the moon and the distance from the earth to the moon. Students explore methods that enabled early astronomers to measure important ratios—the ratio of the earth-moon distance to the moon's diameter and the ratio of the diameter of the moon to the diameter of the earth. An acquaintance with right triangle trigonometry can expedite students' work in the activity but is not essential to it.

The third activity, How Far Is the Sun? lets students put elements of the strategies from the first two activities together to make estimates of the distance of the sun from the earth. This activity depends on a basic trigonometric fact about right triangles—that the ratio of either acute angle's adjacent side to the triangle's hypotenuse is equal to the cosine of the angle. Thus, students who are acquainted with basic right triangle trigonometry will understand the mathematics that is involved. However, if teachers supply relevant information, students can complete the activity without a grounding in trigonometry.

If the Earth Is Round, How Big Is It?

Goals

- Simulate a classical measurement process
- Analyze measurement error resulting from the measurement instrument

Materials and Equipment

For each student—
- A copy of the activity sheet "If the Earth Is Round, How Big Is It?"

For each group of three or four students—
- One or more Styrofoam balls (4–6 inches in diameter)
- Two straight pins (1.25–2 inches in length)
- A flashlight or pen light. (The class can share a movable bright light, if necessary.)
- A short tape measure (calibrated to millimeters; template provided)
- A 4-by-6-inch index card, scissors, and clear tape

For students who have not studied trigonometry—
- Centimeter grid paper (template provided)
- A protractor

For the class—
- A road atlas of the United States that students can use to find the distance (as the crow flies) from Bozeman, Montana, to Tucson, Arizona

Discussion

In the third century B.C., the Greek mathematician Eratosthenes devised a method for measuring the circumference of the earth. He measured the angle of the shadow cast by a vertical stick in Alexandria at noon on the summer solstice. The measure that he found for this angle (shown as $\angle TBA$ in fig. 4.1) was approximately 7.2 degrees. (The system of degrees that we use today to measure angles came into being after the time of Eratosthenes, but we can use degrees in carrying out his reasoning and method.)

Eratosthenes happened to know that on the same day, at the same time, due south in the town of Syene (S in fig. 4.1), the sun was directly overhead, and a vertical stick would cast no shadow. He also knew that the distance from Alexandria to Syene was approximately 5000 stades, or about 500 miles. Eratosthenes assumed that the sun's rays were parallel; hence, $\overline{TB} \parallel \overline{OS}$ in figure 4.1. Thus, $m\angle ABT = m\angle BOS \approx 7.2°$ Since 7.2° is equal to 1/50 of a circle, Eratosthenes concluded that the earth's circumference was 50 × 5000 stades, or 250,000 stades.

The size of a stade has been a subject of debate. One possible value, 559 feet to a stade (Eves 1990), would make Eratosthenes' estimate of the earth's circumference 139,750,000 feet, or 26,468 miles—a value

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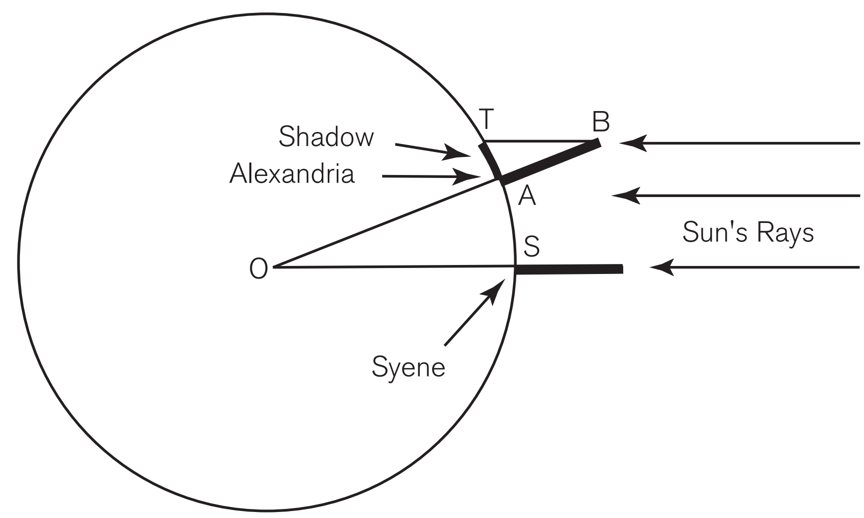
The template "Measuring Tape" on the CD-ROM enables you to print and cut out short paper tape measures, calibrated to millimeters, for your students' use.

You can print grid paper for this activity from the template "Centimeter Grid Paper" on the CD-ROM.

Fig. **4.1.**

The geometry of Eratosthenes' measurement of the earth's circumference (not to scale)



that is within a few percentage points of 25,000 miles, often given today as the earth's circumference. Using 559 feet per stade would make Eratosthenes' estimate of the earth's diameter 8,425 miles as compared with the contemporary measurement of the earth's polar diameter as 7,912 miles. The accuracy of Eratosthenes' estimates is rather amazing, considering the simplicity of his method and the absence of accurate instrumentation and sophisticated measurement systems in the third century B.C.

Like many geometry textbooks, the activity sheet presents a historical note explaining how Erastothenes (ca. 250 B.C.) measured the circumference and diameter of the earth. Using Erastothenes' ideas and data as starting points, students replicate his method and obtain his measurements in step 1 of the activity.

In step 2, students transfer Eratosthenes' method to the task of measuring the circumference and diameter of a ball. Equipped with a Styrofoam ball, a short paper tape measure, two straight pins, a bright light, an index card, scissors, and tape (and perhaps some grid paper and a protractor), students design a simulation of Eratosthenes' method. Their simulation should lead them to an accurate estimate of the diameter of the ball.

By asking students to design their own simulations, the exploration forces the students to think more deeply about the questions that the measurements raise than they would be likely to if they had a step-by-step procedure to follow. Students can be inventive in fashioning the materials to suit their simulations. For example, they might cut a strip from the index card and attach their paper measuring tape to it, stiffening and straightening the tape, as in figure 4.2.

What do the students gain by this? In the simulation pictured in figure 4.2, the measuring tape's straight segment allows the students to obtain a direct measurement of the shadow. By positioning their light source (not shown) so that its rays strike the top of the left-hand pin directly, the students allow the shadow from the right-hand pin to fall onto the straight segment of the ruler. The students can then inspect the right triangle whose legs are formed by the right-hand pin and its shadow. If the students happen to know right triangle trigonometry,

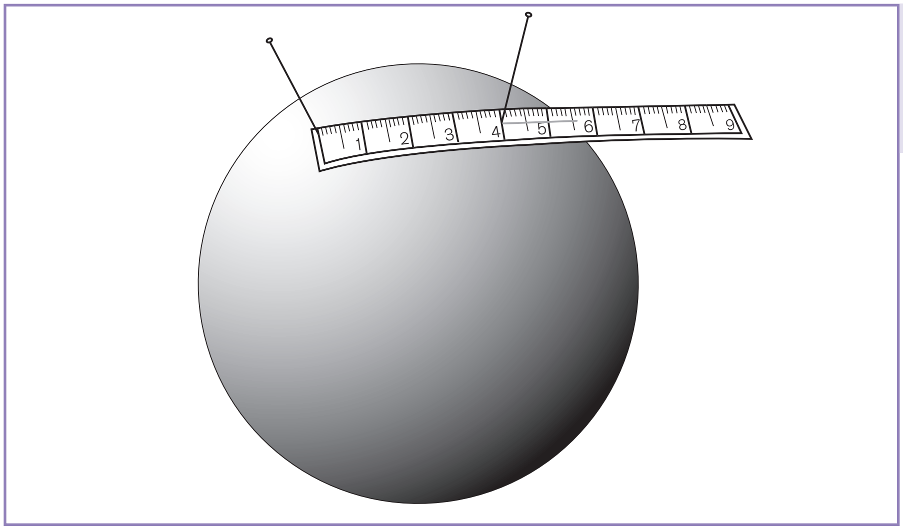


Fig. **4.2.**

A measuring tape extending on a tangent from a pin on a Styrofoam ball in a student simulation

they can compute the angle at the top of the triangle as the arctangent of the ratio of the shadow's length to the pin's height.

Students who have not studied trigonometry might also profitably set up their simulation as pictured in figure 4.2. They could use grid paper (or software for geometric drawing) to create a right triangle congruent to the one whose legs are the pin and its shadow, and then they could measure the angle directly with a protractor.

The activity challenges students to identify sources of error as well as to discuss alternative approaches. For example, if the light source is too close to the ball—say, within five feet—it will not be at all reasonable to assume that its rays are parallel, and any direct measurements that the students make can greatly distort their final, indirect measurement.

As students work with their simulations and make their measurements, they must also focus on the error that arises in working with an instrument (the paper measuring tape) with a particular degree of precision. The ruler provided for the activity allows students to make measurements that are precise to the nearest millimeter—that is, with a round-off error of less than 0.5 mm. This potential error has implications for all the measurements that the students derive from those that they make with the ruler. The solutions in the appendix include a sample showing how students might work out these implications.

Step 3 of the activity extends the students' investigation of Eratosthenes' method by asking them to consider a real-world case in which vertical sticks in two locations *both* cast shadows. Using a road atlas together with the information that Bozeman, Montana, is due north of Tucson, Arizona, students figure out how they could use simultaneous measurements of shadows in the two cities to estimate the circumference of the earth. The situation is sketched in figure 4.3.

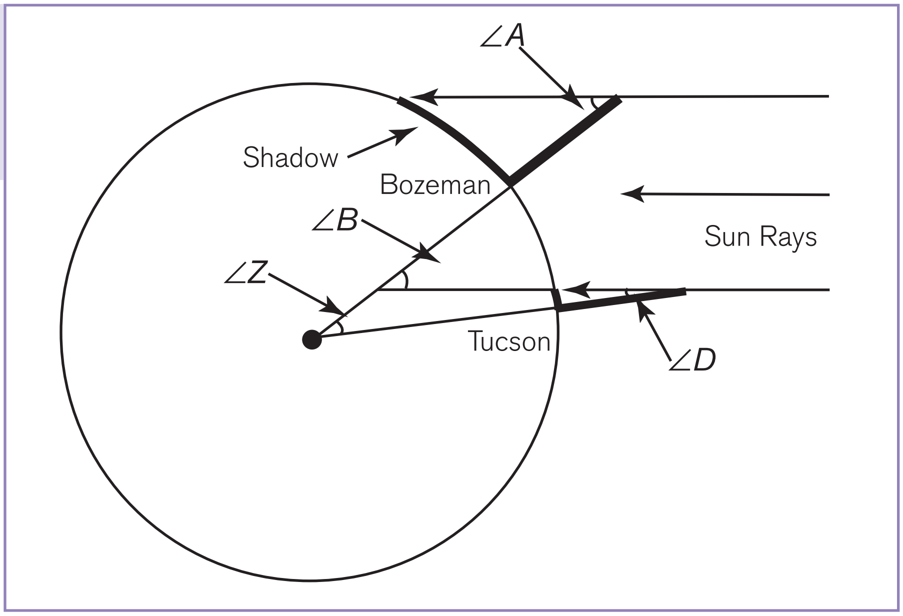
Students must find a relationship between the central angle, labeled as $\angle Z$ in figure 4.3, and the angles of the shadows, shown as $\angle A$ and $\angle D$. Using the Exterior Angle Theorem, students can deduce that $m\angle Z = m\angle A - m\angle D$.

As an alternative approach to the two-shadow situation in step 3, you might prefer to have your students collaborate with those in another mathematics class in a city located on the same longitude line as your city but hundreds of miles away. The students in the two classes could then do the experiment "live" with telephone connections.

"High school students should be able to make reasonable estimates and sensible judgments about the precision and accuracy of the values they report."
(NCTM 2000, p. 322)

Fig. **4.3.**

A variation on Eratosthenes' method of measuring the earth's circumference, with vertical metersticks in two locations casting shadows.



Because $m\angle A = m\angle B$, $m\angle Z = m\angle A - m\angle D$.

Assessment

During the activity, be sure to assess your students' abilities to use the properties of circles and parallel lines to derive angles and arc lengths. You should also check their computation of the uncertainty intervals for their estimates of the circumference of the ball.

Students should gain two important insights from their work:

(1) Indirect measurements of a phenomenon typically depend on the development of a mathematical model, such as those shown in figures 4.1 and 4.3.

(2) These mathematical models usually depend on assumptions about the phenomenon, such as that the earth is a sphere and that the sun's rays are parallel.

Eratosthenes' effort to measure the circumference of the earth counters the myth that everyone thought the earth was flat until the age of Christopher Columbus.

To assess your students' understanding of these essential ideas, you might ask them to write journal entries evaluating the importance of the assumptions that they made in the simulation with the Styrofoam ball in step 2.

Encourage your students to reflect on what would happen to the shadows if various features of the situation were different. For example, how would a light that was very close to a ball affect the shadows of pins in the ball?

Where to Go Next in Instruction

In the history of mathematics, the measurement of the ratio of two quantities was sometimes just as important as, and often a prelude to, the measurement of a specific quantity. One very important ratio that students encounter is π. To the ancients, π was not a number but a ratio of two lengths related to a circle: π = *Circumference* : *Diameter*. The next activity examines some historically interesting ratios whose measurement involved keen observations and basic mathematics.