Introduction

This book focuses on ideas about proof and proving. These are ideas that you need to understand thoroughly and be able to use flexibly to be highly effective in your teaching of mathematics in grades 9–12. The book discusses many mathematical ideas that are common in high school curricula, and it assumes that you have had a variety of mathematics experiences that have motivated you to delve into—and move beyond—the mathematics that you expect your students to learn.

The book is designed to engage you with these ideas, helping you to develop an understanding that will guide you in planning and implementing lessons and assessing your students' learning in ways that reflect the full complexity of proof and proving. A deep, rich understanding of ideas about these aspects of mathematics will enable you to communicate their importance to your students, showing them how these ideas permeate the mathematics that they have encountered—and will continue to encounter—throughout their school mathematics experiences.

The understanding of proof and proving that you gain from this focused study supports the vision of *Principles and Standards for School Mathematics* (NCTM 2000): "Imagine a classroom, a school, or a school district where all students have access to high-quality, engaging mathematics instruction" (p. 3). This vision depends on classroom teachers such as you, who "are continually growing as professionals" (p. 3) and routinely engage their students in meaningful experiences that help them learn mathematics with understanding.

Why Proof and Proving?

Like the topics of all the volumes in NCTM's Essential Understanding Series, proof and proving are major aspects of school mathematics that are crucial for students to learn but challenging for teachers to teach. Students in grades 9–12 need to engage in proving activities if they are to succeed in these grades and in their subsequent mathematics experiences. Learners often struggle with proving. For example, how do students determine whether a mathematical statement is true or false? Many students rely on a teacher, a textbook, or testing various examples to make this determination. Other students might base their arguments on authority, perception, or popular consensus. To know how statements can be justified (or refuted), it is essential for teachers of grades 9–12 to understand the role of proof in mathematics themselves.

Your work as a teacher of mathematics in these grades calls for a solid understanding of proof and the proving practices that you—and your school, your district, and your state curriculum—expect your students to learn. Your work also requires you to know how this mathematics relates to other mathematical ideas that your students will encounter in the lesson at hand, the current school year, and beyond. Rich mathematical understanding guides teachers' decisions in much of their work, such as choosing tasks for a lesson, posing questions, selecting materials, ordering topics and ideas over time, assessing the quality of students' work, and devising ways to challenge and support their thinking.

Understanding Proof and Proving

Teachers teach mathematics because they want others to understand it in ways that will contribute to success and satisfaction in school, work, and life. In working on any mathematical topic, students need to engage in activities that are part of proving—such as developing conjectures, considering the general case, exploring with examples, looking for structural similarities across cases, and searching for counterexamples. Helping your students to develop their capacity to engage in such activities requires that you understand mathematical reasoning deeply. But what does this mean?

It is easy to think that an understanding of proof and facility in proving as a mathematical process mean knowing why certain things are mathematically appropriate and being able to justify particular theorems. For example, you might be expected to prove particular theorems, understand why a single example does not always prove that a statement is true, or refute claims with a counterexample. You are likely to have encountered tasks that require you to write two-column proofs, paragraph proofs, or proofs by contradiction. You are expected to be skillful in determining whether mathematical statements about the content that you teach are true and whether relevant mathematical terms are used appropriately.

Obviously, facts, vocabulary, and proof techniques are not all that you are expected to know. For example, in your ongoing work with students, you have undoubtedly discovered that not only do you need to know why particular theorems are true, but you are also expected to be able to follow your students' reasoning and justify or refute their claims when appropriate.

It is also easy to focus on a very long list of mathematical ideas that all teachers of mathematics in grades 9–12 are expected to know and teach about proof and proving. Curriculum developers often devise and publish such lists. However important the individual items might be, these lists cannot capture the essence of a rich understanding of the topic. Understanding proof and proving deeply requires you not only to know important mathematical ideas but also to recognize how these ideas relate to one another. Your understanding continues to grow with experience and as a result of opportunities to embrace new ideas and find new connections among familiar ones.

Furthermore, your understanding of proof and proving should transcend the content intended for your students. Some of the differences between what you need to know and what you expect them to learn are easy to point out. For instance, your understanding of the topic should include a grasp of how proof in mathematics differs across algebra, geometry, statistics, and so on, and how proof in mathematics differs from proof in science and in other disciplines.

Other differences between the understanding that you need to have and the understanding that you expect your students to acquire are less obvious, but your experiences in the classroom have undoubtedly made you aware of them at some level. For example, how many times have you been grateful to have an understanding of mathematics that enables you to recognize the merit in a student's unanticipated mathematical question or claim? How many other times have you wondered whether you missed an opportunity to help students refine their arguments because of a gap in your knowledge?

As you have almost certainly discovered, knowing and being able to do familiar mathematics are not enough when you're in the classroom. You also need to be able to identify and justify or refute novel claims. These claims and justifications might draw on ideas or techniques that are beyond the mathematical experiences of your students and current curricular expectations for them, but you might be able to draw on your own knowledge to help students see the merit or flaw in a particular claim.

Big Ideas and Essential Understandings

Thinking about the many particular ideas that are part of a rich understanding of proof and proving can be an overwhelming task. Articulating all of those mathematical ideas and their connections would require many books. To choose which ideas to include in this book, the authors considered a critical question: What is *essential* for teachers of mathematics in grades 9–12 to know about proof and proving to be effective in the classroom? To answer this question, the authors drew on a variety of resources, including personal experiences, the expertise of colleagues in mathematics and mathematics education, and the reactions of reviewers and professional development providers, as well as ideas from curricular materials and research on mathematics learning and teaching.

As a result, the mathematical content of this book focuses on essential ideas for teachers about proof and proving. In particular, chapter 1 is organized around five big ideas related to this important area of mathematics. The big ideas are supported by smaller, more specific mathematical ideas, which the book calls *essential understandings*.

Benefits for Teaching, Learning, and Assessing

Understanding mathematical reasoning can help you implement the Teaching Principle enunciated in *Principles and Standards for School Mathematics*. This Principle sets a high standard for instruction: “Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well” (NCTM 2000, p. 16). As in teaching about other critical topics in mathematics, teaching about proof and proving requires knowledge that goes “beyond what most teachers experience in standard preservice mathematics courses” (p. 17).

Chapter 1 comes into play at this point, offering an overview of proof and proving that is intended to be more focused and comprehensive than many discussions of the topic that you are likely to have encountered. This chapter enumerates, expands on, and gives examples of the big ideas and essential understandings related to reasoning, with the goal of supplementing or reinforcing your understanding. Thus, chapter 1 aims to prepare you to implement the Teaching Principle fully as you support and challenge your students in developing more robust proving practices.

Consolidating your understanding in this way also prepares you to implement the Learning Principle outlined in *Principles and Standards*: “Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge” (NCTM 2000, p. 20). To support your efforts to help your students learn about proof and proving in this way, chapter 2 builds on the understanding of reasoning that chapter 1 communicates by pointing out specific ways in which the big ideas and essential understandings connect with mathematics that students typically encounter earlier or later in school. This chapter supports the Learning Principle by emphasizing longitudinal connections in students’ learning about proof and proving. For example, as their mathematical experiences expand, students understand that there is no mathematical content strand in which proving is not an appropriate activity. The subjects of their proofs and the types of proofs that they produce become more sophisticated as they move through the grades.

The understanding that chapters 1 and 2 convey can strengthen another critical area of teaching. Chapter 3 addresses this area, building on the first two chapters to show how an understanding of proof and proving can help you select and develop appropriate tasks, techniques, and tools for assessing your students' reasoning, facility in conjecturing, and skill in constructing arguments. An ownership of the big ideas and essential understandings related to proof and proving, reinforced by an understanding of students' past and future experiences with mathematical reasoning, can help you ensure that assessment in your classroom supports the development of understanding of proof and skill in proving.

Such assessment satisfies the first requirement of the Assessment Principle set out in *Principles and Standards*: "Assessment should support the learning of important mathematics and furnish useful information to both teachers and students" (NCTM 2000, p. 22). An understanding of proof and proving can also help you satisfy the second requirement of the Assessment Principle, by enabling you to develop assessment tasks that give you specific information about how your students are reasoning and what they understand. For example, a proof task might ask students to evaluate two arguments. One argument might be an algebraic proof, and the other argument might be based on examples, with the prompt worded in such a way to leave open the possibility that both arguments were valid. Students could discuss not only whether the given arguments were valid proofs but also provide reasons why or why not. They could be asked to write a paragraph discussing whether the arguments were valid proofs, with the possibility that both arguments were valid. This would increase the likelihood that students would not choose an argument just because it had more words or more symbols.

Ready to Begin

This introduction has painted the background, preparing you for the big ideas and associated essential understandings related to proof and proving that you will encounter and explore in chapter 1. Reading the chapters in the order in which they appear can be a very useful way to approach the book. Read chapter 1 in more than one sitting, allowing time for reflection. Absorb the ideas—both big ideas and essential understandings—related to proof and proving. Appreciate the connections among these ideas. Carry your newfound or reinforced understanding to chapter 2, which guides you in seeing how the ideas related to reasoning are connected to the mathematics that your students have encountered earlier or will encounter later in school. Then read about teaching, learning, and assessment issues in chapter 3.

Alternatively, you may want to take a look at chapter 3 before engaging with the mathematical ideas in chapters 1 and 2. Having the challenges of teaching, learning, and assessment issues clearly in mind, along with possible approaches to them, can give you a different perspective on the material in the earlier chapters.

No matter how you read the book, let it serve as a tool to expand your understanding, application, and enjoyment of proof and proving.