NAVIGATING *through* MATHEMATICAL CONNECTIONS

Introduction

Connections are at the heart of learning mathematics with understanding. The Learning Principle articulated in *Principles and Standards for School Mathematics* (NCTM 2000) stresses that "students' understanding of mathematical ideas can be built throughout their school years if they actively engage in tasks and experiences designed to deepen and connect their knowledge" (page 21). One result of such understanding is a readiness on the part of students to solve problems in a variety of settings.

"A comprehensive mathematics experience can prepare students for whatever career or professional path they may choose as well as equip them to solve many problems that they will face in the future."
(NCTM 2006, p. 1)

Helping Middle-Grades Students Make Connections

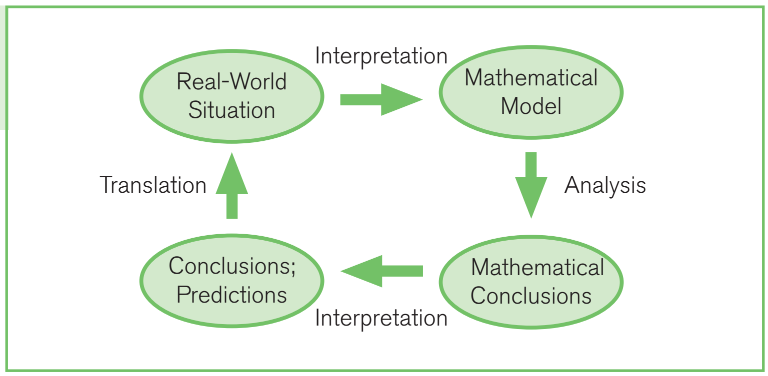
The goal of this book is to share some ideas for helping middle-grades students make connections that will give them the mathematical readiness to think about and solve problems. The book's premise is that activities that engage students in mathematical modeling can promote their discovery of connections and their confidence in themselves as problem solvers.

A brief discussion of *mathematical modeling* can shed light on the indispensable support that modeling gives the Connections Standard. The term *mathematical modeling* has gained popularity and is often applied loosely to any problem-solving activity. However, problem solving and modeling are not identical activities. Although problem solving is at the heart of all mathematical modeling, modeling is not essential to all problem solving.

Mathematical modeling is the use of mathematics to describe an actual phenomenon or event. Modeling is a powerful tool because the resulting mathematical description may be helpful in solving other, related problems based on real-world events. Froelich (2000) summarizes the problem-solving path that mathematical modeling typically takes as “the process of describing real-world phenomena in mathematical terms, obtaining mathematical results from the description, [and] then interpreting and evaluating the mathematical results in the real-world situation” (p. 478).

The activities in this book illustrate these ideas as well as the belief that helping students develop an understanding of mathematical modeling and skills for effective modeling requires that the students make conceptual connections—not only between mathematics and real-world phenomena but also among “big ideas” in mathematics. Figure 0.1 summarizes the steps in the process of mathematical modeling.

Fig. **0.1**.
The process of mathematical modeling



The modeling process begins with the identification of a problem arising in the context of a real-world phenomenon. It continues with the identification of the relevant mathematical factors in the situation and the expression of these factors in mathematical terms. In turn, this mathematical representation, or *model*, allows problem solvers to obtain mathematical results. Next, an analysis of these results in relation to the phenomenon under consideration enables the problem solvers to draw relevant conclusions or predictions about the real-world situation.

Garfunkel, Godbold, and Pollak (2000) discuss the use of this process to solve a problem arising in the lumber industry: A timber company wants to maximize the sustainable yield of lumber for each acre of timber. How does the company do this? Solving this problem calls for developing a model that identifies the best age for harvesting trees and predicts the amount of marketable lumber that any given tree yields, on the basis of its size or volume. Thus, the company needs a quick and efficient way to determine the volume of a tree from particular measurements. The company could use a measurement of the tree’s diameter at the base or at a specified height to develop a model that would give a good estimate of the volume of the tree as a function of the measurement of this dimension. Once the company has a model, data can be selected to test the model and verify its appropriateness. The company would make refinements if the model failed to produce anticipated results with reasonable accuracy.

Froelich (2000; available on the CD-ROM) describes mathematical modeling and offers a modeling activity that lets students investigate ways of improving the efficiency of soft-drink packaging.

Activities such as this one support the three expectations of NCTM’s Connections Standard, which states that “instructional programs from prekindergarten through grade 12 should enable all students to—

- recognize and use connections among mathematical ideas;
- understand how mathematical ideas interconnect and build on one another to produce a coherent whole;
- recognize and apply mathematics in contexts outside of mathematics" (NCTM 2000, p. 64).

The following sections consider each of these expectations in turn, setting each in the context of mathematics instruction for students in grades 6–8.

Students Should "Recognize and Use Connections among Mathematical Ideas"

Mathematical modeling is a process of investigating a problem to discern a mathematical core that provides information about the problem (Garfunkel, Godbold, and Pollak 2000). This mathematical core may merge ideas from several realms of mathematics, including number, geometry, algebra, data analysis, and measurement. Activities that connect multiple mathematical concepts can help students see relationships that transport them beyond a view of mathematics as a collection of unrelated concepts and skills. You can use modeling experiences to make these connections explicit to your middle-grades students.

For example, investigations of length-area relationships can help students make rich connections among mathematical ideas. Teachers often ask middle-grades students to investigate problems that involve them in establishing a maximum area or volume under certain constraints. In one common modeling activity, for instance, students investigate how to maximize the area that a given length of fencing can enclose. Investigating the problem requires the students to understand basic geometric concepts relating to area and perimeter. The collection of data and the development of a model require work with multiple representations, such as graphs and symbols. These representations reinforce the students' understanding of geometric properties, and teachers can use the representations to develop their students' thinking about domain, range, maximum and minimum values of a function, and limits (Day 1995).

See the discussion "Contrasting Linear Relationships with Other Kinds of Relationships" in *Navigating through Algebra in Grades 6–8* (Friel, Rachlin, and Doyle 2001, pp. 52–55) and the activity "Minimizing Perimeter" in *Navigating through Geometry in Grades 6–8* (Pugalee et al. 2002, pp. 73–76) for ideas related to length-area functions.

Students Should "Understand How Mathematical Ideas Interconnect and Build on One Another to Produce a Coherent Whole"

When students lack a strong conceptual understanding, they tend to view mathematics as a set of arbitrary rules. To overcome this tendency, students need experiences that involve them directly in connecting mathematical processes with mathematical concepts. As students progress through school, their work in mathematics should strengthen their abilities to see how similar ideas build on one another and how mathematical constructs apply in different settings.

For example, *equivalence* is a major mathematical concept that acquires more and more meaning as students advance through the mathematics curriculum. Children begin to develop a concept of equivalence in the early years as they explore numeric expressions, shapes, and symbolic expressions. In elementary school, they encounter the equals sign, which they may at first think of simply as a signal to perform an operation, such as addition or subtraction. However, their understanding should quickly extend to the equivalence of the quantities on either side of the sign. In other words, they should soon begin to understand that equivalence is a *relationship*—not an operation.

Students' concepts of equivalence deepen when they explore the equivalence of numerical representations and geometric objects, and their ideas expand again when they move to mathematical justifications and proofs. An understanding of equivalence is central to the development of algebraic thinking and plays a major role in skill in mathematical modeling. Students need many experiences with relationships involving quantities before they begin to grasp equivalence in more abstract contexts dealing with variables and symbolic expressions.

To help your students develop their understanding of the properties of equivalence, let them work with the activities on the Illuminations Web site at http://illuminations.nctm.org/LessonDetail.aspx?ID=U155.

Students Should "Recognize and Apply Mathematics in Contexts Outside of Mathematics"

It is important for students to appreciate how mathematics shapes the world around us. Mathematical modeling gives life to the study of mathematics by enabling problem solvers to develop dynamic mathematical descriptions of events. These descriptions are powerful demonstrations of the role of mathematics in daily experiences and real-world phenomena. Focusing on the applications of mathematics provides opportunities for teaching mathematical concepts that students will need in the workplace and a host of other contexts throughout their lives.

Mathematical modeling activities in middle school should build on mathematical explorations that the students undertook in elementary school. Elementary school mathematics provides numerous opportunities for students to make conjectures or predictions related to real-world events. Students in grades 3–5 are beginning to formulate mathematical descriptions of events that will provide the necessary foundation for increasingly advanced experiences with modeling in grades 6–8.

For example, elementary students frequently study weather patterns, collect data, and make predictions on the basis of their observations. Consider the processes involved when students describe how to predict the best week in the summer for an outdoor camping trip. The development of the students' abilities to reason about relationships is essential to their understanding of how multiple variables interact. Experiences such as this can lay a foundation on which the students can construct mathematical models that capture the complexity of events. Such building blocks are necessary if middle-grades students are to deal with increasingly complex systems that require more and more abstraction and symbolic representation.

Mathematical Modeling and the Teacher

As a teacher of middle school mathematics, you undoubtedly watch vigilantly for opportunities to engage your students in explorations that will foster the development of their ideas about mathematics. Modeling problems provide multiple entry points for learners at various levels, enabling all learners to engage in meaningful mathematics. Although you cannot convert all problems into modeling tasks, some guidelines may assist you in identifying or adapting problems for modeling. Koellner-Clark and Newton (2003) offer the following tips:

- A modeling problem should encourage the development of a model that sufficiently answers the question posed. This model should be based on prior knowledge and should incorporate a sufficient degree of mathematical depth.
- The context of the problem should be realistic and should tie into students' prior knowledge and experiences.
- The model or derived solution should be useful for others in similar situations and should be assessable to the extent that students can determine whether their answers are appropriate and realistic.
- Products and models are actually processes, such that students' mathematical thinking of the concept or idea is usually transparent.
- The solution can be used as a general model for solving similar problems. (p. 430)

Koellner-Clark and Newton developed the modeling problem shown in figure 0.2.

Koellner-Clark and Newton (2003; available on the CD-ROM) describe a modeling problem that motivated high school students to consider multiple solutions, real-life connections, and the usefulness of mathematics.

Fig. **0.2.**

Koellner-Clark and Newton's (2003) formulation of a real-world problem for students to solve by mathematical modeling

The Flower Power Problem

The football team at your high school wants to hire a group to make all the flowers for its float in the school parade. Your service club is interested in this project. The football team is taking bids for the job, and the job will go to the lowest bidder. The work must be done to the team's satisfaction and with quality materials.

For the float, the team wants to cover chicken wire with carnations made as described [at the right]. The area to be covered is 126 square feet. Two-ply Kleenex tissues cost $1.19 for a box of 100 tissues. A package of 100 pipe cleaners costs $2.00.

Your club has twenty members. If you get the bid, you will have a month to work on the project. Some of the members of the club can work on the project after school, but no one can work past 9:00 p.m. on school nights. None of the club members can start work until 4:00 p.m. on school days. No one will work on Friday nights, but some will work on Saturday nights. Everyone agrees to work whenever necessary on Saturday and any time on Sunday from noon until 9:00 p.m. Ten students can work on any weekday afternoon, and five can work on Saturday evenings. All twenty will work from 9:00 a.m. until 5:00 p.m. on Saturdays and from noon until 9:00 p.m. on Sundays, but no one will work for longer than six hours on any given day. You have four weeks and three weekends to do the project. You can hire additional students to work for $7.00 an hour.

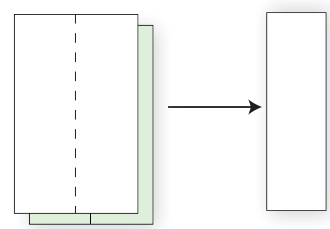
With your group, determine whether the club can do the project and what your bid should be. You want to put in as low a bid as possible, but the club wants to make enough to pay its expenses (materials, $5.00 an hour for members' time, and $7.00 an hour for any outside helpers who might work) and to earn a small profit. Of course, the members will donate their time, so their wage figured into the bid is actually money made for the club.

You can use stopwatches, as well as graph paper and calculators.

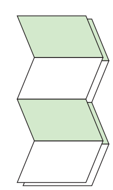
You get a letter from your friend in another town. Her club is thinking of taking on the job of washing windows for the school. She is unsure how to determine what their price should be and whether the club should do the job. Write a letter to your friend, explaining how your club arrived at conclusions regarding the flower project.

How to Make the Flower

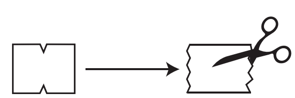
Use two 2-ply Kleenex tissues for each flower. Fit the two tissues on top of each other precisely and fold the two together lengthwise.



Take the folded tissues and fold back and forth from bottom to top, in an accordian pleat.



Now cut small notches on either side at the center and cut ends in a zig-zag (see below).



Then tightly twist a pipe cleaner over and around the center. Carefully pull the layers of tissue apart to make petals.

Teachers can modify many problems to provide mathematical modeling explorations. These problems may lead to short investigations or long ones, depending on the complexity of the problem. It is important to remember that problems that invite modeling involve students in applying multiple mathematical concepts and processes. Therefore, these problems have the potential to address several mathematics learning goals and objectives at once, while allowing students to develop and deepen their understanding of multiple ideas. Because modeling activities have rich applications in mathematics, they hold the potential to engage students more actively in thinking about important ideas than do conventional problems requiring the use of procedures in purely numerical contexts.

Teaching mathematical modeling calls for important changes in classroom practice and approaches to problem solving. Research with elementary students has identified three aspects of conventional classroom problem solving that teachers need to modify to engage students successfully in solving context-based mathematics problems (Verschaffel et al. 2000).

First, the story problems that mathematics lessons typically present follow standard, predictable patterns, and their solutions tend to depend on computations that are straightforward. To overcome these shortcomings, teachers need to offer their students more problems that are nonstandard and embed complex and subtle relationships between mathematics and the physical world.

Heuristic skills enable people to make discoveries and learn for themselves. Mathematical modeling activities invite techniques and processes that have the potential to foster heuristic and metacognitive skills.

Second, to help students approach such problems, teachers must use powerful interactive techniques and processes. These include modeling, coaching, articulating, reflecting, exploring, and "scaffolding"—that is, erecting a structure for learning that allows students to climb with maximum independence from the level that they have mastered to the next level that they are capable of attaining. Mathematical modeling activities invite these kinds of techniques and processes, all of which have the potential to foster heuristic and metacognitive skills.

Third, the classroom culture must move away from traditional classroom rituals and practices that reinforce fragmented views of mathematics. These practices include encouraging students to apply prescribed solution strategies more or less mindlessly to given problems.

Mathematical modeling offers a very powerful and effective response to all these concerns. This book provides some ideas, strategies, and activities that can help you implement such practices, empowering you to guide your students to the kinds of robust mathematical learning experiences that can make them flexible, versatile problem solvers. Moreover, such experiences are essential to giving your students a sophisticated grasp of mathematics as a highly interconnected field, with a myriad of links to other areas of experience as well.