NAVIGATIONS SERIES

GRADES 9–12

NAVIGATING *through* MEASUREMENT

Introduction

Measurement is one of the most fundamental of all mathematical processes, permeating not only all branches of mathematics but many kindred disciplines and everyday activities as well. It is an area of study that must begin early and continue to develop in depth and sophistication throughout all levels of learning.

In its most basic form, measurement is the assignment of a numerical value to an attribute or characteristic of an object. Familiar elementary examples of measurements include the lengths, weights, and temperatures of physical things. Some more advanced examples might include the volumes of sounds or the intensities of earthquakes. Whatever the context, measurement is indispensable to the study of number, geometry, statistics, and other branches of mathematics. It is an essential link between mathematics and science, art, social studies, and other disciplines, and it is pervasive in daily activities, from buying bananas or new carpet to charting the heights of growing children on the pantry doorframe or logging the gas consumption of the family automobile. Throughout the pre-K–12 mathematics curriculum, students need to develop an understanding of measurement concepts that increases in depth and breadth as the students progress. Moreover, they need to become proficient in using measurement tools and applying measurement techniques and formulas in a wide variety of situations.

Components of the Measurement Standard

Principles and Standards for School Mathematics (NCTM 2000) summarizes these requirements, calling for instructional programs from prekindergarten through grade 12 that will enable all students to—

- understand measurable attributes of objects and the units, systems, and processes of measurement; and
- apply appropriate techniques, tools, and formulas to determine measurements.

Understanding measurable attributes of objects and the units, systems, and processes of measurement

Measurable attributes are quantifiable characteristics of objects. Recognizing which attributes of physical objects are measurable is the starting point for studying measurement, and very young children begin their exploration of measurable attributes by looking at, touching, and comparing physical things directly. They might pick up two books to see which is heavier or lay two jump ropes side by side to see which is longer. Parents and teachers have numerous opportunities to help children develop and reinforce this fundamental understanding by asking them to pick out the smallest ball or the longest bat or to line up the teddy bears from shortest to tallest. As children develop an understanding of measurement concepts, they should simultaneously develop the vocabulary to describe them. In the early years, children should have experience with different measurable attributes, such as weight (exploring *heavier* and *lighter*, for example), temperature (*warmer* and *cooler*), or capacity (discerning the glass with the *most* milk, for instance), but the emphasis in the early grades should be on length and linear measurements.

As children measure length by direct comparison—placing two crayons side by side to see which is longer, for example—they learn that they must align the objects at one end. Later, they learn to measure objects by using various units, such as a row of paper clips laid end to end. They might compare each of several crayons to the row and use the results to decide which crayon is longest or shortest. Another time, they might use a row of jumbo paper clips to measure the same crayons, discovering in the process that the size of the measuring unit determines how many of those units they need. Their experiences also should lead them to discover that some units are more appropriate than others for a particular measurement task—that, for example, paper clips may be fine for measuring the lengths of crayons, but they are not practical for measuring the length of a classroom. As their experience with measuring things grows, students should be introduced to standard measuring units and tools, including rulers marked in inches or centimeters.

Children in prekindergarten through grade 2 should have similar hands-on experiences to lay a foundation for other measurement concepts. Such experiences should include using balance scales to compare the weights of objects, filling various containers with sand or water and transferring their contents to containers of different sizes and shapes to explore volume; and working with fundamental concepts of time and learning how time is measured in minutes, hours, days, and so forth—although actually learning to tell time may wait until the children are a bit older. By the end of the pre-K–2 grade band, children should understand that the fundamental process of measurement is to identify a measurable attribute of an object, select a unit, compare that unit to the object, and report the number of units. In addition, they should have had ample opportunities to apply that process through hands-on activities involving both standard and nonstandard units, especially in measuring lengths.

As children move into grades 3–5, their understanding of measurement deepens and expands to include the measurement of other attributes, such as angle size and surface area. They learn that different kinds of units are needed to measure different attributes. They realize, for example, that measuring area requires a unit that can cover a surface, whereas measuring volume requires a unit that can fill a three-dimensional space. Again, they frequently begin to develop their understanding by using convenient nonstandard units, such as index cards for covering the surface of their desks and measuring the area. These investigations teach them that an important attribute of any unit of area is the capacity to cover the surface without gaps or overlaps. Thus, they learn that rectangular index cards can work well for measuring area, but circular objects, such as CDs, are not good choices. Eventually, the children also come to appreciate the value of standard units, and they learn to recognize and use such units as a square inch and square centimeter.

Instruction during grades 3–5 places more emphasis on developing familiarity with standard units in both customary (English) and metric systems, and students should develop mental images or benchmarks that allow them to compare measurements in the two systems. Although students at this level do not need to make precise conversions between customary and metric measurements, they should form ideas about relationships between units in the two systems, such as that one centimeter is a little shorter than half an inch, that one meter is a little longer than one yard or three feet, that one liter is a little more than one quart, and that one kilogram is a little more than two pounds. They should also develop an understanding of relationships within each system of measurement (such as that twelve inches equal one foot or that one gallon is equivalent to four quarts). In addition, they should learn that units within the metric system are related by factors of ten (e.g., one centimeter equals ten millimeters, and one meter equals one hundred centimeters or one thousand millimeters). Students should clearly understand that in reporting measurements it is essential to give the unit as well as the numerical value—to report, for example, "The length of my pencil is 19 centimeters" (or 19 cm)—not simply 19.

During these upper elementary grades, students should also encounter the notion of precision in measurement and come to recognize that all measurements are approximations. They should have opportunities to compare measurements of the same object made by different students, discussing possible reasons for the variations. They should also consider how the chosen unit affects the precision of measurements. For example, they might measure the length of a sheet of paper with both a ruler calibrated in millimeters and a ruler calibrated only in centimeters and compare the results, discovering that the first ruler allows for a more precise approximation than the second. Moreover, they should gain experience in estimating measurements when direct comparisons are not possible—estimating, for instance, the area of an irregular shape, such as their handprint or footprint, by covering it with a transparent grid of squares, counting whole squares where possible and mentally combining partial squares to arrive at an estimate of the total area. In their discussions, they should consider how precise a measurement or estimate needs to be in different contexts.

Measurement experiences in grades 3–5 also should lead students to identify certain relationships that they can generalize to basic formulas. By using square grids to measure areas of rectangles, students might begin to see that they do not need to count every square but can instead determine the length and width of the rectangle and multiply those values. Measurement experiences should also help students recognize that the same object can have multiple measurable attributes. For example, they might measure the volume, surface area, side length, and weight of a wooden cube, expressing each measurement in the appropriate units. From the recognition that multiple attributes belong to the same object come questions about how those attributes might be related. If the side length of a cube were changed, for instance, what would be the effect on the cube's volume or its surface area? Similar questions arise in comparisons between various objects. Would two rectangles with equal perimeters necessarily have the same area? What about the converse? Would two rectangles with equal areas necessarily have the same perimeter? All these measurement lessons should help students appreciate how indispensable measurement is and how closely it is tied to number and operations, geometry, and the events of daily life.

Understanding of and proficiency with measurement should flourish in the middle grades, especially in conjunction with other parts of the mathematics curriculum. As students develop familiarity with decimal numeration and scientific notation and facility in computation with decimals, applications involving metric measurements provide a natural context for learning. As students develop proportional reasoning and learn to evaluate ratios, comparisons between measurements, such as the perimeters or areas of similar plane figures, become more meaningful. Their study of geometry requires students to measure angles as well as lengths, areas, and volumes and lets students see how measurements underlie classifications of geometric figures. For example, they identify triangles as acute, right, or obtuse by evaluating measurements of their angles or classify them as equilateral, isosceles, or scalene by comparing measurements of their sides. Proportional reasoning, geometry, and measurement converge when students create or analyze scale drawings or maps. Algebraic concepts of function that develop in the middle grades have applications in relationships such as that linking distance, velocity, and time. In science classes, students use both measurement and ratios to develop concepts such as density (the ratio of mass to volume) and to identify substances by determining their densities. Through experimentation, they discover that water freezes at 0° Celsius or 32° Fahrenheit and boils at 100° Celsius or 212° Fahrenheit, and from these data they can develop benchmarks for comparing the two scales. (For example, they can see that a ten-degree change in the Celsius temperature corresponds to an eighteen-degree change in the Fahrenheit temperature or that a forecast high temperature of 30° Celsius signals a hot day ahead.)

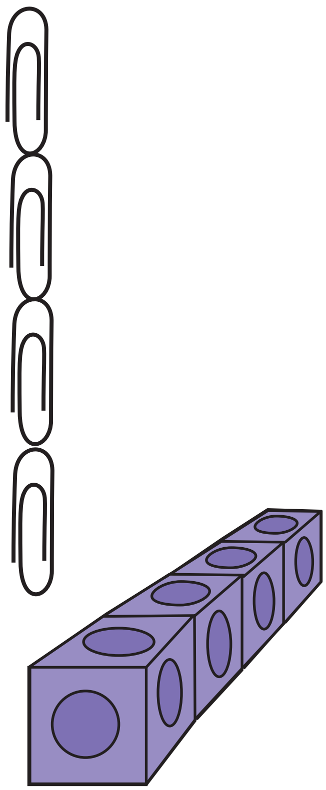
Middle-grades students should become proficient in converting from one unit to another within a system of measurement; they should know equivalences and convert easily among inches, feet, and yards or among seconds, minutes, hours, and days, for example. They should develop benchmarks for both customary and metric measurements that can serve as aids in estimating measurements of objects. For example, they might estimate the height of a professional basketball player as about two meters by using the approximate height of a standard doorframe as a benchmark for two meters, or they might use a right angle as a basis for approximating other angle measurements like 30, 45, or 60 degrees. Although students do more computations of measurements such as areas and volumes during the middle grades than in the earlier years, they still need frequent hands-on measurement experiences, such as tiling a surface with square tiles, making shapes on a geoboard, or building a prism with blocks or interlocking cubes, to solidify their understanding of measurement concepts and processes.

By the time students reach high school, they should be adept at using the measurement concepts, units, and instruments introduced in earlier years, and they should be well grounded in using rates, such as miles per hour or grams per cubic centimeter, to express measurements of related attributes. As they engage in measurement activities during grades 9–12, students are increasingly likely to encounter situations in which they can effectively employ powerful new technologies, such as calculator-based labs (CBLs), graphing calculators, and computers, to gather and display measurements. Such instruments can report measurements, often with impressive precision, but students do not always understand clearly what is measured or how the technology has made the measurement. How a measurement of distance is obtained when a tape measure is stretched between two points is obvious; it is not so obvious when an electronic instrument reflects a laser beam from a surface. Thus, students need a firm foundation both in measurement concepts and in how to interpret representations of measurements and data displayed on screens.

Also during the high school years, students encounter new, nonlinear scales for measurement, such as the logarithmic Richter scale used to report the intensity of earthquakes (a reading of 3 on the Richter scale signifies an earthquake with ten times the intensity of an earthquake with a Richter-scale measurement of 2). Especially in their science classes, students learn about derived units, such as the light-year (the distance that light travels in one year, moving at the rate of $3(10^8)$ meters per second, or about 186,000 miles per second) or the newton (N) (the unit of force required to give an acceleration of 1 m/sec^2 to a mass of 1 kilogram). Students also extend ideas of measurement to applications in statistics when they measure certain characteristics of a sample and use those data to estimate corresponding parameters of a population. Students preparing for a more advanced study of mathematics begin to consider smaller and smaller iterations—infinitesimals, limits, instantaneous rates of change, and other measurement concepts leading to the study of calculus.

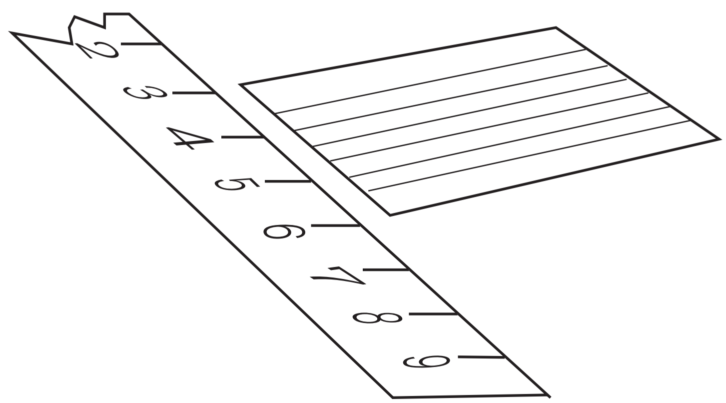
Applying appropriate techniques, tools, and formulas to determine measurements

To learn measurement concepts, students must have hands-on experiences with concrete materials and exposure to various techniques,



such as counting, estimating, applying formulas, and using measurement tools, including rulers, protractors, scales, clocks or stopwatches, graduated cylinders, thermometers, and electronic measuring instruments.

In the pre-K–2 years, students begin to explore measurement with a variety of nonstandard as well as standard units to help them understand the importance of having a unit for comparison. Such investigations lead them to discoveries about how different units can yield different measurements for the same attribute and why it is important to select standard units. For young children, measurement concepts, skills, and the vocabulary to describe them develop simultaneously. For example, children might learn to measure length by comparing objects to "trains" made from small cubes, discovering as they work that the cubes must be placed side by side in a straight row with no gaps, that all the cubes must be the same size (though not necessarily the same color), and that one end of the object that they want to measure must be aligned with one end of the cube train. Later, when they learn to use rulers to measure length, they must learn how to locate the zero on the ruler's scale and align it with one end of the object that they are measuring. When they attempt tasks of greater difficulty, such as measuring an attribute with a unit or instrument that is smaller than the object being measured—the width of their desks with a 12-inch ruler or a large index card, for instance—they must learn how to iterate the unit by moving the ruler or card and positioning it properly, with no gaps or overlaps from the previous position. Furthermore, they must learn to focus on the number of units and not just the numerals printed on the ruler—counting units, for example, to determine that the card shown in the illustration is three inches wide, not six inches.

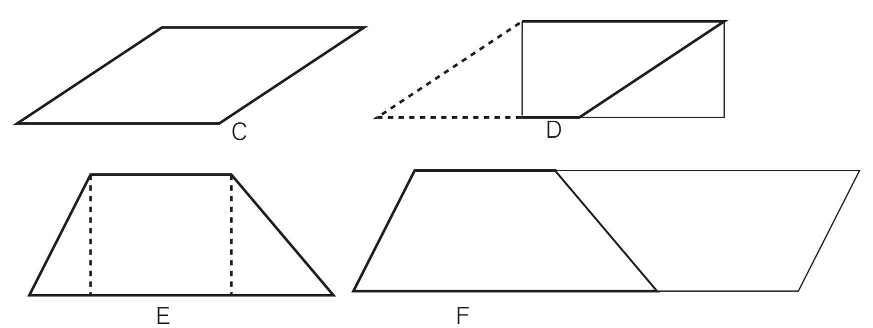


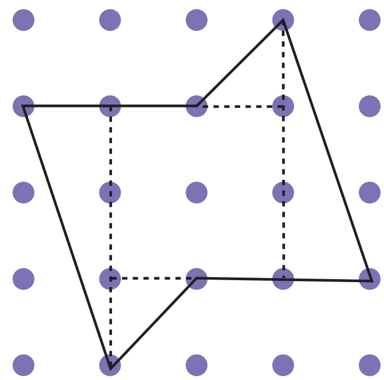
While students in prekindergarten through grade 2 are becoming acquainted with simple measuring tools and making comparisons and estimating measurements, students in grades 3–5 should be expanding their repertoires of measurement techniques and their skills in using measuring tools. In addition to becoming adept at using standard tools like rulers, protractors, scales, and clocks, third- through fifth-grade students should also encounter situations that require them to develop new techniques to accomplish measurement tasks that cannot be carried out directly with standard instruments. For example, to measure the circumference of a basketball, they might decide to wrap a string around the ball and then measure the length of the string; to measure the volume of a rock, they might submerge it in a graduated cylinder containing a known volume of water to obtain the total volume of water plus rock; to measure the weight of milk in a glass, they might weigh the empty glass as well as the glass and milk together.

As students in grades 3–5 hone their estimation skills, they should also be refining their sense of the sizes of standard units and the reasonableness of particular estimates. They might recognize 125 centimeters as a reasonable estimate for the height of a third grader but know that 125 meters or 1.25 centimeters could not be, or that a paper clip could weigh about a gram but not a kilogram. Students also should discuss estimation strategies with one another and compare the effectiveness of different approaches. In so doing, they should consider what degree of precision is required in a given situation and whether it would be better to overestimate or underestimate.

In grades 3–5, students also learn that certain measurements have special names, like *perimeter*, *circumference*, or *right angle;* and, as discussed earlier, they should look for patterns in measurements that will lead them to develop simple formulas, such as the formulas for the perimeter of a square, the area of a rectangle, or the volume of a cube. Through hands-on experience with objects, they should explore how different measurements might vary. For instance, by rearranging the seven tangram pieces to form a square, trapezoid, parallelogram, triangle, or nonsquare rectangle, they should find that the areas of all the shapes are the same, since they are made from the same seven pieces, but that the perimeters are different.

During middle school, students should apply their measurement skills in situations that are more complex, including problems that they can solve by decomposing or rearranging shapes. For example, they might find the area of an irregular shape on a geoboard by partitioning it into rectangles and right triangles (A) or by inscribing it in a rectangle and subtracting the areas of the surrounding shapes (B). Extending the strategy of decomposing, composing, or rearranging, students can arrive at other formulas, such as for the area of a parallelogram (C) by transforming it into a rectangle (D), or the formula for the area of a trapezoid either by decomposing it into a rectangle and two triangles (E) or by duplicating it to form a parallelogram with twice the area of the trapezoid (F). Other hands-on explorations that guide students in deriving formulas for the perimeter, area, and volume of various two- and three-dimensional shapes will ensure that these formulas are not just memorized symbols but are meaningful to them.





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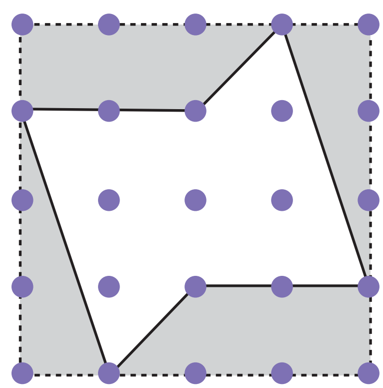

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Students in grades 6-8 should become attentive to precision and error in measurement. They should understand that measurements are precise only to one-half of the smallest unit used in the measurement (for example, an angle measured with a protractor marked in degrees has a precision of ±0.5 degree, so a reported angle measurement of 52° indicates an angle between 51.5° and 52.5°). Students in the middle grades also spend a great deal of time studying ratio, proportion, and similarity—concepts that are closely tied to measurement. Students should conduct investigations of similar triangles to help them realize, for example, that corresponding angles have equal measures; that corresponding sides, altitudes, perimeters, and other linear attributes have a fixed ratio; and that the areas of the triangles have a ratio that is the square of the ratio of their corresponding sides. Likewise, in exploring similar three-dimensional shapes, students should measure and observe that corresponding sides have a constant ratio; that the surface areas are proportional to the square of the ratio of the sides; and that the volumes are proportional to the cube of the ratio of the sides.

Through investigation, students should discover how to manipulate certain measurements. For example, by holding the perimeter constant and constructing different rectangles, they should learn that the area of the rectangle will be greatest when the rectangle is a square. Conversely, by holding the area constant and constructing different rectangles, they should discover that the perimeter is smallest when the rectangle approaches a square. They can apply discoveries like these in constructing maps and scale drawings or models or in investigating how the shape of packaging, such as cracker or cereal boxes, affects the surface area and volume of the container. They also should compare measurements of attributes expressed as rates, such as unit pricing (e.g., dollars per pound or cents per minute), velocity (e.g., miles per hour [MPH] or revolutions per minute [rpm]), or density (e.g., grams per cubic centimeter). All these measurements require proportional reasoning, and they arise frequently in the middle school mathematics curriculum, in connection with such topics as the slopes of linear functions.

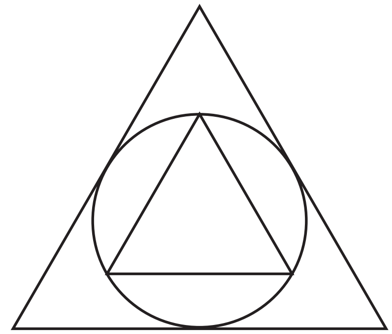
High school students should develop an even more sophisticated understanding of precision in measurement as well as critical judgment about the way in which measurements are reported, especially in the significant digits resulting from calculations. For example, if the side lengths of a cube were measured to the nearest millimeter and reported as 141 mm or 14.1 cm, then the actual side length lies between 14.05 cm and 14.15 cm, and the volume of the cube would correctly be said to be between 2773 cm^3 and 2834 cm^3, or $(14.05 \text{ cm})^3$ and $(14.15 \text{ cm})^3$. It would not be correct to report the volume as 2803.221 cm^3—the numerical result of calculating $(14.1 \text{ cm})^3$. Students in grades 9–12 also should develop a facility with units that will allow them to make necessary conversions among units, such as from feet to miles and hours to seconds in calculating a distance in miles (with the distance formula $d = v \cdot t$), when the velocity is reported in feet per second and the time is given in hours. Building on their earlier understanding that all measurements are approximations, high school students should also explore how some measurements can be estimated by a series of successively more accurate approximations. For example, finding the perimeter of inscribed and circumscribed n-gons as n increases ($n = 3, 4, 5, \ldots$) leads to approximations for the circumference of a circle.

High school students can use their mathematical knowledge and skills in developing progressively more rigorous derivations of important measurement formulas and in using those formulas in solving problems, not only in their mathematics classes but in other subjects as well. Students in grades 9–12 should apply measurement strategies and formulas to a wider range of geometric shapes, including cylinders, cones, prisms, pyramids, and spheres, and to very large measurements, such as distances in astronomy, and extremely small measurements, such as the size of an atomic nucleus or the mass of an electron. Students should also encounter highly sophisticated measurement concepts dealing with a variety of physical, technological, and cultural phenomena, including the half-life of a radioactive element, the charge on an electron, the strength of a magnetic field, and the birthrate of a population.

Measurement across the Mathematics Curriculum

A curriculum that fosters the development of the measurement concepts and skills envisioned in *Principles and Standards* needs to be coherent, developmental, focused, and well articulated. Because measurement is pervasive in the entire mathematics curriculum, as well as in other subjects, it is often taught in conjunction with other topics rather than as a topic on its own. Teaching measurement involves offering students frequent hands-on experiences with concrete objects and measuring instruments, and teachers need to ensure that students develop strong conceptual foundations before moving too quickly to formulas and unit conversions.

The *Navigating through Measurement* books reflect a vision of how selected “big ideas” of measurement and important measurement skills develop over the pre-K–12 years, but they do not attempt to articulate a complete measurement curriculum. Teachers and students who use other books in the Navigations Series will encounter many of the concepts presented in the measurement books there as well, in other contexts, in connection with the Algebra, Number, Geometry, and Data Analysis and Probability Standards. Conversely, in the *Navigating through Measurement* books, as in the classroom, concepts related to this Standard are applied and reinforced across the other strands. The four *Navigating through Measurement* books are offered as guides to help educators set a course for successful implementation of the very important Measurement Standard.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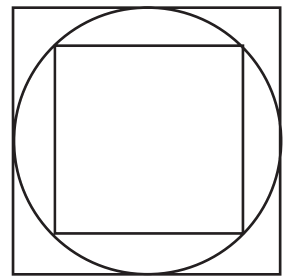
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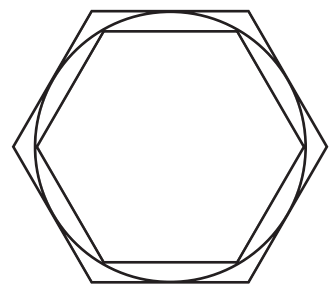
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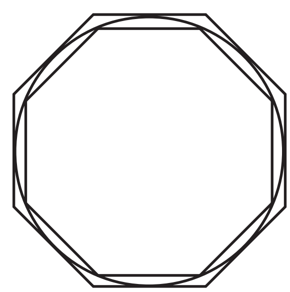
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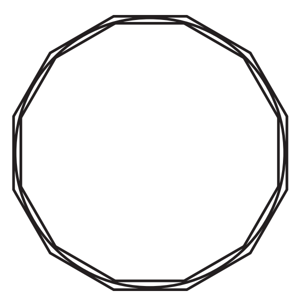
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