NAVIGATIONS SERIES

GRADES 6–12

NAVIGATING *through* DISCRETE MATHEMATICS

Introduction

"Discrete mathematics should be an integral part of the school mathematics curriculum."
(NCTM 2000, p. 31)

Discrete mathematics is an important branch of contemporary mathematics that is widely used in business and industry. Elements of discrete mathematics have been around as long as mathematics itself. However, discrete mathematics emerged as a distinct branch of mathematics only in the middle of the twentieth century, when it began expanding rapidly, primarily because of the computer revolution, but also because of the need for mathematical techniques to help plan and implement such monumental logistical projects as landing a man on the moon. Discrete mathematics has grown to be even more important and pervasive today.

Principles and Standards for School Mathematics (NCTM 2000) recommends that discrete mathematics be "an integral part of the school mathematics curriculum" (p. 31). In the area of discrete mathematics, *Principles and Standards* features two major changes from NCTM's 1989 *Curriculum and Evaluation Standards for School Mathematics*, which included a Discrete Mathematics Standard for grades 9–12. First, *Principles and Standards* recommends including discrete mathematics in the curriculum for all grades, from prekindergarten through grade 12. Second, *Principles and Standards* does not include a separate standard for discrete mathematics. Rather, it recommends that the main topics of discrete mathematics be distributed across all the Standards, since "these topics naturally occur throughout the other strands of mathematics" (p. 31).

The goal of this book is to elaborate on the vision of discrete mathematics presented in *Principles and Standards.* In this introduction, we give an overview of discrete mathematics and guidelines for integrating discrete mathematics topics into a curriculum that is based on the NCTM Standards. Because discrete mathematics may be unfamiliar to many readers, we begin by considering a fundamental question: What is discrete mathematics?

What Is Discrete Mathematics?

Descriptions of discrete mathematics often list the topics that it includes, such as vertex-edge graphs, systematic counting, and iteration and recursion. Other topics relevant to the school curriculum include matrices, voting methods, fair division, cryptography, coding theory, and game theory. In general, discrete mathematics deals with finite processes and whole-number phenomena. Sometimes described as the mathematical foundation of computer science, discrete mathematics has even broader application, since the social, management, and natural sciences also use it. Discrete mathematics contrasts with continuous mathematics, such as the mathematics underlying most of calculus. However, this association gives the impression that discrete mathematics is only for advanced high school students, although elements of discrete mathematics are actually accessible and important for all students in all grades.

For further discussion of discrete mathematical modeling and algorithmic problem solving, see Hart (1997, 1998 [available on the CD-ROM]).

Broad definitions of discrete mathematics identify it as "the mathematics of decision making for finite settings" (NCTM 1990, p. 1) and the mathematics for optimizing finite systems. Common themes in discrete mathematics include the following:

- Discrete mathematical modeling—using discrete mathematical tools such as vertex-edge graphs and recursion to represent and solve problems
- Algorithmic problem solving—designing, using, and analyzing step-by-step procedures to solve problems
- Optimization—finding the best solution

Kenney (1991), Maurer (1997), and Rosenstein (2007) offer more answers to the question, What is discrete mathematics?

Which Discrete Mathematics Topics Does *Principles and Standards* Include?

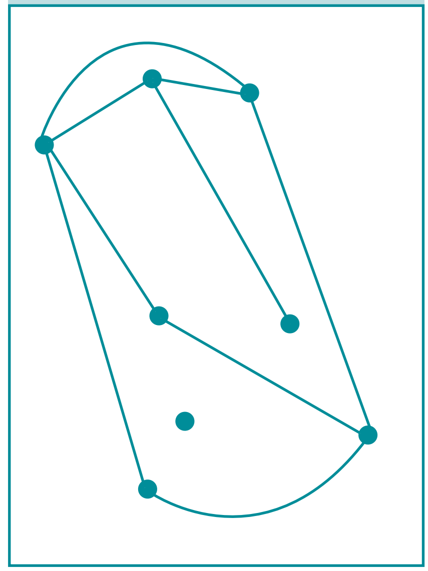
Principles and Standards integrates three important topics of discrete mathematics: combinatorics, iteration and recursion, and vertex-edge graphs.

- Combinatorics is the mathematics of systematic listing and counting. It facilitates solving problems such as determining the number of different orders for picking up three friends or counting the number of different computer passwords that are possible with five letters and two numbers.
- Iteration and recursion can be used to represent and solve problems related to sequential step-by-step change, such as the growth of a population or an amount of money from year to year. To iterate

means to repeat, so iteration involves repeating a procedure, process, or rule over and over. Recursion is the method of describing the current step of a process in terms of the previous steps.

- Vertex-edge graphs, like the one pictured in figure 0.1, consist of points (called *vertices*) and line segments or arcs (called *edges*) that connect some of the points. Such graphs provide models for, and lead to solutions of, problems about paths, networks, and relationships among a finite number of elements.

Fig. **0.1.** A vertex-edge graph



Principles and Standards focuses on integrating discrete mathematics with other areas of the mathematics curriculum. For example, vertex-edge graphs are an important part of geometry. Recursion occurs in all the content strands but is particularly instrumental in algebra. Concepts of systematic listing and counting appear throughout the curriculum. Matrices, which many consider to be part of discrete mathematics, are addressed throughout *Principles and Standards*.

Other discrete mathematics topics that may receive attention in the school curriculum include the mathematics of information processing (such as error-correcting codes and cryptography) and the mathematics of democratic and social decision making (for example, voting methods, apportionment, fair division, and game theory). This book focuses on the three discrete mathematics topics that *Principles and Standards* emphasizes (p. 31): combinatorics, iteration and recursion, and vertex-edge graphs. First, however, let's consider why the school curriculum should include discrete mathematics.

Why Should the School Curriculum Include Discrete Mathematics?

Instructional time is valuable, and the mathematics curriculum has limited space, so educators must make careful choices about what to include in the curriculum. *Principles and Standards* recommends that discrete mathematics be an integral part of the school mathematics curriculum because it is useful, contemporary, and pedagogically powerful, in addition to being a substantial and active field of mathematics.

"As an active branch of contemporary mathematics that is widely used in business and industry, discrete mathematics should be an integral part of the school mathematics curriculum, and these topics naturally occur throughout the other strands of mathematics."
NCTM 2000, p. 31)

Discrete Mathematics Is Useful Mathematics

Discrete mathematics has many uses in business, industry, and daily life. Rosenstein, Franzblau, and Roberts (1997) enumerate a variety of applications, asserting that discrete mathematics topics are "used by decision-makers in business and government; by workers in such fields as telecommunications and computing that depend upon information transmission; and by those in many rapidly changing professions involving health care, biology, chemistry, automated manufacturing, transportation, etc. Increasingly, discrete mathematics is the language of a large body of science and underlies decisions that individuals will have to make in their own lives, in their professions, and as citizens" (p. xiii–xiv).

Discrete Mathematics Is Contemporary Mathematics

Discrete mathematics is a rapidly expanding field of mathematics. It is particularly relevant in today's digital information age. For example, it underlies many aspects of the Internet, from secure encryption of consumers' credit card numbers when they make purchases online to effective compression and decompression of the music, photos, and videos that users download. Moreover, many solved and unsolved problems at the frontiers of discrete mathematics are not only relevant to today's students but also accessible to them. Students can understand the problems and some partial solutions, such as the problem of finding the shortest circuit through a network (the traveling salesman problem) or finding a more secure method for transmitting data between computers. Furthermore, since discrete mathematics has strong links to technology and today's schoolchildren are tomorrow's technological workforce, it is important for their futures, as well as for the future of our nation, that they become more familiar with the topics of discrete mathematics.

Discrete Mathematics Is Pedagogically Powerful

Not only does discrete mathematics include important mathematical content, but it is also a powerful vehicle for teaching and learning mathematical processes and engaging students in doing mathematics. Because discrete mathematics is useful and contemporary, it often motivates and interests students. Discrete mathematics topics can engage and provide success for students who have previously been unsuccessful or alienated from mathematics. Many of these topics are accessible to students in all grades, whether they are engaged in sorting different types of buttons in the early grades, counting different flag patterns in middle school, or using vertex-edge graphs and the critical path method to plan a dance in high school.

Discrete mathematics provides an effective context for developing the skills addressed in the Process Standards.

Furthermore, discrete mathematics is an effective context for addressing NCTM's Process Standards. In working with discrete mathematics, students strengthen their skills in reasoning, proof, problem solving, communication, connections, and representation in many ways. For example, they reason about paths in the visual context of vertex-edge graphs and justify whether certain circuits must exist. They argue about why a recursive formula is better than an explicit formula, or vice versa, in a particular situation. They learn new methods of proof, including proof by mathematical induction. They develop new types of reasoning, such as combinatorial reasoning, which they can use to reason about the number of different possibilities that can arise in counting situations (for example, the number of different pizzas that are possible when they choose two out of five toppings). Students exercise their problem-solving skills when they solve problems in a variety of accessible yet challenging settings. They develop new problem-solving strategies, such as algorithmic problem solving, and new ways of thinking, such as recursive thinking. Students acquire and apply new tools—including recursive formulas and vertex-edge graphs—for representing problems. Thus, students learn important mathematical content and powerful mathematical processes while they study discrete mathematics.

Recent History and Resources

Discrete mathematics surfaced as a curricular issue in the 1980s, when the Mathematical Association of America (MAA) began debating the need for more instruction in discrete mathematics during the first two years of college. This debate culminated in a report released in 1986 (MAA 1986). Although the recommendations of this report were not implemented in full, educators instituted more discrete mathematics courses, which continue to be taught in colleges around the world. In particular, discrete mathematics is now a standard course in collegiate computer science programs and a required course for many mathematics majors.

This discussion about discrete mathematics in college made its way down to the high school level a few years later, when the National Council of Teachers of Mathematics recommended a Discrete Mathematics Standard for grades 9–12 in its seminal Standards publication, *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989). As a result of these Standards, discrete mathematics expanded rapidly in the school curriculum. The National Science Foundation (NSF) funded teacher-enhancement projects to help implement the Discrete Mathematics Standard.

Eric W. Hart and Harold L. Schoen worked on NSF-funded teacher enhancement projects from 1987 to 1994, Margaret J. Kenney's projects extended from 1992 to 1997, the work of Joseph G. Rosenstein and Valerie A. DeBellis extended from 1990 until 2005, and James T. Sandefur's projects extended from 1992 to 1995.

High schools began offering courses in discrete mathematics, and many states added discrete mathematics to their state frameworks. Discrete mathematics courses play an increasingly important role in the high school curriculum, providing essential mathematics for the technology- and information-intensive twenty-first century, particularly since more students are required to take more mathematics, and the traditional calculus-preparatory high school curriculum does not serve all students well.

Several NSF-funded Standards-based curriculum development projects have integrated discrete mathematics into new high school textbook series. These series include *Core-Plus Mathematics* (Hirsch et al. 2008), *Interactive Mathematics Program* (Alper et al. 2003), *Mathematics: Modeling Our World* (COMAP 1999), and *SIMMS Integrated Mathematics* (SIMMS 2006). In addition, several new textbooks are available for high school courses in discrete mathematics (see COMAP [2006], Crisler and Froelich [2006], and Tannenbaum [2007], for example). Some teacher education textbooks address discrete mathematics, including *Making Math Engaging: Discrete Mathematics for K–8 Teachers* (DeBellis and Rosenstein 2008). Finally, many articles about discrete mathematics and activities for teaching it are available.

"A Bibliography of Print Resources for Discrete Mathematics" on the accompanying CD-ROM indicates the topics and grade bands covered by the identified resources.

Thus far, we have considered what discrete mathematics is, along with some history and resources, and we have described why discrete mathematics should be part of the curriculum. In the remainder of this introduction, we present an overview across the grades of the three main topics: combinatorics (systematic listing and counting), vertex-edge graphs, and iteration and recursion. In developing these topics across the grade levels, we have in mind two important progressions in the prekindergarten–grade 12 curriculum—from concrete to abstract and from informal to more formal reasoning.

The chapters that follow address these topics one at a time, for grades 6–8 and 9–12 in the present book, and for prekindergarten through grade 2 and grades 3–5 in the companion book, *Navigating through Discrete Mathematics in Prekindergarten–Grade 5* (DeBellis et al. forthcoming). In fact, we can use discrete mathematics to reason that since each book includes three topics and two grade bands, each book must include 3×2, or 6, chapters—which brings us to the first topic, systematic listing and counting.

Overview of Systematic Listing and Counting in Prekindergarten–Grade 12

Students at all grade levels should be able to solve counting problems. Examples of appropriate problems at different levels follow:

- In elementary school: "How many different outfits can someone put together with three shirts and two pairs of shorts?"
- In middle school: "How many different four-block towers can a person build with red and blue blocks?"
- In high school: "What is the number of possible computer passwords that use six letters and three digits?"

Knowledge of counting strategies helps lay a foundation for understanding ideas of probability.

The key to answering such questions is to develop strategies for listing and counting, in a systematic manner, all the ways of completing the task. As students advance through the grade levels, the tasks change—the objects to be counted become abstract as well as concrete, the numbers of objects increase, the representations become more algebraic, and the reasoning becomes more formal, culminating in proof—but the common thread is that the students need to do the counting systematically. If students have enough opportunities to explore counting problems at all grade levels, then these transitions will be smooth, and they will acquire a deep understanding. In addition, knowledge of the counting strategies helps lay the necessary foundation for understanding ideas of probability.

All the NCTM Standards integrate concepts and methods of systematic listing and counting. In support of this integration, the following recommendations suggest how to develop systematic listing and counting throughout the grades.

Recommendations for Systematic Listing and Counting in Prekindergarten–Grade 12

In prekindergarten–grade 2, all students should—

- sort, organize, and count small numbers of objects;
- informally use the addition principle of counting;
- list all possibilities in counting situations;
- sort, organize, and count objects by using Venn diagrams.

In grades 3–5, all students should—

- represent, analyze, and solve a variety of counting problems by using arrays, systematic lists, tree diagrams, and Venn diagrams;
- use and explain the addition principle of counting;
- informally use the multiplication principle of counting;
- understand and describe relationships among arrays, systematic lists, tree diagrams, and the multiplication principle of counting.

In grades 6–8, all students should—

- represent, analyze, and solve counting problems that do or do not involve ordering and that do or do not involve repetition;
- understand and apply the addition and multiplication principles of counting and represent these principles with algebra, including factorial notation;
- solve counting problems by using Venn diagrams and use algebra to represent the relationships shown by a Venn diagram;
- construct and describe patterns in Pascal's triangle;
- implicitly use the pigeonhole principle and the inclusion-exclusion principle.

In grades 9–12, all students should—

- understand and apply permutations and combinations;
- use reasoning and formulas to solve counting problems in which repetition is or is not allowed and ordering does or does not matter;
- understand, apply, and describe relationships among the binomial theorem, Pascal's triangle, and combinations;
- apply counting methods to probabilistic situations involving equally likely outcomes;
- use combinatorial reasoning to construct proofs as well as solve a variety of problems.

Overview of Vertex-Edge Graphs in Prekindergarten–Grade 12

Another discrete mathematics topic that *Principles and Standards* recommends for study is vertex-edge graphs. Vertex-edge graphs are mathematical models that consist of points (*vertices*), with curves or line segments (*edges*) connecting some of the points (see fig. 0.1 on p. 3). Such diagrams aid in solving problems related to paths, circuits, and networks. For example, vertex-edge graphs can help in optimizing a telecommunications network, planning the most efficient circuit through cities that a salesperson visits, finding an optimal path for plowing snow from city streets, or determining the shortest route for collecting money from neighborhood ATM machines.

Vertex-edge graphs aid in solving problems related to paths, circuits, and networks.

More abstractly, vertex-edge graphs may be useful in analyzing situations that involve relationships among a finite number of objects. Vertices represent the objects, and the relationship among the objects is shown by edges that connect some vertices. The relationships may be very concrete, such as airline routes that connect cities in the salesman example; or they may be more abstract, as in vertex-edge graphs depicting conflicts or prerequisites. For instance, a vertex-edge graph facilitates scheduling committee meetings without conflicts (where an edge links two committees that cannot meet at the same time because of a shared member) or finding the earliest completion time for a large construction project consisting of many tasks (where directed edges are used to link a task to its prerequisite tasks).

Graph theory is the formal study of vertex-edge graphs. The term *vertex-edge graph* distinguishes these diagrams from other types of graphs, such as graphs of functions or graphs used in data analysis. Nevertheless, a commonly used term is simply *graphs*. In this volume, we employ both terms, as appropriate.

Size, shape, and position are not essential characteristics of vertex-edge graphs.

Graph theory is part of discrete mathematics, but it is also part of geometry, since graphs are geometric diagrams that consist of vertices and edges. Graphs share some characteristics with other geometric objects in school mathematics—for example, both polyhedra and graphs have vertices and edges. But in contrast with most of school geometry—which focuses on the size and shape of figures—size, shape, and position are not essential characteristics of vertex-edge graphs. In vertex-edge graphs, it does not really matter whether the graph is large or small or whether the edges are straight or curved. All that really matters are the number of vertices and edges and how the vertices are connected by edges.

The school mathematics curriculum should include several fundamental graph-theory topics. Table 0.1 summarizes these topics. Chapters 3 and 4 furnish more detail and explanation about vertex-edge graphs.

The analysis and representation of all these problems are concrete at the early grades and become more formal and abstract as a student moves upward through the grades. In this volume, chapters 3 and 4 elaborate on specific recommendations for grades 6–8 and 9–12, respectively. However, a common set of goals exists for all grades. All students should—

- use vertex-edge graphs to model and solve a variety of problems related to paths, circuits, networks, and relationships among a finite number of objects;
- understand and apply properties of graphs;
- devise, describe, and analyze algorithms to help solve problems related to graphs;
- use graphs to understand and solve optimization problems.

Important themes at all grade levels include mathematical modeling, applications, optimization, and algorithmic problem solving. Mathematical modeling is a multistep process of solving a real-world problem by using mathematics to represent the problem, finding a mathematical solution, translating that solution into the context of the original problem, and finally interpreting and judging the reasonableness of the result. Optimization problems are important throughout mathematics and in many applications. The goal is to find the best solution—for example, the shortest path, the most efficient strategy, the fewest

Table 0.1.
Fundamental Topics in Graph Theory for the School Curriculum

Optimal Paths and Circuits		
Graph Topic	Basic Problem	Sample Application
Euler paths	Find a route through a graph that uses each edge exactly once.	Determine routes for a snowplow.
Hamilton paths	Find a route through a graph that visits each vertex exactly once.	Rank players in a tournament.
Shortest paths	Find a shortest path from here to there.	Measure the degree of influence among people in a group.
Critical paths	Find a longest path or critical path.	Schedule large projects.
Traveling salesman problem (TSP)[1]	Find a circuit through a graph that visits all vertices, that starts and ends at the same location, and that has minimum total weight.	Determine the least expensive circuit through cities that a sales representative visits.

Optimal Spanning Networks		
Graph Topic	Basic Problem	Sample Application
Minimum spanning trees	Find a network within a graph that joins all vertices, has no circuits, and has minimum total weight.	Create an optimal computer or road network.

Optimal Graph Coloring		
Graph Topic	Basic Problem	Sample Application
Vertex coloring	Assign different colors to adjacent vertices, and use the minimum number of colors.	Avoid conflicts—for example, in meeting schedules or in chemical storage.

[1]When this problem was formulated, there were very few female sales representatives, so the historic name for the problem is the *traveling salesman problem*. We will often call the problem the *TSP*.

conflicts, or the earliest completion time. Algorithmic problem solving is the process of devising, using, and analyzing algorithms—step-by-step procedures—for solving problems.

When teaching these vertex-edge graph topics and themes, don't become bogged down in formal definitions and algorithms. Use the visual nature of graphs to make this material engaging, accessible, and fun. In fact, if you present vertex-edge graphs in this lively manner, many students who have previously experienced difficulty or apathy in mathematics may discover that the study of graphs is refreshing and interesting, and they may experience success in learning this topic, thereby gaining confidence about digging into other topics.

The following recommendations suggest how to develop the topic of vertex-edge graphs throughout the grades in a manner that is consistent with *Principles and Standards for School Mathematics*.

Many students who have previously experienced difficulty or apathy in mathematics may discover that the study of graphs is refreshing and interesting.

Recommendations for Vertex-Edge Graphs in Prekindergarten–Grade 12

In prekindergarten–grade 2, all students should—

- build and explore vertex-edge graphs by using concrete materials;
- explore simple properties of graphs, such as the numbers of vertices and edges, neighboring vertices and the degree of a vertex, and whole-number weights on edges;
- use graphs to solve problems related to paths, circuits, and networks in concrete settings;
- color simple pictures by using the minimum number of colors;
- follow and create simple sets of directions related to building and using graphs;
- concretely explore the notion of the shortest path between two vertices.

In grades 3–5, all students should—

- draw vertex-edge graphs to represent concrete situations;
- investigate simple properties of graphs, such as vertex degrees and edge weights, and explore ways to manipulate two graphs physically to determine whether they are the "same";
- use graphs to solve problems related to paths, circuits, and networks in concrete and abstract settings;
- color maps and color the vertices of a graph by using the minimum number of colors as an introduction to the general problem of avoiding conflicts;
- follow, devise, and describe step-by-step procedures related to working with graphs;
- analyze graph-related problems to find the "best" solution.

In grades 6–8, all students should—

- represent concrete and abstract situations by using vertex-edge graphs and represent vertex-edge graphs with adjacency matrices;
- describe and apply properties of graphs, such as vertex degrees, edge weights, directed edges, and isomorphism (whether two graphs are the "same");
- use graphs to solve problems related to paths, circuits, and networks in real-world and abstract settings, including explicit use of Euler paths, Hamilton paths, minimum spanning trees, and shortest paths;
- understand and apply vertex coloring to solve problems related to avoiding conflicts;
- use algorithmic thinking to solve problems related to vertex-edge graphs;
- use vertex-edge graphs to solve optimization problems.

In grades 9–12, all students should—

- understand and apply vertex-edge graph topics, including Euler paths, Hamilton paths, the traveling salesman problem (TSP), minimum spanning trees, critical paths, shortest paths, vertex coloring, and adjacency matrices;

- understand, analyze, and apply vertex-edge graphs to model and solve problems related to paths, circuits, networks, and relationships among a finite number of elements in real-world and abstract settings;
- devise, analyze, and apply algorithms for solving vertex-edge graph problems;
- compare and contrast topics in terms of algorithms, optimization, properties, and types of problems that can be solved;
- extend work with adjacency matrices for graphs through such activities as interpreting row sums and using the nth power of the adjacency matrix to count paths of length n in a graph.

Overview of Iteration and Recursion in Prekindergarten–Grade 12

Iteration and recursion constitute the third main discrete mathematics topic that *Principles and Standards for School Mathematics* recommends. Iteration and recursion are powerful tools for representing and analyzing regular patterns in sequential step-by-step change, such as day-by-day changes in the chlorine concentration in a swimming pool, year-by-year growth of money in a savings account, or the rising cost of postage as the number of ounces in a package increases.

As previously mentioned, to iterate means to repeat, so iteration is the process of repeating the same procedure or computation over and over again, like adding 4 each time to generate the next term in the sequence 4, 8, 12, 16, …. Recursion is the method of describing a given step in a sequential process in terms of the previous step or steps. A recursive formula provides a description of an iterative process. For example, the recursive formula NEXT = NOW + 4, or $s_{n+1} = s_n + 4$, with $n \geq 1$ and $s_1 = 4$, describes the pattern in the preceding sequence. The cluster of symbols s_n, which we read as "s sub n," provides a name for an arbitrary term, the nth term, of sequence s. Thus, s_4 is the fourth term, s_{10} is the tenth term, and so on. The equation $s_{n+1} = s_n + 4$ indicates that the $(n + 1)$st term of the sequence is 4 more than the nth term of the sequence; it has the same meaning as the equation NEXT = NOW + 4.

"Mathematics topics such as recursion, iteration, and the comparison of algorithms are receiving more attention in school mathematics because of their increasing relevance and utility in a technological world." (NCTM 2000, p. 16)

Iteration and recursion are two sides of the same coin. You can think of recursion as moving backward from the current step to previous steps, whereas iteration moves forward from the initial step. Both iteration and recursion are powerful tools for analyzing regular patterns of sequential change. (Computer science uses precise technical definitions for *iteration* and *recursion*, but this volume uses the terms in the more informal sense just described.)

As in the case of other topics, the students' work with iteration and recursion in the early grades is concrete and exploratory. The representation and analysis become more abstract and formal as the students progress through the grades. For example, in prekindergarten–grade 2, they should explore sequential patterns by using physical, auditory, or pictorial representations, like a pattern of handclaps that increases by two each time. In grades 3–5, students might describe a pattern of adding two each time as NEXT = NOW + 2. In middle school, they can begin using subscripts in a very basic way—for example, to describe the add-2 pattern with the recursive formula $T_{n+1} = T_n + 2$. In high school, students can take a recursive view of functions, recognizing, for example, that NEXT = NOW + 2 can represent a linear function with slope 2.

Both recursive and explicit representations have merit. A recursive formula gives the next term as a function of the current term. An explicit formula gives any term in the sequence without requiring knowledge of the previous term.

In middle school and high school, students should also compare and contrast recursive formulas and explicit, or closed-form, formulas. For example, they might describe the sequence 5, 8, 11, 14, 17, … by using the recursive formula $s_{n+1} = s_n + 3$, with the initial term $s_0 = 5$, or by using the explicit formula $s_n = 5 + 3n$, for $n \geq 0$. The recursive formula describes the step-by-step change and gives a formula for the next term, s_{n+1}, in terms of the current term, s_n. In contrast, the explicit formula gives a formula for any term s_n in the sequence, without requiring knowledge of the previous term. Both representations have merit. The recursive formula more clearly shows the pattern of adding 3 each time, but the explicit formula is more efficient for computing a term far along in the sequence, such as s_{50}.

All the NCTM Standards include the ideas of iteration and recursion. In support of this integration, the following recommendations suggest how to develop iteration and recursion across the grades.

Recommendations for Iteration and Recursion in Prekindergarten–Grade 12

In prekindergarten–grade 2, all students should—

- describe, analyze, and create a variety of simple sequential patterns in diverse concrete settings;
- explore sequential patterns by using physical, auditory, and pictorial representations;
- use sequential patterns and iterative procedures to model and solve simple concrete problems;
- explore simple iterative procedures in concrete settings by using technology, such as Logo-like environments and calculators.

In grades 3–5, all students should—

- describe, analyze, and create a variety of sequential patterns, including numeric and geometric patterns, such as repeating and growing patterns, tessellations, and fractal designs;
- represent sequential patterns by using informal notation and terminology for recursion, such as NOW, NEXT, and PREVIOUS;
- use sequential patterns, iterative procedures, and informal notation for recursion to model and solve problems, including those in simple real-world contexts, such as growth situations;
- describe and create simple iterative procedures by using technology, such as Logo-like environments, spreadsheets, and calculators.

In grades 6–8, all students should—

- describe, analyze, and create simple additive and multiplicative sequential patterns (in which a constant is added or multiplied at each step), as well as more complicated patterns, such as Pascal's triangle (in which each row of numbers, except the first two rows, is constructed from the previous row) and the Fibonacci sequence 1, 1, 2, 3, 5, 8, … (in which each term, except the first two terms, is the sum of the previous two terms);

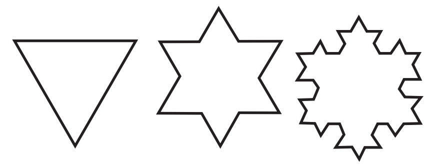
- use iterative procedures to generate geometric patterns, including fractals like the Koch snowflake and Sierpinski's triangle;
- use informal notation such as NOW and NEXT, as well as subscript notation, to represent sequential patterns;
- find and interpret explicit (closed-form) and recursive formulas for simple additive and multiplicative sequential patterns and translate between formulas of these types;
- use iterative procedures and simple recursive formulas to model and solve problems, including those in simple real-world settings;
- describe, create, and investigate iterative procedures by using technology, such as Logo-like environments, spreadsheets, calculators, and interactive geometry software.

In grades 9–12, all students should—

- describe, analyze, and create arithmetic and geometric sequences and series;
- create and analyze iterative geometric patterns, including fractals, with an investigation of self-similarity and the areas and perimeters of successive stages;
- represent and analyze functions by using iteration and recursion;
- use subscript and function notation to represent sequential patterns;
- investigate more complicated recursive formulas, such as simple nonlinear formulas; formulas in which the added quantity is a function of n, such as $S(n) = S(n-1) + (2n+1)$; and formulas of the form $A(n+1) = rA(n) + b$, recognizing that the resulting sequence is arithmetic when $r = 1$ and geometric when $b = 0$;
- use finite differences tables to find explicit (closed-form) formulas for sequences that can be represented by polynomial functions;
- understand and carry out proofs by mathematical induction, recognizing a typical situation for induction proofs, in which a recursive relationship is known and used to prove an explicit formula;
- use iteration and recursion to model and solve problems, including those in a variety of real-world contexts, particularly applied growth situations, such as population growth and compound interest;
- describe, analyze, and create iterative procedures and recursive formulas by using technology, such as computer software, graphing calculators, and programming languages.

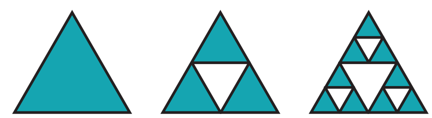
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Rows 0–5 of Pascal's triangle



Stage 0 Stage 1 Stage 2

Stages 0–2 of the Koch snowflake



Stage 0 Stage 1 Stage 2

Stages 0–2 of Sierpinski's triangle

Conclusion

This introduction has presented an overview of the three topics of discrete mathematics that *Principles and Standards* recommends for study in prekindergarten–grade 12: combinatorics (systematic listing and counting), vertex-edge graphs, and iteration and recursion. It has also provided specific recommendations for developing these important topics across the grades from prekindergarten–grade 12. The chapters that follow discuss each of these topics as they might be presented in grades 6–8 and grades 9–12. The development of an understanding of discrete mathematics in the earlier years is the focus of the companion volume, *Navigating through Discrete Mathematics in Prekindergarten–Grade 5* (DeBellis et al. forthcoming).