suppose a student simply memorizes the formula for the surface area of a cube (6 times the area of the base, or 6 × length × width). He or she may then think that this formula works for the surface area of all rectangular solids (Gilliland 2002). If instead of being presented up front with the formula to measure the surface area of a cube, the student experiences various activities to investigate the surface area of different sized boxes, the formula for the surface area of a cube can be derived in a meaningful way from a cube's relationship to other rectangular solids so that students will know when to apply which formulas in future problems. See Task 2 on page 9 for a sample investigation on finding the surface area of various rectangular solids. Students in middle school will learn and use formulas and procedures to solve problems; however, investigations that lead to understanding these procedures and discussing why they work is needed so students gain facility in their use.

Instruction to Support a Focused Curriculum

Questions to Reflect On

- What does instruction that supports depth of understanding and connections among mathematical ideas "look like"?
- How can questioning be used to support the development of depth of understanding and connections in a focused curriculum?
- What is the role of practice in a focused curriculum?
- What impact does instruction that supports a focused curriculum have on time management?

Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well.

—The Teaching Principle, Principles and Standards for School Mathematics

Although NCTM's Curriculum Focal Points can help prioritize and organize mathematics content, teachers and the instruction they provide are crucial to using focal points to improve student learning. Focusing mathematics instruction on a few central ideas at each grade requires skilled teachers who know the content well and can connect mathematical ideas and teach for depth of understanding.

Use of the Process Standards

It is essential that teachers incorporate the Process Standards of Problem Solving, Reasoning and Proof, Communication, Connections, and Representation as described in *Principles and Standards for School Mathematics* (NCTM 2000) into classroom instruction. Teachers should create a climate that supports mathematical thinking and communication. In this kind of

Copyright © 2010 by the National Council of Teachers of Mathematics, Inc. www.nctm.org. All rights reserved. This material may not be copied or distributed electronically or in other formats without written permission from NCTM.

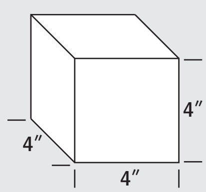
classroom, students are accustomed to reasoning about a mathematical problem and justifying or explaining their results, representing mathematical ideas in multiple ways, and building new knowledge as well as applying knowledge through problem solving. Brief descriptions of the Process Standards can be found in the table that follows. More detailed descriptions can be found in Principles and Standards for School Mathematics.

NCTM Process Standards

Problem Solving. Through problem solving, students can not only apply the knowledge and skills they have acquired but can also learn new mathematical content. Problem solving is not a specific skill to be taught, but should permeate all aspects of learning. Teachers should make an effort to choose "good" problems—ones that invite exploration of an important mathematical concept and allow students the chance to solidify and extend their knowledge. Compare the two versions of a volume-and-surface-area task below; whereas task 1 requires students to do little more than correctly apply formulas, task 2 engages them intellectually because it challenges them to search for something and is not immediately solvable. The instructional strategies used in the classroom should also promote collaborative problem solving. Students' learning of mathematics is enhanced in a learning environment that is a community of people collaborating to make sense of mathematical ideas (Hiebert et al. 1997).

Task 1

Find the volume and surface area of a cube with length, width, and height each 4 inches.



Task 2

A manufacturer wants to package 1-inch cubes as classroom manipulatives. The company wants to design an inexpensive box in the shape of a rectangular prism that exactly fits the 64 cubes. Find all the ways that 64 cubes can be arranged into a rectangular prism. Which arrangement would require the least amount of material for creating the box?

Source: "Why Not Just Use a Formula?" by Kay Gilliland (Mathematics Teaching in the Middle School 9, no. 7 [May 2002]: 510–11).

Reasoning and Proof. For students to learn mathematics with understanding, it must make sense to them. Teachers can help students make sense of the mathematics they are learning by encouraging them to always explain and justify their solutions and strategies as well as evaluate other students' ideas. Questions such as "Why?" and "How do you know?" should be a regular part of classroom discussions. The teacher should respond in ways that focus on thinking and reasoning rather than only on getting the correct answer. Incorrect answers should not simply be judged wrong. Instead, teachers can help students identify the parts of their thinking that may be correct, often leading to new ideas and solutions that are correct.

Communication. Reasoning and Proof goes hand in hand with the process of Communication. Students should have plenty of opportunities and support for speaking, writing, reading, and listening in the mathematics classroom. Communicating one's ideas orally and in writing helps to solidify and refine learning. Listening to others' explanations can also sharpen learning by providing multiple ways to think about a problem. The teacher plays an important role in developing students' communication skills by modeling effective oral and written communication of mathematical ideas as well as giving students regular opportunities to communicate mathematically. Precise mathematical vocabulary and definitions are important, and teachers need to help students articulate these ideas as well as to make sure students understand these ideas during class discussions.

Connections. As students move through the grades, they should be presented with new mathematical content. Students' abilities to understand these new ideas depend greatly on connecting the new ideas with previously learned ideas. Mathematics is an integrated field of study and should be presented in this way instead of as a set of disconnected and isolated concepts and skills. Instruction should emphasize the interconnectedness of mathematical ideas both within and across grade levels and should be presented in a variety of contexts.

Representation. Mathematical ideas can be represented in a variety of ways: pictures, concrete materials, tables, graphs, numerical and alphabetical symbols, spreadsheet displays, and so on. Such representations should be an essential part of learning and doing mathematics and serve as a tool for thinking about and solving problems. Teachers should model representing mathematical ideas in a variety of ways and discuss why some representations are more effective than others in particular situations.

Facilitating Classroom Discourse

The Process Standards, especially the Communication Standard and the Reasoning and Proof Standard, are related to the discourse in the mathematics classroom. "The discourse of a classroom—the way of representing, thinking, talking, agreeing, and disagreeing—is central to what and how students learn mathematics" (NCTM 2007, p. 46). The teacher plays an important role in initiating and facilitating this discourse and can do so in the following ways:

- posing questions and tasks that elicit, engage, and challenge each student's thinking;
- listening carefully to students' ideas and deciding what to pursue in depth from among the ideas that students generate during a discussion;
- asking students to clarify and justify their ideas orally and in writing and by accepting a variety of presentation modes;
- deciding when and how to attach mathematical notation and language to students' ideas;
- encouraging and accepting the use of multiple representations;
- making available tools for exploration and analysis;
- deciding when to provide information, when to clarify an issue, when to model, when to lead, and when to let students wrestle with a difficulty; and
- monitoring students' participation in discussions and deciding when and how to encourage each student to participate. (NCTM 2007, p. 45)

The following classroom vignette illustrates a teacher's use of effectively facilitated discourse in the mathematics classroom.

Vignette

The students began by working collaboratively in pairs to solve the following problem, adapted from Bennett, Maier, and Nelson (1998):

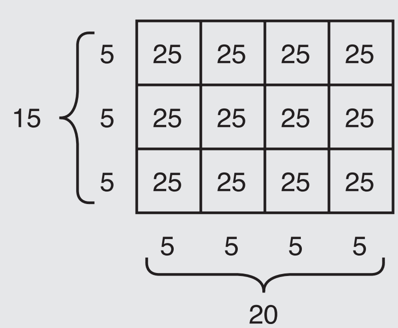
> A certain rectangle has length and width that are whole numbers of inches, and the ratio of its length to its width is 4 to 3. Its area is 300 square inches. What are its length and width?

As the students worked on the problem, the teacher circulated around the room, monitoring the work of the pairs and responding to their questions. She also noted different approaches that were used by the students and made decisions about which students she would ask to present solutions. After most students had a chance to solve the problem, the teacher asked Lee and Randy to present their method. They proceeded to the overhead projector to explain their work. After briefly restating the problem, Lee indicated that 3 times 4 is equal to 12 and that they needed "a number that both 3 and 4 would go into." The teacher asked why they had multiplied 3 by 4. Randy replied that the ratio of the length to the width was given as "4 to 3" in the problem. Lee went on to say that they had determined that "3 goes into 15 five times and that 4 goes into 20 five times." Since 15 times 20 is equal to 300, the area of the given rectangle, they concluded that 15 inches and 20 inches were the width and length of the rectangle.

The teacher asked if there were questions for Lee or Randy. Echoing the teacher's query during the presentation of the solution, Tyronne said that he did not understand their solution, particularly where the 12 had come from and how they knew it would help solve the problem. Neither Lee nor Randy was able to explain why they had multiplied 3 by 4 or how the result was connected to their solution. The teacher then indicated that she also wondered how they had obtained the 15 and the 20. The boys reiterated that they had been looking for a number "that both 3 and 4 went into." In reply, Darryl asked how the boys had obtained the number 5. Lee and Randy responded that 5 was what "3 and 4 go into." At this point, Keisha said "Did you guys just guess and check?" Lee and Randy responded in unison, "Yeah!" Although Lee and Randy's final answer was correct and although it contained a kernel of good mathematical insight, their explanation of their solution method left other students confused.

To address the confusion generated by Lee and Randy, the teacher decided to solicit another solution. Because the teacher had seen Rachel and Keisha use a different method, she asked them to explain their approach. Keisha made a sketch of a rectangle, labeling the length 4 and the width 3. She explained that the 4 and 3 were not really the length and width of the rectangle but that the numbers helped remind her about the ratio. Then Rachel explained that she could imagine 12 squares inside the rectangle because 3 times 4 is equal to 12, and she drew lines to subdivide the rectangle accordingly. Next she explained that the area of the rectangle must be equally distributed in the 12 "inside" squares. Therefore, they divided 300 by 12 to determine that each square contains 25 square inches. At the teacher's suggestion, Rachel wrote a 25 in each square in the diagram to make this point clear. Keisha then explained that in order to find the length and width of the rectangle, they had to determine the length of the side of each small square. She argued that since the area of each square was 25 square inches, the side of each square was 5 inches. Then, referring to the diagram [below], she explained that the length of the rectangle was 20 inches, since it consisted of the sides of four squares. Similarly, the width was found to be 15 inches. To clarify their understanding of the solution, a few students asked questions, which were answered well by Keisha and Rachel.

Rachel's and Keisha's method



Source: Principles and Standards for Mathematics (NCTM 2000, pp. 268–70). Reprinted with permission.

In this classroom example, the teacher guided the classroom discussion without taking over. Various students shared their ideas and solutions, and the teacher encouraged students to seek clarification if they did not understand a particular solution. The teacher also helped steer the discussion in certain directions by calling on certain students to share solutions that would contribute to and advance the discussion and by asking the students questions to encourage further explanation. To advance the discussion further and focus on main ideas, the teacher might ask the students whether Keisha and Rachel's method would work for problems in which the ratio was not 4 to 3 or the area was something other than 300 square inches. Next the teacher might ask the students to come up with equations that could be used to solve the problem in a more systematic and algebraic way by letting L represent the length and W represent the width. The students should eventually come up with something such as the following: Since L and W are in the ratio 4:3, we know that $\frac{L}{W} = \frac{4}{3}$ so 3L = 4W or L = $(\frac{4}{3})$W. Since the area is LW and LW = 300, $(\frac{4}{3})W \cdot W = 300$. Solving for W, we get that the width is 15 inches and therefore that the length is 20 inches.

Use of Questioning to Focus Learning and Promote Connections

As described in the Introduction and the "Focusing Curriculum" section of this guide, using focal points to organize instruction does not mean that you are teaching less or more content, but instead means directing the majority of your instruction at a smaller number of core areas with the goal of students' gaining a deeper mathematical understanding of those mathematical ideas and the connections among them. To teach for depth of understanding, teachers need to understand what their students are thinking and be able to support and extend that thinking. A teacher's use of questioning plays a vital role in focusing learning on foundational mathematical ideas and promoting mathematical connections. Such reasoning questions as "Why?" and "How do you know that?" posed during a lesson are great starters, but teachers also need to incorporate questioning techniques into their planning by thinking about specific questions to ask related to the particular topic. When planning instruction, teachers must also anticipate the kinds of answers they might get from students in response to the questions posed.

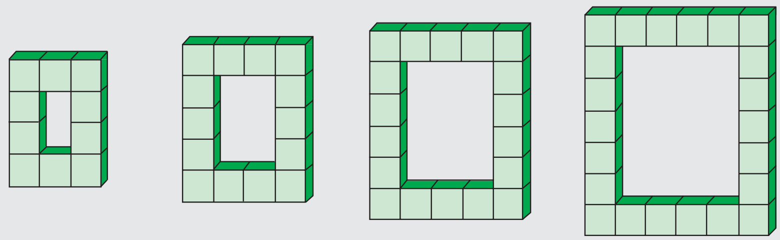
A teacher's use of questioning plays a vital role in focusing learning on foundational mathematical ideas and promoting mathematical connections.

Let us look at the following classroom example to show a teacher's use of questioning related to a lesson on generating algebraic expressions for an increasing pattern of connecting cubes arranged in a rectangle.

Vignette

Juan Rodriguez is beginning a unit on algebraic expressions with his ninth-grade students. He wants to use a problem that he saw discussed at a professional conference to determine whether it will help his students make more sense of algebraic expressions and better understand how different expressions can be equivalent.

Mr. Rodriguez distributes a bag of connectable unit cubes to each group of three students. He shows a large-image projection of the following stages for everyone to see.



Stage 1 **Stage 2** **Stage 3** **Stage 4**

Mr. Rodriguez: In your groups, use your unit cubes to construct exact replicas of stage 1, stage 2, stage 3, and stage 4. Everyone at the table should check to make sure you have used the correct number of cubes in each stage.

The students enjoy the variety of activities that Mr. Rodriguez uses, so they quickly become engaged in the task. Mr. Rodriquez moves around the room to assure that students are focused on the task.

Mr. Rodriquez: On your mini white boards, please write the number of cubes that would be needed to construct the stage-5 figure. Raise your board so I can see your number.

The students discuss the question. Some students start building the stage-5 model while other students make calculations with paper and pencil. Eventually, all groups raise their boards.

Mr. Rodriguez: Sandra, can you tell us how your group got twenty-six?

Sandra: We used our cubes to build the next stage, and then we counted how many cubes it took to make it.

Mr. Rodriguez: Did any other groups build the stage-5 model to answer the question?

Students in several groups raise hands.

Mr. Rodriguez: Roger, how did your group decide how many cubes are required to build stage 5?

Roger: Well, each stage is like a rectangle with the middle cut out. And the next one is one cube longer and one cube wider than the stage before.