Preface

> I had a scheme, which I still use today when someone is explaining something that I'm trying to understand: I keep making up examples. For instance, the mathematicians would come in with a terrific theorem, and they're all excited. As they are telling me the conditions of the theorem, I construct something which fits all the conditions. You know, you have a set (one ball)—disjoint (two balls). Then the balls turn colors, grow hairs, or whatever, in my head as they put more conditions on. Finally, they state the theorem, which is some dumb thing about the ball which isn't true for my hairy green ball, so I say, "False!"
>
> —Richard Feynman,
> *Surely You're Joking, Mr. Feynman!*

When mathematics teachers at any level get together to talk about what they do, two questions are almost sure to come up:

- What do we teach?
- How do we teach?

Questions about content and pedagogy are central to what we do. It is right that these two questions are so important; thinking about them leads to improved curricula and teaching methods.

But there's a third important question, one that occupies the careers of many educational theorists, that is beginning to make its way into discussions in teachers' lounges and department meetings:

- How do students learn?

In many ways this is a much more difficult question. It requires that we look into the minds of our students and that we think about things from *their* perspectives. It is very hard for an adult, experienced in mathematics, to assume the perspectives of a beginner. But many teachers, mathematicians, and educators are realizing that smart decisions about content and pedagogy require that we understand much more about the ways our students learn mathematics, how they come to develop mathematical habits of mind, and even how they develop misunderstandings about our discipline.

This yearbook is about one aspect of how students learn mathematics. More precisely, this book is about how students learn to build mathematical representations of phenomena. This year marks the twenty-fifth anniversary of this series of yearbooks, and it is appropriate that we take up this timely theme.

All of us have an intuitive idea of what it means to represent a situation; we do it all the time when we teach or do mathematics. We represent numbers by points on a line or by rows of blocks. We use equations and geometric figures to represent each other. We talk about numerical, visual, tabular, and algebraic representations. And we *think* about things using "private" representations and mental images that are often difficult to describe.

But what do we mean, precisely, by "representation," and what does it mean to represent something? These turn out to be hard philosophical questions that get at the very nature of mathematical thinking.

I believe that as mathematics itself evolves, new methods and results shed light on such questions—that mathematics is its own mirror on the very thinking that creates it. And sure enough, there is a *mathematical* discipline called representation theory. In representation theory, one attempts to understand a mathematical structure by setting up a structure-preserving map (or correspondence) between it and a better-understood structure. There are two features of this mathematical use of the word *representation* that mirror uses of "representation" in this book:

- The representation is the *map*. It is neither the *source* of the representation (the thing being represented) nor its *target* (the better-understood object). When a child sets up a correspondence between numbers and points on a line, the points are not the representation; the representation lives in the setting up of the correspondence.
- Representations don't just match things; they preserve *structure*. Entering on a calculator an algebraic expression that stands for a physical interaction is not, all by itself, a representation. If algebraic operations on the expression correspond to transformations of the physical situation, *then* we have a genuine representation. Representations are "packages" that assign objects and their transformations to other objects and *their* transformations.

The articles in this book present a wide array of perspectives about the nature of representations, how students create them, and how they learn to use them. The book is divided into four parts, each a collection of articles that deal with a related circle of ideas.

The first part, "Roles for Representations," sets the stage by providing a discussion of two central dialectics in the educational theory of representations:

- *Internal and external representations.* External representations are the representations we can easily communicate to other people; they are the marks on the paper, the drawings, the geometry sketches, and the equations. Internal representations are the images we create in our minds for mathematical objects and processes—these are much harder to describe.

Gerald Goldin and Nina Shteingold discuss this distinction in their opening article. They present an overview of the theoretical issues and discuss an approach that integrates the research on internal and external representations.

- "*Invented" and "presented" representations.* Fran Curcio first used these words to describe these types of representations; she means the difference between representations that students invent and those passed down from teachers. We hear a great deal these days about student-invented representations. These are often quite different from the classical representations that have evolved in mathematics. The article by Constance Kamii, Lynn Kirkland, and Barbara Lewis makes a strong argument for the importance of allowing students to develop their own representations. However, centuries of evolution have produced standard mathematical representations that have been used to solve extremely deep problems. Mark Saul struggles in his article with the challenges of helping high school students understand the "standard" representational systems—the symbols and operations —of algebra. And the article by Rina Zazkis and Karen Gadowsky looks at the difficulties undergraduates have exploiting the "hidden meaning" in representations built up from the ordinary representations of arithmetic.

The second part, "Tools for Thinking," discusses representations as devices people use to help them gain insights into mathematical phenomena. Irene Miura describes a fascinating connection between one's natural language and how one thinks about numbers and numeration. Michelle Stephan, Paul Cobb, Keono Gravemeijer, and Beth Estes give an approach to the "invent or present" tension that introduces standard measuring tools in response to students' needs. Carmel Diezmann and Lyn English look at the role of diagrams in doing mathematics and discuss strategies for helping students become proficient at inventing and using diagrams. Marty Schnepp and Ricardo Nemirovsky describe the tools they use to represent some subtle ideas treated in AP calculus; one byproduct of their approach is that students see the computational techniques of calculus as tools for solving problems about rate and accumulations. Mark and Maxine Bridger develop an alternative to the rule of three for describing real-valued functions of a real variable; their "mapping diagram" representation highlights some important features of functions that are often hidden by tabular, symbolic, and graphical representations. Daniel Scher and Paul Goldenberg take us on a dynamic tour of the law of cosines, and they show how interactive geometry environments can be used to represent, illustrate, and even discover this important theorem. And Larry Lesser catalogs some beautiful representations that help make sense of a counterintuitive situation in statistics called Simpson's paradox.

Part 3 is called "Symbols and Symbol Systems." Mathematics is full of symbols—symbols that stand for numbers, functions, geometric objects, even other symbols. But the symbols of mathematics aren't just aliases. They are part of symbol *systems* that allow people to act on and transform the symbols in meaningful ways. Susan Lamon describes her research into effective ways for children to represent, use, and calculate with rational numbers. A great deal of research has gone on around the use of algebraic symbol systems to represent and transform algebraic functions. Wendy Coulombe and Sarah Berenson describe their work with beginning algebra students in this area. Alex Friedlander and Michal Tabach describe their work around multiple representations, using algebraic symbolism as one of several mechanisms for describing functions. Deborah Franzblau and Lisa Warner investigate different symbol systems for describing recursively defined phenomena, contrasting, for example, subscript and functional notation. Finally, Regina Kiczek, Carolyn Maher, and Robert Speiser tell the story of a student with whom they worked over the course of several years and who developed some creative ways for using the binary system of enumeration to solve a combinatorial problem.

The last part, "The Role of Context," looks at the interplay between modeling and representation. Kristine Reed Woleck offers an insightful look at her young students' work, showing how their representations of mathematical situations as pictures evolve into symbolic representations. Phyllis and David Whitin describe their work using literature with children to elicit mathematical thinking. Margaret Meyer picks up the story at the middle school level and shows how pictures and icons can be incorporated into symbol systems that closely approximate the classical system of algebra. Michal Yerushalmy and Beba Shternberg describe their approaches to strengthening what they call the "fragile link" between the visualization of situations and the concept of function and to helping students develop skill at using classical algebraic symbolism. And Josh Abrams takes us inside his high school modeling course, describing the techniques he uses to teach explicitly the skills of modeling and representation.

Assembling this collection of articles and helping authors revise their drafts were the work of an expert Editorial Panel:

- Hyman Bass, University of Michigan
- Carolyn Kieran, Université du Québec à Montréal
- Arthur B. Powell, Rutgers, The State University of New Jersey—Newark
- Jesse Solomon, City on a Hill Public Charter School, Boston, Massachusetts

Frances Curcio, the general editor for the 1999–2001 yearbooks, helped us through the entire process. Fran participated in the editorial deliberations, helped us stay on task, furnished us with context and background, and dealt with every detail at every level, all at once, all the time. Fran, the four panelists, and I quickly formed a team; we built on one another's ideas, learned from one another, and collaborated in ways that I'll very much miss.

Several other people worked behind the scenes to make this book possible. Helen Lebowitz and Sara Kennedy worked with me to communicate with authors and to manage the substantial amount of correspondence involved in producing this book. They provided exactly the help I needed with everything from scheduling meetings to editing articles. Wayne Harvey offered support, advice, and expert editing suggestions. Charles Clements and the NCTM staff worked incredibly hard on this project, editing and advising along the way and contributing to every aspect of the book. And my wife, Micky, helped me manage the details (a task at which I'm notoriously inept), listened to several hundred variations of my saying "It's almost done," and (almost) never complained about the meetings and the late nights and the piles of manuscripts on the floor of our study.

This book would not have been possible without the contributions of everyone who submitted a manuscript. The hardest part of this job was selecting the final manuscripts; given more pages, I would have liked to include much more than what is here. As I read the drafts, I was struck at how much knowledge, insight, creativity, and common sense are distributed across our field. And all this expertise—mathematics, pedagogy, and epistemology—gets integrated, synthesized, and applied every day in thousands of classrooms all over the country by classroom teachers—teachers who know how to take the ideas in this book and turn them into classroom experiences that make young people see the beauty and excitement in mathematics.

Albert A. Cuoco
2001 Yearbook Editor