Navigations Series

Grade 1

Problem Solving and Reasoning

Introduction

In three landmark publications—*Agenda for Action* (NCTM 1980), *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989), and *Principles and Standards for School Mathematics* (NCTM 2000)—the National Council of Teachers of Mathematics has consistently identified learning to solve problems as the major goal of school mathematics. Each of these publications highlights the importance of giving students opportunities to apply the mathematical concepts and skills that they are learning—together with various problem-solving strategies and methods of reasoning—to the solution of challenging problems. The hope is that students will gain a greater appreciation for the power of mathematics and for their abilities to wrestle with important mathematical ideas. Neither the mathematical knowledge nor the reasoning strategies can be developed in isolation. They must be learned and used concurrently. Furthermore, problem-solving strategies and reasoning methods are rarely applied in isolation from each other; they, too, are normally applied together in solving mathematical problems.

Problem-Solving Strategies and Reasoning Methods

Students begin to develop a variety of problem-solving strategies and reasoning methods in prekindergarten through grade 2. These strategies and methods are illustrated here with examples from the investigations in this book.

Problem-Solving Strategies and Reasoning Methods

- Identification of relationships
- Inference
- Generalization
- Representation
- Guess, check, and revise
- Analogy
- Verification

Identification of mathematical relationships

Determining how numbers, shapes, and mathematical concepts are related is central to understanding mathematics. Early in the learning of mathematics, students identify the characteristics of shapes in order to make comparisons. They look for similarities and differences among objects and numbers, and they sort, categorize, rank, or sequence them on the basis of attributes. Later, students differentiate among problems by noting their structural similarities and dissimilarities. At the most abstract level, students identify mathematical relationships presented symbolically or in tables, graphs, diagrams, models, or text.

Grade 1 students gain experience in identifying mathematical relationships in five investigations in this book. In Fit the Facts, students use clues to identify relationships and fit numbers into a story so that the story makes sense. The clues are related to age (older, younger, oldest) number (pairs, half a dozen), time (start of school, lunchtime, noon), or money. In Inside or Outside? students identify attributes common to the geometrical figures in a set, including size, number of sides, type of angles, and number of lines of symmetry. Creature Features requires students to identify proportional relationships among the numbers of eyes, ears, and teeth needed to make two, three, four, or five identical creatures. In Which Town Is Which? students identify towns on the basis of the distances of the towns from one another. Students determine relationships among data presented in graphs in the investigation Changeable Graphs.

Inference

Inference is the strategy of deducing unstated information from observed or stated information. Students use inferential reasoning when they formulate conjectures or hypotheses or draw conclusions from their analyses of a problem.

Grade 1 students practice inferential reasoning in four of the activities. In Fit the Facts, students use clues about age and time to figure out the age of the oldest or youngest or what time is reasonable for the start of school. In Inside or Outside? students draw conclusions about the unidentified attribute possessed by some shapes but not by others. In Changeable Graphs, students draw inferences from data obtained in consecutive years. For example, given a graph that shows that six children could tie their shoes last year and another graph that shows that eight children can tie their shoes this year, students can infer that two children learned to tie their shoes during the year. In Which Town Is Which? students draw inferences about distances from word clues and from measurements on a map. For example, given that the round-trip distance between two towns is ten miles, students must infer that the distance between the towns is $10 \div 2$, or five, miles.

Generalization

Generalization is the strategy of identifying a pattern of information or events and then using the pattern to formulate conclusions about other like situations. Students generalize when they—

- identify and continue shape, number, rhythm, color, and pitch patterns;
- describe these patterns with rules in words or symbols;
- predict from a sample; and
- identify trends from sets of data.

In Inside or Outside? students generalize relationships among shapes when they identify the attribute that characterizes an entire group of shapes.

Representation

Representation is the process of using symbols, words, illustrations, graphs, and charts to characterize mathematical concepts and ideas. It involves creating, interpreting, and linking various forms of information and data displays, including those that are graphic, textual, symbolic, three-dimensional, sketched, or simulated. The process also involves identifying the most appropriate display for a particular situation, purpose, and audience, and it requires the ability to translate among different representations of the same relationship.

In the activity Which Town Is Which? students learn to interpret data presented in a map. Changeable Graphs requires students to interpret data that are displayed in a graph. In Creature Features, students might draw pictures to represent creatures in order to determine the total number of eyes, mouths, and noses that will be needed for multiple creatures.

Guess, check, and revise

This strategy involves using one or more conditions of a problem to identify a candidate for the solution to the problem, checking the candidate against all the problem conditions, and revising the candidate appropriately if it does not meet all the conditions. The revised candidate for the solution is then checked against the problem conditions. The process continues until a solution that matches all the problem conditions is found.

In Fit the Facts, Inside or Outside? and Which Town Is Which? students practice making, checking, and revising their guesses. In Fit the Facts, they guess which numbers to use to fill in the blanks. As they continue, they may find that a previously used number is needed for a different blank. They must then revise their original guess to complete the story. In Inside or Outside? students identify the attribute shared by the shapes in a circle and choose a label for the circle. They must then check to make sure that all the shapes outside the circle do not possess the attribute and revise their guess if necessary. In Which Town Is Which? students identify towns on the basis of clues about their distances from other towns. As they consider new clues, students may have to revise their original guesses.

Analogy

Analogy is a method of identifying structural similarities and important elements in problems without regard to the particular contexts. Analogy facilitates the solution process because known or easily identified solutions to a simpler problem can be applied to a more complex problem. For instance, if students recognize that two problems are structurally alike and they know how to solve one of the problems, they can apply the same solution method to the other problem. In another example, when students are confronted with a complex mathematical problem, they may construct a simpler problem that preserves the essential features or properties of the more difficult problem. By solving the simpler problem first, the students may discover a solution method that can be applied to the more complex problem.

Students use analogical reasoning in the activity Creature Features. They reason, for example, that if one creature requires three eyes, then two creatures will require 3 + 3, or six, eyes and four creatures will require twelve eyes.

Verification

Verification is the process of checking, proving, or confirming a conclusion or point of view. Verification occurs when students—

- identify information that is relevant to, and has value for, the solution of a problem (and when they disregard irrelevant information);
- identify fallacies and unwarranted assumptions;
- recognize that solutions are reasonably close to estimates and make sense within the contexts of problems;
- justify the use of particular solution strategies by convincing arguments or—at a later age—proofs;
- formulate counterexamples.

Students also verify their own solutions when they identify gaps, inconsistencies, or contradictions in another person's line of reasoning.

Four of the activities in this book involve verification. In Fit the Facts, students verify that the story makes sense after they have filled in all the blanks. In Inside or Outside? students justify the placement of shapes inside or outside a circle on the basis of a common attribute, and they justify their identification of the shared attribute of shapes on the basis of the placement of the shapes inside or outside the circle. In Which Town Is Which? students can check that the clues and the data agree with their placement of the names of the towns on a map. Students can draw copies of the creatures they have made in Creature Features and then count to verify that they have drawn their creatures with the correct number of eyes or ears.

Developing Mathematical Dispositions

It is hoped that these investigations, which emphasize problem solving and reasoning, and other challenging mathematical activities will develop in students a love of mathematics and dispositions to—

- enjoy solving difficult problems;
- make sense of seemingly nonsensical situations or fix or "salvage" vague problems by rephrasing them and eliminating ambiguities;

- persist until they find a solution to a problem or until they determine that no solution exists;
- reflect on their solutions and solution methods and make adjustments accordingly;
- recognize that to solve some problems, they must learn more mathematics;
- generate new mathematical questions for a given problem;
- listen to others and analyze and verify their peers' lines of reasoning.

The Role of the Teacher

To strengthen students' mathematical reasoning and problem-solving abilities, teachers must create classroom environments that are mathematically "safe"—that is, ones in which every child feels free to make conjectures, to explore different ways of thinking, and to share his or her ideas with classmates. Teachers must be able to assess students' thinking and adjust mathematical tasks on the basis of assessment data. Most important, teachers must facilitate classroom discourse and ask probing questions in order to deepen students' understanding of the mathematics and of the reasoning methods and problem-solving strategies that the students employ.

Facilitate classroom discourse

Classroom discourse gives students opportunities to communicate their mathematical reasoning. In such discourse, students explore conjectures and clarify their understanding of problem-solving strategies. Informal discussions among pairs or small groups of students can enhance students' commitments to a task and assist less able learners in understanding the nature of a task, the meaning of the terminology, and the appropriate vocabulary to use in a response. Whole-class discussions serve as forums for students to share their findings, make generalizations, and explore alternative approaches. Classroom discourse also gives teachers important insights into their students' thinking.

Students in prekindergarten to grade 2 often share their mathematical thinking in pairs or small groups quite naturally, with little or no intervention by the teacher. Most young children are comfortable talking aloud as they solve problems. It can be challenging, however, to sustain a whole-class discussion among young students. Nonetheless, teachers can foster such discussions in a variety of ways:

- *Extend wait time.* Students need time to ponder important ideas and to formulate their responses. Don't be concerned if your students don't comment immediately. When teachers wait a bit longer than they are accustomed to doing, students often do respond.
- *Allow students to correct one another*. It can be difficult not to respond to every incorrect comment. Constant correction by the teacher, however, leads students to rely on the teacher as the authority rather than on their own mathematical knowledge, reasoning, and verification methods.

- *Ask more questions.* Instead of always responding to a student's contribution with a direct comment, encourage student-to-student interaction by asking such questions as these: "Did anyone else find this solution?" "Can anyone help with this question?" "What do you think we should do about this?"
- *Support reticent speakers.* Afford students who rarely comment or ask questions opportunities to practice what they intend to share with their group or class so that they may become more confident. Inquire if they would like to speak first so that they don't need to wait anxiously for their turns. You can also bring these students into discussions by asking, "Would anyone else like to add something or give another opinion?"
- *Encourage the use of recording sheets.* For very young children, recording may take the form of making simple drawings of solutions, strategies, or merely something about the problem. As students' abilities to record their thinking develop, drawings become more sophisticated, and recordings may include written explanations and symbolic representations. More-mature students may depict more than one solution strategy. The recording sheets give all students something to share and can help young children recall their investigative work.
- *Summarize ideas.* Recording students' ideas on the chalkboard or on large easel paper helps focus discussions and lets the students know that their ideas are important.

Students' discourse is an invaluable resource. It can lead to a deeper understanding of the mathematics embedded in problems and may launch new investigations. It offers opportunities for students to develop their reasoning abilities as they challenge and defend ideas. Finally, it gives teachers insights into students' thinking that can in turn be valuable in making instructional decisions.

Ask probing questions

The questions that a teacher asks during an investigation can help students understand their own thinking. In responding to these questions, the students make links among problems, strategies, and representations, and they check their logic and make generalizations.

Good problem solvers know what they are doing and why they are doing it. They know when they need help or should change strategies. Teachers' questions help young students develop good metacognitive habits. The following are examples of questions that prompt students' reflection:

- "What did you do first? Why?"
- "Why did you change your mind?"
- "What were you thinking when you recorded this?"
- "Which clue did you think was the most (least) helpful? Why?"
- "What made this investigation easy (or difficult) for you?"
- "What do you plan to do next?"
- "What hint would you give to a friend who was stuck?"

Discovering connections among problems, strategies, and representations deepens mathematical thinking and strengthens problem-solving abilities. To help students make such connections, ask questions such as these:

- "Does this problem remind you of another problem that you have already solved?"
- "Is there another way to solve this problem?"
- "Can you create a problem that could also be solved this way?"
- "Can you represent this information in a different way?"

Rich mathematical investigations give students opportunities to develop their reasoning skills further. Students can make predictions, generalize ideas, and recognize logical inconsistencies. Questions such as the following can help students enhance their reasoning abilities:

- "What do you think will happen next? Why?"
- "Do you think this pattern will continue? Why?"
- "Would this still be true if you began with an odd number [*or other counterexample*]?"
- "Can you state a general rule you have discovered?"
- "What will never happen when you do this?"

Finally, through your example, you can strengthen your students' problem-solving and reasoning abilities. Throughout the school day, teachers as well as students have numerous opportunities to exhibit curiosity about how things work and what generalizations can be made, to exemplify good reasoning and the use of varying problem-solving strategies, and to affirm the belief that mathematical thinking is an elegant and exciting problem-solving tool.

"Good problem solvers monitor their thinking regularly and automatically."
(Van de Walle 2004, p. 54)