Measurement and Fair-Sharing Models for Dividing Fractions

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Van de Walle (2007) describes dividing one fraction by another in this way: "Invert the divisor and multiply is probably one of the most mysterious rules in elementary mathematics" (p. 326). Tirosh (2000) concurs and cites research suggesting that "division of fractions is often considered the most mechanical and least understood topic in elementary school" (p. 6) and that students' performance on tasks involving division of fractions is typically very poor. These claims are reflected in the difficulties that college students experience in courses for mathematics for elementary teachers when they try to explain why the invert-and-multiply algorithm works. See the following problem.

> A new machine can polish 1/2 of the floors in 3/4 of an hour. What fraction of the floors can be polished per hour?

When solving such a problem, we often find that a student will write

$$\frac{1}{2}\times\frac{4}{3}=\frac{4}{6}=\frac{2}{3}.$$

When asked why this procedure works, he or she usually explains, "Well, it's really 1/2 ÷ 3/4, but I flipped the second fraction and then multiplied." When pressed to explain why it is possible to "flip the second fraction and multiply" to obtain the answer to 1/2 ÷ 3/4, the student usually responds, "Because it's a division problem."

One goal in our mathematics courses for elementary teachers is for students to develop a conceptual understanding of the standard algorithms for adding, subtracting, multiplying, and dividing whole numbers, fractions, and decimals. These courses are taught using an "inquiry approach." Class sessions are devoted to small-group work on challenging tasks intended to promote mathematical discussion among peers, followed by whole-class discussions of students' thinking about the tasks. Our role as instructors is to guide these discussions by introducing conventional terminology, symbols, and notation by posing "What if?" questions and counterexamples; by asking students to think about what they have done, about how others have done it, and about how they could have done it differently; and by asking them to consider why what they have done has or has not worked. With regard to helping students understand division of fractions, the challenge has been to develop sequences of activities that will help students (a) appropriately interpret situations that could involve division of fractions, and (b) make sense of algorithmic procedures for dividing fractions.

Using discussions and sample problems in van de Walle (2007) and Fosnot and Dolk (2002) as our starting point, we developed a sequence of activities for what van de Walle calls the "common-denominator algorithm" and a sequence of activities for the "invert-and-multiply algorithm." As van de Walle points out, these two algorithms are related to the two different interpretations of division—measurement and fair sharing. We highlight these interpretations with students when discussing whole-number division. Recall that in the measurement model of division, we know the size of each group and must find the number of groups of that size that can be made from the dividend. A problem that fits this model is the following:

> Ms. Wright has 28 students in her class. She wants to divide them into groups, with 4 students in each group. How many groups will she have?

In other words, "How many 4s are in 28?" In contrast, in the fair-sharing (or partitive) model of division, we know the number of groups to be formed and must determine the size of each group. A problem that fits this model would be this:

> Ms. Wright has 28 students in her class. She wants to divide them into 4 groups. How many students will be in each group?

In the remainder of this article, we will describe the two fraction division sequences we have developed, relate them to the two interpretations of division, and explain the rationale behind them.

The Common-Denominator Algorithm Sequence

We begin this sequence by introducing the idea of serving sizes using the nutrition facts label from the sides of various containers, noting that the serving size is not always a whole number (e.g., the serving size may be 1 1/2 cookies). The first set of problems that we present in the serving-size context is shown in **figure 1**. We explain in problem 2, for instance, that students should express any leftover cookies in terms of the fraction of a serving that they comprise. Students usually do not find this task to be too difficult for problems such as 2, but it becomes decidedly more challenging in problems such as 6 and 7. A typical solution for problem 6 is shown in **figure 2**. Students take one 3/4 serving from each cookie and then three more 1/4 pieces to make a sixth serving. They are left with two 1/4-cookie pieces. The dilemma is how to express the leftover amount. Many students initially say the answer is 6 1/2, which almost always leads to a rich discussion about the units to which the 6 and the 1/2 refer (cf. Perlwitz 2005). It is incorrect to say 6 1/2 servings, but many students struggle initially with viewing the two leftover pieces as 2/3 of a serving.

Note that these problems fit with the measurement interpretation of division because they are asking, "How many 1/2s are in 5?" and "How many 3/4s are in 5?" and so on. We continue working with problems in which both the serving size and the amount given are fractions (see **fig. 3**). We then move to a page of similar problems that contain no illustrations. Students are permitted to use drawings to help them solve the problems, but to move toward a computational algorithm for solving these problems, we encourage them to try to solve the problems without using drawings. The first three problems on this page are the following:

1. A serving is 3 cookies. How many servings can I make from 7 cookies?
2. A serving is 3/8 cookie. How many servings can I make from 7/8 cookie?
3. A serving is 3/11 cookie. How many servings can I make from 7/11 cookie?

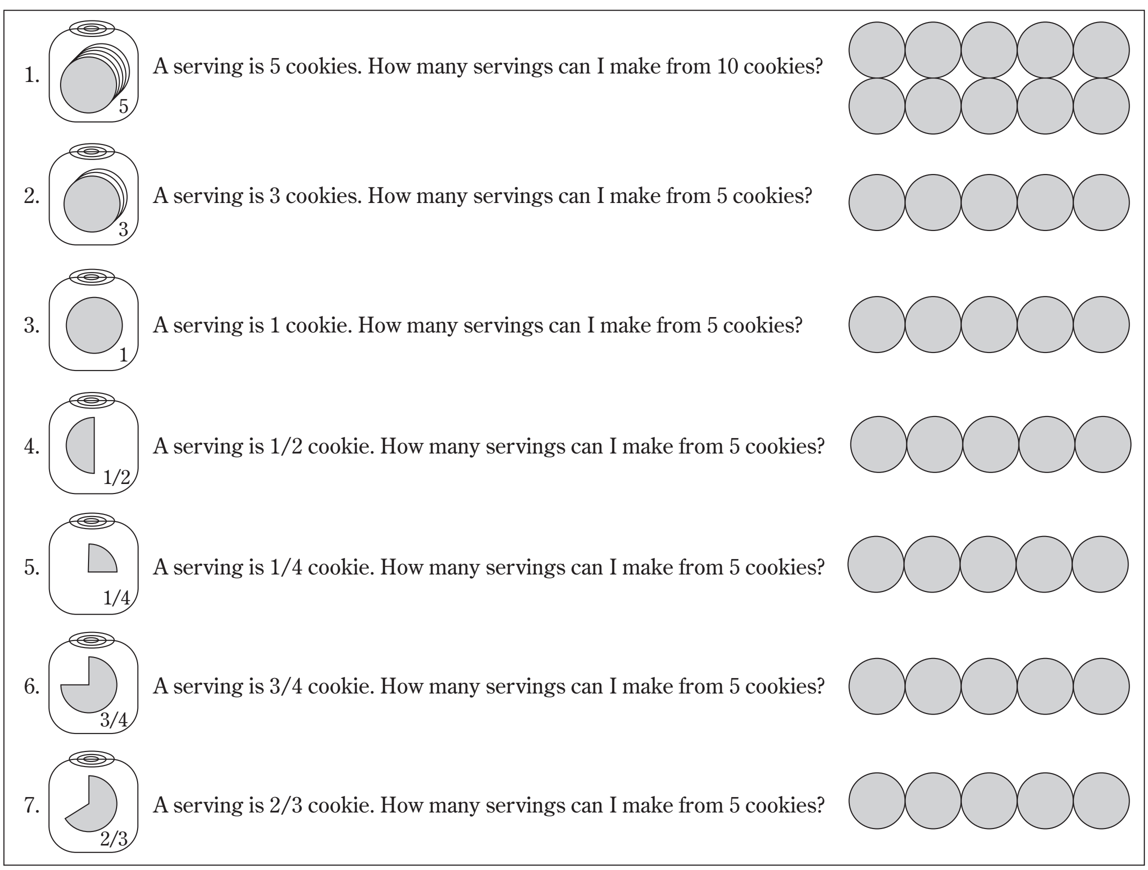


Fig. 1. A collection of problems in the serving-size context

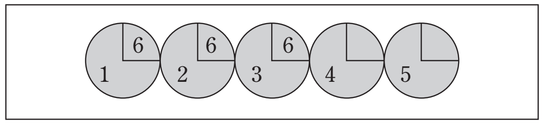


Fig. 2. A typical solution to problem 6 from figure 1

Fig. 3. Problems in which the serving size and the amount given are fractions

This sequence is intended to help students realize that as long as the serving size and the given amount are expressed in the same-sized pieces (e.g., whole cookies, eighth of a cookie, eleventh of a cookie), then the size of the pieces (as expressed by the denominator) is irrelevant. In each case, a serving consists of 3 things of a certain size, and we have 7 things of that same certain size, so how many servings can we make?

At this point, we have not discussed with students that these items may be viewed as division problems, so we present one more page of problems without illustrations. The first five problems on this page are the following:

1. A serving size is 6 cookies. How many servings can I make from 30 cookies?
2. A serving size is 7 cookies. How many servings can I make from 30 cookies?
3. A serving size is 1/2 cookie. How many servings can I make from 30 cookies?

4. A serving size is 1/4 cookie. How many servings can I make from 30 cookies?

5. A serving size is 1/2 cookie. How many servings can I make from 3/4 cookie?

After discussing students' solutions to these problems, we ask, "How can we view these problems as division problems? Can you write a division number sentence for each of these problems?" Students have little difficulty writing the number sentences 30 ÷ 6 and 30 ÷ 7, respectively, for the first two problems. For problem 3, many students figure out that the answer is 60 and many write the number sentence 30 ÷ 1/2, but some students think 30 ÷ 1/2 should be 15. At this point, we discuss the measurement interpretation of division: How many 7s are in 30? How many 1/2s are in 30? (as opposed to how many 2s are in 30?) and so on. Students are then able to interpret problem 5 as being 3/4 ÷ 1/2, or how many 1/2s are in 3/4? We also discuss the idea that the question asked in problem 5 is exactly the same as that asked in problem 1. The only difference is the size of a serving and the amount of cookies we have from which to make servings.

Next we return to a discussion of the units associated with the answer to a problem such as 3/4 ÷ 1/2. Students have little difficulty with the cookie/serving-size context since they have previously used diagrams (as shown in **fig. 4**) to solve such problems. But what about the number sentence 3/4 ÷ 1/2 = 1 1/2? To what does the 1 1/2 refer? The students' drawings and the measurement interpretation of division are helpful when exploring this issue. If the question is "How many 1/2s are in 3/4?" then the answer, 1 1/2, must mean that there are one and a half 1/2s in 3/4. **Figure 4** illustrates this solution if we replace "1 serving" by "1/2 serving." We discuss with our preservice teachers the subtle yet significant challenge that students face in making sense of 3/4 ÷ 1/2 = 1 1/2 in a measurement context: The dividend and the divisor refer to the same-sized unit (e.g., 3/4 of a cookie, 1/2 of a cookie), but the quotient refers to a unit that is the size of the divisor (e.g., 1 1/2 *half* cookies).

We are ready to move toward the common-denominator algorithm for dividing fractions and present the problems shown in **figure 5**. Tim is a pseudonym for an eighth-grade student who constructed this method as he participated in a series of lessons taught by one of the authors. We extend Tim's strategy notationally by writing the following:

$$\frac{3}{4} \div \frac{1}{3} = \frac{9}{12} \div \frac{4}{12} = 9 \div 4 = 2\frac{1}{4}$$

We relate this to the previously discussed idea that if both the serving size and the given amount are expressed in the same-sized pieces, then the denominator is irrelevant. One must focus on the number of pieces in the serving size and the given amount (i.e., the numerator). We also relate the process of getting a common denominator in this algorithm to the need, when solving the problem pictorially, to cut the representations of both the dividend and the divisor into pieces that are the same size (see **fig. 4**). Note that this algorithm is essentially the same as that invented by a seventh grader whom Perlwitz (2004) interviewed. In fact, in many of our classes, the "Tim's Method" page (see **fig. 5**) is not needed because several students have already invented a comparable strategy by the time we reach this point in the sequence.

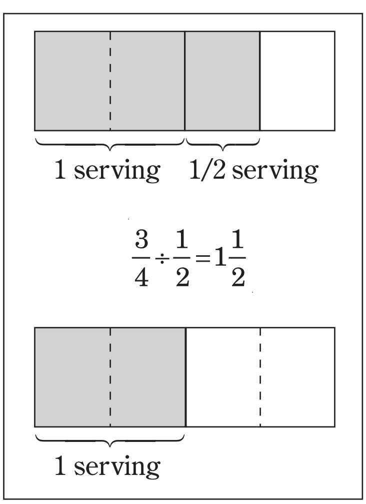


Fig. 4. There are one and a half 1/2s in 3/4.

Tim devised the following method for figuring out the "How many servings?" problems.

Tim said: To figure out a problem like *My serving size is 1/3 cookie. How many servings can I make from 3/4 cookie?* you can first figure out a common denominator for these numbers. By making the 1/3 = 4/12 and the 3/4 = 9/12, the problem is much easier to solve. From the 9/12 I can get 2 whole servings of 4/12 and have 1/12 leftover. The 1/12 that is leftover is 1/4 of a serving, so my answer is 2 1/4. This works with all division with fraction problems.

Do you think Tim's method is valid? Test Tim's method on the tasks below. Then explain why you think this method works with division of fraction problems or why it does not work.

a. 5/8 ÷ 1/2 b. 2 1/4 ÷ 3/8 c. 3/8 ÷ 1/2 d. 9/8 ÷ 2/3

Fig. 5. Tim's method

The Invert-and-Multiply Algorithm Sequence

After applying the measurement interpretation of division to make sense of division of fractions situations, we ask our students if we could apply the fair-sharing interpretation of division to division of fractions. They initially respond no. If one starts by considering a division sentence such as 3/4 ÷ 2/3, it is not clear how such an interpretation might apply. We want to divide 3/4 of a cake equally among 2/3 of a group. What does that mean? We take a similar approach in designing this sequence as in the common-denominator algorithm sequence, that is, we start with situations involving whole numbers and work our way toward those involving fractions.

In particular, we start with problems involving whole-number divisors and unit-fraction dividends followed by whole-number divisors and non-unit-fraction dividends. The following is a typical initial sequence:

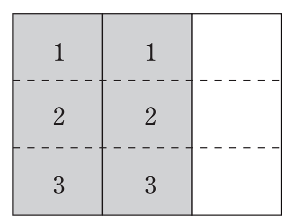
1. I have 1/3 of a whole cake. I want to divide it equally into 3 containers. How much cake will be in each container?
2. I have 1/3 of a whole cake. I want to divide it equally into 4 containers. How much cake will be in each container?
3. I have 1/3 of a whole cake. I want to divide it equally into 8 containers. How much cake will be in each container?
4. I have 2/3 of a whole cake. I want to divide it equally into 2 containers. How much cake will be in each container?
5. I have 2/3 of a whole cake. I want to divide it equally into 3 containers. How much cake will be in each container?
6. I have 3/4 of a whole cake. I want to divide it equally into 2 containers. How much cake will be in each container?

Students often use diagrams to help solve these problems. A typical solution for problem 5 is shown in **figure 6**. Students begin by drawing a cake and shading 2/3. Then they cut the 2/3 into three equal parts (represented by the horizontal dashed lines) and determine what fraction of a whole cake each of the three equal parts comprises.

Some students develop nonpictorial strategies for problem 6:

$$\frac{3}{4}=\frac{6}{8}=\frac{3}{8}+\frac{3}{8}$$

This equation shows that each container holds 3/8 of a whole cake. As we discuss students' solutions to these problems, we ask, "What division number sentence could we write for this problem?" Few students have difficulty interpreting these problems as 2/3 ÷ 3, 3/4 ÷ 2, and so on. We discuss that we are now using the fair-sharing interpretation of division, since we are distributing (sharing) a certain amount of cake among some number of containers and want to know how much cake will be in one container.



There will be 2/9 of a cake in each container.

Fig. 6. I have 2/3 of a whole cake. I want to divide it equally into 3 containers. How much cake will be in each container?

Next we move to problems with unit-fraction divisors:

1. I have 1/3 of a whole cake. It fills up exactly 1/2 of my container. How much cake will fit in 1 whole container?
2. I have 1/3 of a whole cake. It fills up exactly 1/4 of my container. How much cake will fit in 1 whole container?
3. I have 3/4 of a whole cake. It fills up exactly 1/2 of my container. How much cake will fit in 1 whole container?

For these problems, many students apply a repeated-addition or multiplicative strategy: If 3/4 of a cake fills up 1/2 of the container, then the whole container must hold 3/4 + 3/4 = 2 × 3/4 = 1 1/2 cakes. Although these problems are not difficult for students, the key discussion point is to connect the problems to division and to the problems with whole-number divisors discussed previously. What would be an appropriate division number sentence for problem 3 above? Much as we did in the common-denominator algorithm sequence, we ask students to compare problems with whole-number divisors to problems with fractional divisors. For example, consider the following:

1. I have 3/4 of a whole cake. I want to divide it equally into 2 containers. How much will be in each container?
2. I have 3/4 of a whole cake. It fills up exactly 1/2 of my container. How much cake will fit in 1 whole container?

In both cases, there is an amount of cake that fits into a certain space, and the problem is to determine how much *1 container* will hold. If the first problem is 3/4 ÷ 2, then the second one must be 3/4 ÷ 1/2. Note the ratio aspect:

$$\frac{3/4 \text{ cake}}{2 \text{ containers}} = \frac{\text{how much cake}}{1 \text{ container}}$$

and

$$\frac{3/4 \text{ cake}}{1/2 \text{ container}} = \frac{\text{how much cake}}{1 \text{ container}}$$

We conclude the sequence by moving to cake problems with non-unit-fraction divisors, first using a whole-number, then a unit-fraction, and finally a non-unit-fraction amount of cake:

1. I have 3 whole cakes. They fill up exactly 2/3 of my container.
   a. How much cake will fit in 1/3 of my container?
   b. How much cake will fit in 1 whole container?
2. I have 1/2 of a cake. It fills up exactly 3/4 of my container.
   a. How much cake will fit in 1/4 of the container?
   b. How much cake will fit in 1 whole container?
3. I have 3/4 of a cake. It fills up exactly 2/3 of my container.
   a. How much cake will fit in 1/3 of the container?
   b. How much cake will fit in 1 whole container?

A pictorial solution for problem 3 is shown in **figure 7**. Each question is broken into 2 parts in an effort to foster solution strategies that can be related to the invert-and-multiply algorithm. For example, to find how much cake will fit in 1/3 of a container in problem 3, we can divide 3/4 by 2. Knowing how much fits in 1/3 of a container, we can then multiply by 3 to determine how much 1 container holds. We describe this process notationally as

$$\frac{3}{4} \div \frac{2}{3} = \left(\frac{3}{4} \div 2\right) \times 3.$$

However, when dividing some amount into 2 equal parts (recall that we are using the fair-sharing interpretation of division), each of those 2 parts is 1/2 of the total amount. So dividing by 2 is the same as multiplying by 1/2 (3/4 is *two*-thirds; 1/2 of that amount will be *one*-third). We write

$$\frac{3}{4} \div \frac{2}{3} = \left(\frac{3}{4} \div 2\right) \times 3 = \left(\frac{3}{4} \times \frac{1}{2}\right) \times 3 = \frac{3}{4} \times \frac{3}{2}.$$

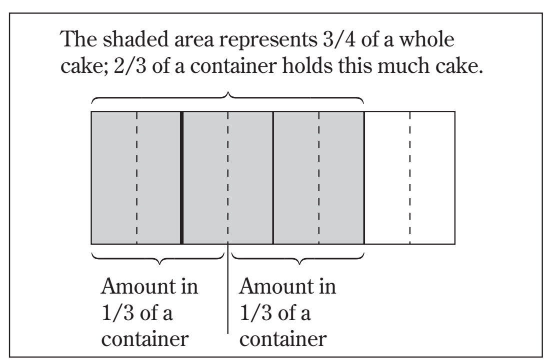


Fig. 7. If 1/3 of the container holds 3/8 of a cake, 1 container holds 3/8 + 3/8 + 3/8 = 3 × 3/8 = 9/8 = 1 1/8 cakes.

Similarly, if 2/5 of a cake fills 3/4 of a container, and we wanted to know how much 1 container would hold, we could divide 2/5 by 3 to find how much fits in 1/4 of a container and then multiply by 4. We would write

$$\frac{2}{5} \div \frac{3}{4} = \left(\frac{2}{5} \div 3\right) \times 4 = \left(\frac{2}{5} \times \frac{1}{3}\right) \times 4 = \frac{2}{5} \times \frac{4}{3}.$$

Conclusion

We have used the two sequences of problems described here with preservice teachers in both methods and mathematics content courses, and one of the authors has used them in conjunction with the classroom teacher in both a sixth-grade and an eighth-grade class. For both groups, the common-denominator algorithm stemming from the measurement interpretation of division seemed more accessible in terms of students being able to construct the algorithm with a conceptual grounding. Flores, Turner, and Bachmann (2005) describe two teachers who "made the connection between [their] previous understanding of division of fractions in terms of measurement and the standard rule of multiplying by the inverse" (p. 118). However, these teachers simply noted that the result obtained by dividing the numerators in the common-denominator algorithm was the same as that obtained when multiplying the dividend by the inverse of the divisor in the invert-and-multiply algorithm. They did not explain why the invert-and-multiply rule works. Thus, we believe that some sort of invert-and-multiply algorithm sequence of problems is needed.

We interspersed the cake problems in our invert-and-multiply sequence with problems like this:

> 2/5 of a room can be painted in 3/4 of an hour. How much can be painted in 1 hour?

These problems engendered more ratio interpretations on the part of students. In other words, students reasoned as follows:

$$\frac{\frac{2}{5}\text{ room}}{\frac{3}{4}\text{ hour}} = \frac{?\text{ room}}{1\text{ hour}}$$

We had done considerable work with ratio tables in which students generated equivalent ratios. So the students could reason that to get from 3/4 hour to 1 hour they would need to multiply by 4/3. (To our delight, one student explained that you would need to multiply by 1 1/3 because in 1 hour there is one 3/4 hour and 1/3 of another 3/4 hour.) To keep the ratio the same, the 2/5 must also be multiplied by 4/3. This gives

$$\frac{\frac{2}{5}}{\frac{3}{4}} = \frac{\frac{2}{5} \times \frac{4}{3}}{\frac{3}{4} \times \frac{4}{3}} = \frac{\frac{2}{5} \times \frac{4}{3}}{1} = \frac{2}{5} \times \frac{4}{3}.$$

Thus, the painting problems resulted in what Tirosh (2000) calls a formal argument for fraction division, one that uses ratios, fraction multiplication, and the principle that the product of reciprocals is 1. We are anxious to continue developing our invert-and-multiply algorithm sequence to examine the influence of the cake problems, the painting problems, and perhaps several other scenarios while helping our students make sense of this algorithm.

References

Flores, Alfinio, Erin E. Turner, and Renee C. Bachmann. "Posing Problems to Develop Conceptual Understanding: Two Teachers Make Sense of Division of Fractions." *Teaching Children Mathematics* 12 (October 2005): 117–21.

Fosnot, Catherine Twomey, and Maarten Dolk. *Young Mathematicians at Work: Constructing Fractions, Decimals, and Percents*. Portsmouth, N.H.: Heinemann, 2002.

Perlwitz, Marcela D. "Two Students' Constructed Strategies to Divide Fractions." *Mathematics Teaching in the Middle School* 10 (October 2004): 122–26.

———. "Dividing Fractions: Reconciling Self-Generated Solutions with Algorithmic Answers." *Mathematics Teaching in the Middle School* 10 (February 2005): 278–83.

Tirosh, Dina. "Enhancing Prospective Teachers' Knowledge of Children's Conceptions: The Case of Division of Fractions." *Journal for Research in Mathematics Education* 31 (January 2000): 5–25.

van de Walle, John A. *Elementary and Middle School Mathematics: Teaching Developmentally*. Boston: Allyn & Bacon, 2007.