Convey the Many Meanings of $\frac{a}{b}$

Elementary and middle school programs must provide students with adequate time and experiences to develop a deep conceptual understanding of this important area of the curriculum.

John A. Van de Walle, Karen S. Karp, and Jennifer M. Bay-Williams (2010, p. 286)

The ways in which students understand the meaning and concept of fractions have important implications for what they will understand and be able to do later on when faced with new ideas that build on this concept—including quotients, decimals, percents, ratios, rates, proportions, proportionality, and linearity. In high school or college calculus, some students will encounter even more advanced mathematical ideas, such as the notion of derivative as the ratio of differentials $\frac{dx}{dy}$.

By the *meaning* of a fraction $\frac{a}{b}$, I mean the many possible concepts the symbol can represent. And by *understanding*, I mean what results from a student's interpretations of words, symbols, actions, and discussions pertaining to fraction contexts and situations. Students assign meaning according to a web of connections that they build over time, through interactions with their own

interpretations of fractions and through interactions with other students as they, too, struggle to construct new understandings.

This makes the teacher's role in fraction knowledge building all the more crucial: teachers must offer students rich and varied experiences if they are to develop a dense web of meanings around the concept of *fraction*—meanings they can fall back on when they become confused, forget a memorized procedure, or learn a new fraction-related concept (such as ratio). Building a robust *fraction sense* is not simple. It is much more than correctly naming a fraction's components, accurately shading a given fraction of a region, or successfully carrying out a computational algorithm. It is even more than knowing the procedural rules for transforming, say, a fraction to a decimal or a percent to a fraction. The key to a well-grounded fraction sense is *time*. Mathematical knowledge building takes time—and by time, I mean *years*.

A good place to start is with students' own mental images and ideas about fractions. Teachers of course cannot read the minds of their students to see these internal representations. Rather, they must access them indirectly by making inferences from students' discourse about fractions, the outer representations they construct, and their ongoing interactions with fractions through drawing, gesturing, and writing. Through careful observation and conversation, we can detect misconceptions, help students make connections between their existing informal knowledge and the new mathematical constructs we hope to teach them, and help them use their intuition to construct new knowledge.

Roberto's Story

Roberto taught middle school for several years and then left to take a better-paying job in industry. But after a while he found that the good money alone was not fulfilling; he missed teaching, and he especially

missed the kids, so he decided to return to the classroom. He had hoped to go back to his middle school, but there were no openings, so he took a position teaching 5th grade instead. What follows are some key moments I observed during Roberto's introductory lesson on fractions during his first year of teaching 5th grade.

Roberto wrote "$\frac{3}{4}$" on the board and asked his students to write down or draw whatever came to mind when they saw this fraction. After some reflection time, students shared their ideas; Roberto asked two students to record all of the answers on the board.

Many students offered part-whole interpretations of the fraction, represented by the shading of $\frac{3}{4}$ of a square, rectangle, or circle, accompanied by explanations such as Tatiana's: "I have a cookie and eat a quarter of it, so three-fourths is left. I see the picture of the circle with a quarter of it missing."

Another student, Kaleb, used his class instead of a region to explain his idea of $\frac{3}{4}$:

Kaleb: 21 students in our class!

Roberto: How is 21 students an interpretation of $\frac{3}{4}$?

Kaleb: We're 28 students in this class. I split the class in four groups of seven students. Three groups is 21 people, so that's three-fourths of our class.

Roberto: Ah! Now I see. Can anyone explain in what way Tatiana's $\frac{3}{4}$ of a cookie and Kaleb's $\frac{3}{4}$ of you 5th graders are different and in what way they're the same?

After an insightful discussion about the difference between *discrete* quantities (such as students, tables, and marbles) and *continuous* quantities (such as time, money, length, and area), Roberto identified for students their most popular interpretation of $\frac{3}{4}$ as the "part-whole" meaning

of a fraction: "When we partition the whole or set into four equal parts and take three parts, then $\frac{3}{4}$ is one part, $\frac{1}{4}$ is the other part, and $\frac{4}{4}$ is the whole."

Here are some other meanings students shared about $\frac{3}{4}$:

- 0.75 (3 students)

- 45 minutes (2 students)

- 75¢ (1 student)

- $\frac{3}{4}$ of a mile (1 student)

- 75% (1 student)

Only about a quarter of the class expressed an idea other than the part of a region or set.

Roberto posed another question to the class:

Roberto: Suppose I decided to group 75¢, $\frac{3}{4}$ of a mile, and 45 minutes into one category of ideas you all came up with. What might be my criterion behind that grouping? Serena?

Serena: They're all real.

Roberto: They're all real-world examples—is that what you mean, Serena?

Serena: Yes.

Roberto: What do those examples tell us about the real world?

Another enlightening discussion ensued about measuring quantities in the real world, such as money (75¢), distance $\left(\frac{3}{4} \text{ of a mile}\right)$, and time (45 minutes), and how when we measure real-life quantities, we often get non-integer measures, which are called *fractional measures*. Roberto's more subtle goal for this discussion was to make explicit to students the implicit nature of the unit they took for granted in all three interpretations: $1.00 (or 100 cents), 1 mile, and 1 hour (or 60 minutes), respectively.

Roberto ended this discussion on fractions by drawing four squares on the board and shading three.

He said, "First I asked you what images the symbol 3 over 4 conjured up in your mind. Now I'm asking what this set of four squares, with three of them shaded, illustrates to you." In quasi-unison, the class answered, "Three-fourths." Roberto waited a moment and then inquired, "Anything else?"

Finally, one student in the back row shyly asked, "Could it also be a fourth?"

"Explain your thinking," Roberto prompted.

"One white square out of four."

"Absolutely!" He paused. "Anything else?" This question was greeted with silence.

Roberto concluded with a homework assignment: "For tomorrow, think about whether the illustration could represent $\frac{1}{3}$ or $\frac{4}{3}$, in addition to $\frac{1}{4}$ or $\frac{3}{4}$."

This experienced teacher, though new to the elementary school grades, benefited from the perspective on rational numbers expounded in the middle grades. His ability to align students' developing ideas with the new ideas he wanted to teach—not by merely stating definitions or rules but through effective classroom math talk—was evident. He regularly made students' ideas more explicit, for their own benefit as well as others'. He drew powerful connections among the different solutions provided as ways to guide his students to new mathematical territory: the distinction between *discrete* versus *continuous* quantities, which he differentiated as "quantities we can *count*" versus "quantities we can *measure*." Instead of

lamenting the high number of part-whole interpretations, he used them to review the notion of part-to-whole and the importance of the *unit* in measuring.

⬡ Teaching Tip: Revoicing During Whole-Group Discussions

Given the noise and confusion that can occur in a classroom of 20–30 students, there has been a growing interest in the notion of *revoicing* in classroom discourse. Enyedy and colleagues (2008) describe revoicing as a discursive teaching practice that promotes deeper conceptual understanding: teachers position students in relation to one another, which facilitates classroom debate and fosters mathematical argumentation. While that sounds like a sophisticated practice, it's something that teachers do daily. Nevertheless, we can become more effective at revoicing by being more mindful. For example,

- We can respond neutrally. When we revoice a correct answer, is our restatement accompanied by a smile of approval or an affirming tone of voice? Conversely, is our restatement of an incorrect answer tainted by a frown or an interrogative tone of voice? Rather than result in further classroom participation, such revoicing ironically cuts it short. If we add too much of our own thinking to our students' utterances, we discourage them from reasoning further, both on their own thinking and on the contributions of others (O'Connor & Michaels, 1996).

- We can improve the effect of rebroadcasting students' ideas by adding a verification right after a neutral reformulation, such as "Is that what I heard?" or "Is that what you just said?" If the restatement is for the purpose of clarity, try asking, "Is that what you mean?" or "Is that what you're trying to say?" The ball is then back in the student's court, and the discussion progresses.

- We can invite a student to do the revoicing, which ensures neutrality. Asking, "Did everyone hear what Sandra said?" in tones of interest or excitement is one way of

drawing attention to an answer or statement, whether correct or incorrect, that you wish to highlight or use as a teaching tool.

CCSS.Math.Practice.MP3 states that students will "construct viable arguments and critique the reasoning of others" (Common Core State Standards Initiative, 2010). If this is not yet an established mathematical practice in your classroom, then many students may answer that they did not hear Sandra's response—*but there's always at least one student who did*. Ask that student to revoice for the benefit of the whole class: "Can you please restate what you heard Sandra say?'

Recognizing Misconceptions

Beneath the fraction symbol $\frac{3}{4}$ in particular, or the fraction symbol $\frac{a}{b}$ in general, lies a multitude of meanings and interpretations that students develop and come to know over time through a variety of out-of-school experiences and instructional tasks involving thinking, talking, gesturing, doing, operating, and solving. Traditionally, though, elementary school students' experiences with fractions have been restricted to the part-whole interpretation. More recently, mathematics educators and researchers have realized that this limited view of fractions has left students with an impoverished foundation for the complex system of rational numbers and operations. Consequently, teachers such as Roberto are making conscious attempts to discuss and explore all of the meanings of fraction symbols described in this chapter.

Limited Ideas About the Meaning of a Fraction

Rather than having misconceptions about the meaning of fractions, what we observed in Roberto's classroom is what research verifies: students' ideas associated with fractions are limited. About

three-fourths of the 5th graders (20 out of 28) in the vignette gravitated toward the part-whole interpretation of $\frac{3}{4}$ and only one student thought of a fractional part of a *set* or *collection* rather than a fractional part of a *region*. Despite standards and curricula that claim to stress the measure, number, quotient, operator, and ratio meanings of fractions alongside the part-whole metaphor, the latter is most commonly used to introduce fractions to young students because it builds on their grounded understanding of partitioning for equal sharing. However, for students acquainted with only the part-whole construct of $\frac{3}{4}$ ("three parts out of four equal parts"), the fraction $\frac{5}{4}$ will seem senseless: How can we take five parts out of only four equal parts? Further down the line, when studying division with fractions, dividing a whole number by a fraction will seem meaningless as well: The expression $5 \div \frac{2}{3}$, or $\frac{5}{2/3}$ or $\frac{5}{\frac{2}{3}}$ will be interpreted as "5 parts out of $\frac{2}{3}$ parts" or even "5 parts out of (2 parts out of 3 equal parts)," which is even harder for students to grasp than "5 parts out of 4."

Difficulty Conceptualizing a Fraction as a Single Number

The CCSSM introductory standard in the grade 3 category "Number and Operations—Fractions" clearly states, "Develop understanding of fractions as numbers" (Common Core State Standards Initiative, 2010). However, the composite nature of fractions creates a serious obstacle. How is the meaning of 3 combined with the meaning of 4 supposed to give meaning to the numeral $\frac{3}{4}$? Moreover, when students try to represent $\frac{3}{4}$ as a point on the number line, knowing the location of the points representing 3 and 4 on the number line doesn't really serve them. Conceptualizing a fraction as a single entity is expressed in the upper elementary school grades by representing a fraction as a single point on the number line. The fact that not one of Roberto's students mentioned the idea that $\frac{3}{4}$ could be a number, or drew a point on the number line to show their thinking, confirms the

research on children's primary difficulty when beginning their journey into fractions: thinking of a fraction as a single number.

Unpacking the Mathematical Thinking

Deeply understanding fractions involves knowing what the word *fraction* and the symbol $\frac{a}{b}$ mean, appreciating the different mathematical concepts associated with fractions, and knowing how these fit together in a web of related meanings. Two factors infuse a mathematical construct with meaning: the mathematical *theory* behind the construct and the mathematical *applications* of the construct. Kieren (1976), Vergnaud (1979), and Freudenthal (1983) independently suggested that a fully developed rational number construct implies a rich set of integrated *subconstructs*, including part-whole, measure, quotient, ratio, and multiplicative operator. Since understanding fractions in grades 3–5 is a precursor to understanding rational numbers later on, it is essential that we begin in 3rd grade to introduce these related meanings of fractions, and their associated processes, by exposing students to multiple applications. We begin with the part-whole meaning of fraction.

The Part-Whole Meaning of $\frac{a}{b}$

While students have limited out-of-school contexts in which they can construct meaning for fractions, they do have extensive experience partitioning, mostly with the goal of forming equal shares. Young children have a good understanding of constructing fractional parts of a whole, such as 2 *one-halves* of a square sandwich, 3 *one-thirds* of a rectangular chocolate bar, or 4 *one-fourths* of a circular birthday cake. The *parts* are what result when the *whole* is partitioned into equal-size portions or fair shares.

Students are less comfortable with *parts of a set*, as we will see in later chapters. Roberto's three shaded squares is an example of a part of a set of four squares.

The CCSS Math Content.3.NF.A.1 states that students should interpret a fraction $\frac{a}{b}$ as the quantity formed by a parts of size $\frac{1}{b}$ (Common Core State Standards Initiative, 2010) and does not use the traditional phrase heard in classrooms everywhere: "$\frac{a}{b}$ is a parts out of b equal parts." What is behind this subtle linguistic distinction? Namely, it guards students against inferring four ideas that don't serve them as they move through the grades and construct more complex ideas associated with the symbol $\frac{a}{b}$:

- **Inclusion.** When $\frac{a}{b}$ is stated as "a parts out of b equal parts," there is a sense that the a parts are a subset of the b parts. From this, students conclude that the whole and the parts must be *of the same nature*. Consequently, the question "What fraction of the children in the school choir are girls?" makes sense to them because girls *are* children—but when later tackling ratios, questions like "What fraction of the number of boys in the choir is the number of girls?" will confuse them because girls *are not* boys.

  Note: An analogy with a continuous quantity is "What fraction of the salad dressing is oil [part to whole → *fraction*]?" versus "What fraction of the oil volume is the vinegar volume [part to part → *ratio*]?"

- **Size.** Interpreting $\frac{a}{b}$ solely as a parts out of b equal parts infers that a must be smaller than b, since we take the a parts out of the b parts. Thus, fractions $\frac{2}{3}$ and $\frac{4}{5}$ make sense to students, but $\frac{7}{5}$ and $\frac{10}{13}$ do not— how can we take 7 parts out of 5 parts? This language consequently favors "proper" fractions (i.e., those that are less than 1).

  Note: In my opinion, we should gradually do away with the terms *proper* or *improper* to characterize fractions. The non-mathematical meaning of the adjective *improper* only exacerbates students' discomfort with fractions greater than 1. Once students understand the concept of a fraction, and the procedure for

constructing it, placing $\frac{5}{3}$ on a number line is no more difficult for them than placing $\frac{2}{3}$. Treating *all* fractions equally, whether they are less than or greater than 1, will make for a more seamless transition to rational numbers.

- **Separate numbers.** The restricted "*a* parts out of *b* equal parts" interpretation has a delaying effect on the progression toward viewing a fraction as *a single number*. If, when seeing $\frac{3}{5}$, a student thinks exclusively of the actions "I cut the whole into 5 parts and then take 3 parts," then the student is mentally manipulating two different numbers or quantities: 5 parts on the one hand, and 3 parts on the other. This delays the more mature understanding of $\frac{3}{5}$ as a single number, with a representation as a unique point on the number line.

- **Additive thinking.** Perhaps the greatest limitation of the "*a* out of *b*" language is the obstacle it presents to multiplicative thinking, an important precursor to proportional thinking. Consider Roberto's fraction $\frac{3}{4}$. Thinking of it in terms of "the parts I shade or take" and "the parts I don't shade or don't take" is thinking *additively*, because $\frac{3}{4} + \frac{1}{4} = \frac{4}{4}$.

$$\frac{3}{4} + \frac{1}{4} = \frac{4}{4}$$
[Parts] [Whole]
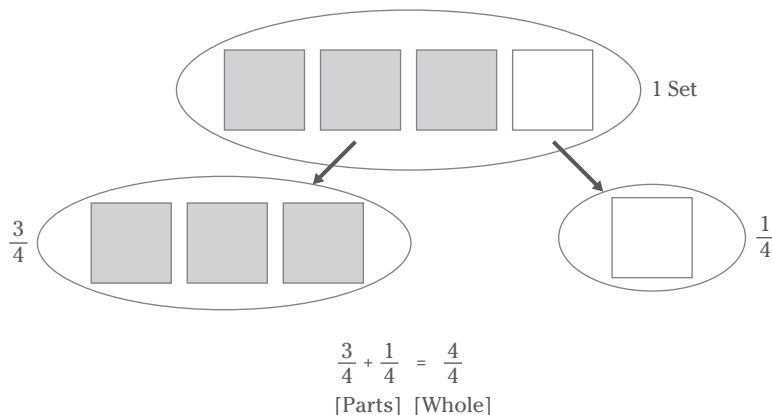
On the other hand, interpreting $\frac{3}{4}$ as "3 parts of size $\frac{1}{4}$" is thinking *multiplicatively*: I first consider the fractional unit $\frac{1}{4}$ of the set (represented by one square), then I count three copies (or iterations) of $\frac{1}{4}$—or better yet, "I multiply 3 times the fractional unit $\frac{1}{4}$ to get $\frac{3}{4}$."

1 Set

represents $\frac{1}{4}$ of the set.

represents 3 parts of size $\frac{1}{4}$.
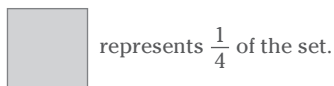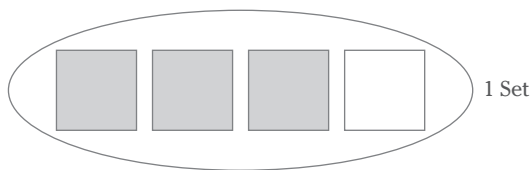
$\frac{3}{4}$ is 3 copies of $\frac{1}{4}$ (or) $3 \times \frac{1}{4}$ (or) $\frac{3}{4}$

With this multiplicative mindset, the fraction $\frac{5}{4}$ is conceptualized no differently: "I first identify the fractional unit $\frac{1}{4}$, then I think of $\frac{5}{4}$ as '5 iterations of $\frac{1}{4}$' or '5 times $\frac{1}{4}$.'"

1 Set

represents $\frac{1}{4}$ of the set.

represents 5 parts of size $\frac{1}{4}$.
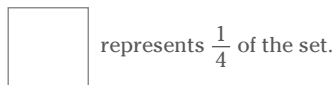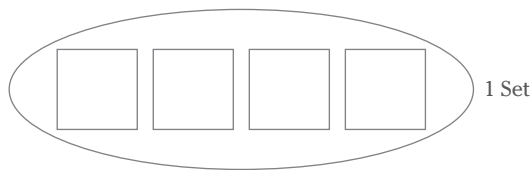
$\frac{5}{4}$ is 5 copies of $\frac{1}{4}$ (or) $5 \times \frac{1}{4}$ (or) $\frac{5}{4}$

The Measure Meaning of $\frac{a}{b}$

The concept of the whole underlies the concept of a fraction.

Merlyn J. Behr & Thomas R. Post (1992, p. 213)

From *part of a whole* to *compared with a whole*. Barnett-Clarke, Fisher, Marks, and Ross (2010) noted, "The interpretation of rational number [fraction] as a measure pushes us beyond our interpretation of a fraction as a *part of a whole* to the broader idea of a fraction as a *quantity compared with a whole*" (p. 23). When a child says, "I ran half a kilometer," the measure $\frac{1}{2}$ km tells us the distance the child ran *compared with the whole*, 1 km, which in this case is a unit of measure for distance. When a child says, "I'm five and a half," the inferred unit of measure (which the child is too young to know!) is one year. The measure $\frac{11}{2}$ years, or the mixed number $5\frac{1}{2}$ years, therefore, tells us the amount of time the child has lived since birth—or the child's age—*compared with the whole*, 1 year. A good way of making children aware from a young age of the importance of associating a number to the unit of measure (which is often not specified but merely implied) in any measurement is to probe with questions such as, "Do you mean five and a half *days* old?"

Measuring is a multiplicative process par excellence. The process of measuring requires multiplicative thinking in two ways:

- **In the unit conversion:** Students "use their knowledge of relationships between units and their understanding of multiplicative situations to make conversions, such as expressing . . . 3 feet as 36 inches" (NCTM, 2000, p. 172). Indeed, if 1 foot is 12 inches, then 3 feet is three times more, or 36 inches.

- **In the nature of the measuring process itself:** Calculating the length of a book in non-standard units, such as staples, means figuring out *how many times* the staple (or *unit of length*) fits in a straight line, end to end, from one end of the desk to the other. When sufficient copies of the unit of measure are available, lining them up offers a nice visual of this multiplicative process.

An important inverse relationship. In measuring, there is an important relationship between the *size* of the unit of measure and the *number* of units it takes to measure a quantity. For example, if the trapezoid in Figure 1.1, which is half the area of the hexagon, is selected to be one unit of area, then the hexagon has an area of two units. On the other hand, if the triangle, which is three times smaller than the trapezoid, is selected as the new unit of area, the same hexagon now has an area of six units, or *three times greater than* the first measurement.
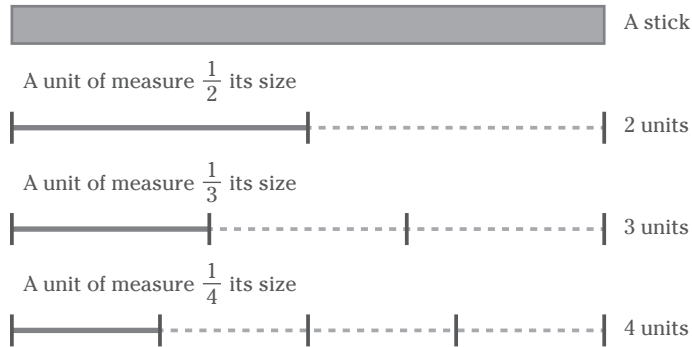
FIGURE 1.1
Area: An Example of Two-Dimensional Measure
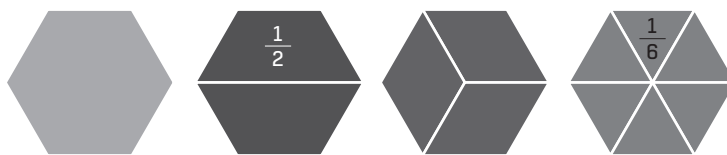
Note: Consider the size of the unit of measure for area: it is inversely proportional to the area measure of the hexagon.

An analogous situation in *linear* measure (length) is illustrated by different measures of the same stick, depending on the choice of unit (Figure 1.2). If the unit of measure is half of the stick's length, then the stick measures two units. But if the unit of measure changes to one-quarter of the stick's length—half the size of the first unit—then the new measure of the stick's length is four units, or twice the first measurement.

FIGURE 1.2

Length: An Example of One-Dimensional Measure

A stick

A unit of measure $\frac{1}{2}$ its size

2 units

A unit of measure $\frac{1}{3}$ its size

3 units

A unit of measure $\frac{1}{4}$ its size

4 units

Note: Consider the size of the unit of measure for length; it is inversely proportional to the length measure of the stick.

At around age 7, students begin to observe this inverse relationship between the size of the unit of measure and the number of units it takes to express the measure of a given quantity. The role of instruction in helping students make their observation explicit is crucial. Use these measuring experiences to help students see why the ordering of unit fractions is the inverse of the ordering of whole numbers.

$$\frac{1}{1} > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} \ \dots \text{yet} \ \ 1 < 2 < 3 < 4 \dots$$

Fractions and measures: the importance of the unit. Another connection between a measure and a fraction, which, when made explicit, helps students better understand the meaning of a fraction, is the stated or implied unit. We saw that any measure, be it 4, $\frac{3}{4}$, or $5\frac{1}{2}$, is always stated in reference to a whole or a unit of measure, say, 4 quarts, $\frac{3}{4}$ of a mile, or $5\frac{1}{2}$ years. If the unit is unknown, then the number stated alone gives no sense of "How much?" This is also true for a fraction, which has no meaning if the whole or unit is unknown. Chapter 3 develops this important concept.

CCSS.Math.Practice.MP2: Reason Abstractly and Quantitatively

In all actions of measuring, we find structural similarities. Measuring is the process of assigning numbers to objects—it is understood that some attribute of the object (e.g., length, area, or volume) is being measured. The process consists of three steps:

1. Assign the number 1 to a selected unit of measure, non-standard or standard.

2. Express the measure of the object's attribute of interest as a certain number of "copies" of this unit (which is almost always a fraction, either less than or greater than 0).

3. Record the measurement as a number followed by the unit of measure.

For students to make sense of the use of fractions in their lives, they should experience multiple instances of measuring—both the process of measuring (the action) and the assignment of number to measurement (the product). They should understand how measurement instruments work so they can learn to use them to describe the physical world in meaningful ways. To construct the concepts of perimeter, area, and volume, they should actually wrap a diameter string around a circle, tile a region with different polygonal two-dimensional (2-D) shapes, or fill a container with unit cubes—and find that the result is almost never a whole number. In the process, students come to understand that *quantities* are actually attributes of objects (or phenomena) that are measurable. It is precisely our capacity to measure them that makes them quantities!

In 3rd and 4th grade, students assign whole numbers to these real-world objects, and then fractions and decimals: a length, a period of time, a price. In 5th and 6th grade, they begin to generalize perimeter, area, volume, and other measurement formulas, many of which contain fractional expressions, such as $\frac{1}{2}bh$ or $\frac{1}{2} \times base \times height$, for the area of a triangle. In 7th and 8th grade, measurement turns to more complex phenomena, such as rates of change (e.g., speed), which are also replete with fractional expressions.

More complex than simple numbers or expressions, the products of the measurement process in these grades are mathematical models (symbolic algebraic expressions, equations, systems of equations, etc.).

Measurement in grades 3–8 is thus a quintessential area of mathematical learning that requires students to juggle *quantitative reasoning* with *abstract reasoning*. We must offer students more quantitative experiences—and fewer computations with numbers disconnected from concrete situations—that help them ground their fractions, fractional expressions, and symbolic and algebraic expressions in the world they live in. If instruction fails to help students connect abstract symbolic representations with the problems or situations from which they emerged, students will not be able to build rich meanings of fractions. The students who do understand the principle of measuring attributes of objects and quantifying phenomena will have acquired insight into the important connection between real-world contexts and abstract mathematical models—the core of all scientific investigation!

The Quotient Meaning of $\frac{a}{b}$

Many students have difficulty conceptualizing division and its relationship to multiplication and fractions. In later chapters, we discuss division with fractions in more detail. Here, we focus on the interpretation of a fraction $\frac{a}{b}$ as referring to the division operation $a \div b$ and its resulting quotient. To this end, we review four situations in which students encounter division with whole numbers, where the dividend is greater than the divisor (as in $20 \div 4$).

The two big division ideas: *partitioning* and *grouping*. Consider the mathematical expression $20 \div 4$. What situations might it model? "We want to share 20 apples equally among 4 students. How many apples does each person get?" would be a classic example of *partitive* division,

because we *partition* the 20 apples into 4 equal or fair shares, modeled by
20 ÷ 4. Partitioning or sharing is the action associated with this mental
image of division; the result of the division action or the quotient $\frac{20}{4}$ is
the equal share of 5. Both children and adults are most comfortable with
this meaning of division; the number of containers is known (given), but
the number or amount contained in each is unknown (sought), as shown
in Figure 1.3.

FIGURE 1.3
Modeling 20 ÷ 4 as Partitive Division
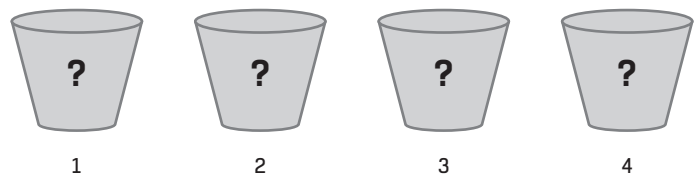
Now suppose we needed to pack all 20 apples in bags of 4 apples each.
The new question would be "How many baskets do I need?" In this case,
the number or amount contained is known (given), but the number of
containers, modeled by 20 ÷ 4, is unknown (sought), as shown in Figure 1.4.
The action associated with this idea of division is grouping, segmenting, or
portioning: making equal groups of 4 apples until all 20 apples are used up.

FIGURE 1.4
Modeling 20 ÷ 4 as Quotative Division
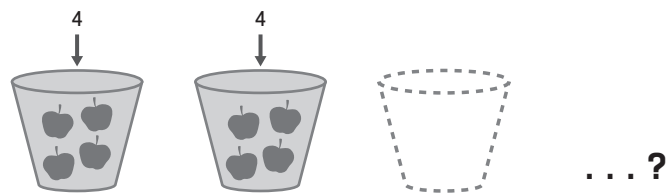
The result of the division action or the quotient of $\frac{20}{4}$ is the number of baskets required, which is 5. This less understood mental image of division is known as *quotative* or *subtractive measurement.*

Two other division ideas: deriving factors and reducing quantities. Even though students can easily parrot the sentence "Division is the inverse of multiplication," using this relationship to solve problems doesn't come naturally. In 3rd grade, students learn through exploration that the total number of square units in the area of a rectangle is the product of its two dimensions, later symbolized algebraically by the area formula $A = l \times w$. They are expected to know how to derive one dimension of a rectangle if they know the area and the other dimension.[1] Say, for example, that the area of a rectangle is 20 square feet and only one dimension, 4 feet, is known. The quotient of 20 ÷ 4 will express the length, in feet, of the rectangle's second dimension. Here, the division action is *deriving* one factor of a multiplicative formula, knowing the other factor and their product. The quotient, $\frac{20}{4}$ or 5, is the numerical value of the target factor.

A similar derivation process is required in Cartesian product situations—yet another conceptualization of multiplication. For example: "Jordan can make 20 different outfits by combining his shirts and his 4 pairs of pants. If an outfit consists of 1 shirt and 1 pair of pants, how many shirts does Jordan have?" Structurally, this problem is identical to the preceding one: O (# of outfits) = s (# of shirts) × p (# of pants). In this case, $20 = s \times 4$. The number of shirts is derived by the division action 20 ÷ 4, and the quotient, $\frac{20}{4}$ or 5, is the number of Jordan's shirts.

---

[1] CCSS.Math.Content.3.MD.D.8: Grade 3, Measurement and Data: Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters (Common Core State Standards Initiative, 2010).

The difference, however, between the products of $l \times w$ and $s \times p$ lies in the nature of the units: $l \times w$ generates *square units*, mathematical units for measuring area (dependent on the generating *linear units*), whereas $s \times p$ creates completely new non-mathematical units called *outfits*.

Let's consider one more problem. Suppose you insert a 20 cm × 13 cm picture into a document on your computer; however, you need to reduce the picture's dimensions to a quarter of their original lengths so that its area is $\frac{1}{16}$ of its original area. Here, 20 ÷ 4 expresses the length of the reduced picture (Figure 1.5).

FIGURE 1.5
Modeling 20 ÷ 4 as Length Reduction
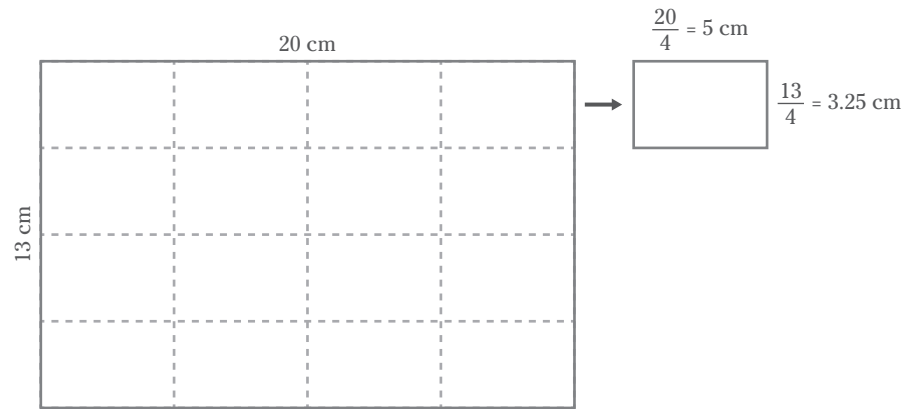
In this case, the division action is *reducing*; the dividend is the quantity being reduced, the divisor is the "shrinking factor" by which it is being reduced, and the quotient, $\frac{20}{4}$ or 5, represents the value of the quantity after the reduction. This interpretation of division involves one quantity rather than two, and there's no action of splitting the original quantity into parts: a single quantity (in this case, length) undergoes a transformation.

This "shrinking" effect of division is developed in the middle grades through the study of similar figures. Multiplication by a *scale factor* is used

to denote the shrinking or stretching effect. In this example, from large to small, the scale factor for the shrinking or reduction of the picture's length is $\frac{1}{4}$, because $20 \times \frac{1}{4} = 5$. If we moved in the opposite direction, from small to large, the stretching or expansion scale factor is 4, because $5 \times 4 = 20$.

Connecting division and quotients to fractions. Operations are the means by which we express, describe, and solve problems in the social and physical world. Division metaphors give meaning to fractions themselves and are the foundation for division with fractions, which we will explore in later chapters.[2] The situations modeled by the same equation, $20 \div 4 = 5$, illustrate four distinct interpretations of the division action $20 \div 4$ and of its corresponding quotient $\frac{20}{4}$, which is a fraction greater than 1. In each case, the significance assigned to dividend, divisor, and quotient are different.

The challenge now is to revisit these metaphors for fractions less than 1 and explore if the meanings of *process* and *product* still hold. For example, consider $3 \div 4$ and $\frac{3}{4}$:

- Can we partition three things equally among four people? How do we express the equal shares?

- Can a rectangle whose area is 3 square feet have a side length of 4 feet? What would be the other side length?

- Can we imagine reducing a rectangle with a side length of 3 cm to $\frac{1}{4}$ of its length? How would we express this reduced side length?

- Do all interpretations work for $3 \div 4$ and $\frac{3}{4}$ as they do for $20 \div 4$ and $\frac{20}{4}$?

Investigate these questions with your students!

---

[2] For an in-depth analysis of multiplication metaphors for grades 3–5, see *Planting the Seeds of Algebra, 3–5: Explorations for the Upper Elementary Grades* (Neagoy, 2014, pp. 111–148).

A Bridge to Algebra:
"How many thirds do you need to make a whole?"

Through the repeated actions of partitioning a whole into b equal parts, students realize on their own that the larger the b value, the smaller the equal parts named $\frac{1}{b}$. They say things like "The more people who share a pizza, the less we each get to eat!" What they are learning from an algebraic perspective is that when a whole is divided by b, the answer is $\frac{1}{b}$. In a few years, they will express this symbolically as "$1 \div b = \frac{1}{b}$." When they are able to abstract this big idea to any number n, it will become $1 \div n = \frac{1}{n}$.

But there is a "companion" big idea that we could help instill simply by routinely posing the right questions as friendly reminders. For instance, when working with thirds, pose this question: "So, how many thirds do you need to make the whole?" When working with eighths, ask, "Remind me, how many eighths would I need to make a whole?" This will help automatize the reaction that it takes b b-ths to make a whole, or $\frac{1}{b} \times b = 1$. Generalizing as we did for any number n other than 0, we obtain $\frac{1}{n} \times n = 1$.

Taken together, these two algebraic equations summarize the process of dividing a whole by any number n, then multiplying the parts by n to get back the whole:

$1 \div n = \frac{1}{n}$ (we partition, decompose, or divide)

$\frac{1}{n} \times n = 1$ (we iterate, recompose, or multiply)

Though this may seem obvious, I can assure you that many middle school students do not have the helpful reflex of thinking, "The number 1 can be written as the quotient of any number n over itself, or any algebraic expression E over itself":

$1 = \frac{n}{n} = \frac{E}{E}$, a conclusion drawn from $\frac{1}{n} \times n = 1$

Routinely asking, "How many thirds do you need to make 1?" will go a long way!

The Ratio Meaning of $\frac{a}{b}$

Another possible interpretation of the symbol $\frac{a}{b}$ is a *ratio*. Although they have been relegated to the middle grades in the past, more and more math programs are including the concept of ratios prior to 6th grade, thanks to the Common Core State Standards. The word *ratio* is scary to some of us, as it conjures up negative emotions associated with complex ratio, rate, or proportion problems from our own middle school years. But it needn't be. Fifth grade students and even some 4th graders can develop an intuitive and informal notion of the ratio concept.

A ratio expresses a relationship, a *multiplicative* comparison, between two or more quantities. It compares their relative counts or measures. For example, suppose that a total of 12 people attended a picnic—3 chaperoning adults and 9 children. At least four ratios can be created from this information, two part-to-whole ratios, similar to fractions, and two part-to-part ratios, which are different from fractions.

- The ratio of children to the total number of people is an example of a part-whole ratio and can be denoted in a variety of ways, including 9 children to 12 people, 9 to 12, 9:12, and $\frac{9}{12}$. This ratio tells us that three out of every four people were children.

- The children-to-adults ratio, on the other hand, is a part-part ratio. This ratio, which can be expressed as 9 children to 3 adults, 9 to 3, 9:3, or $\frac{9}{3}$, tells us that there were three times as many children as adults at the picnic.

3 adults   9 children

Using *A*, *C*, and *P* for the number of adults, children, and people at the picnic, respectively, Figure 1.6 lists all four ratios.

FIGURE 1.6
Four Ratios Derived from the Picnic Problem

Part-to-whole ratios	Part-to-part ratios
$\dfrac{A}{P} = \dfrac{3}{12} = \dfrac{1}{4}$	$\dfrac{A}{C} = \dfrac{3}{9} = \dfrac{1}{3}$
$\dfrac{C}{P} = \dfrac{9}{12} = \dfrac{3}{4}$	$\dfrac{C}{A} = \dfrac{9}{3} = \dfrac{3}{1}$

A ratio need not be expressed in fraction notation, but it can be. When ratios are first introduced, many U.S. math programs use the colon notation to distinguish a ratio from the fraction notation—but then the fraction notation is quickly introduced. We will discuss the differences between ratios and fractions in more depth, but for now here are two distinctions:

- Fractions are always part-whole comparisons, but ratios can be either part-whole or part-part comparisons.
- Fractions that express part-whole relationships, quotients, measures, and multiplicative operators are always rational numbers, but fractions expressing ratios need not be. The four ratios in the picnic problem were all rational numbers. But a famous ratio, familiar to readers, that is not a rational number is π, the Greek equivalent for the letter *p*. Pi represents the ratio of the circumference to the diameter of all circles. The fact that circumference to diameter, $C : D$ or $\dfrac{C}{D}$, is 3.14159. . . simply means that the length of the circumference of any circle is about 3.14 times

the length of its diameter. To express this in a more visual and child-friendly way: it takes three diameters plus a bit more to wrap around any circle.
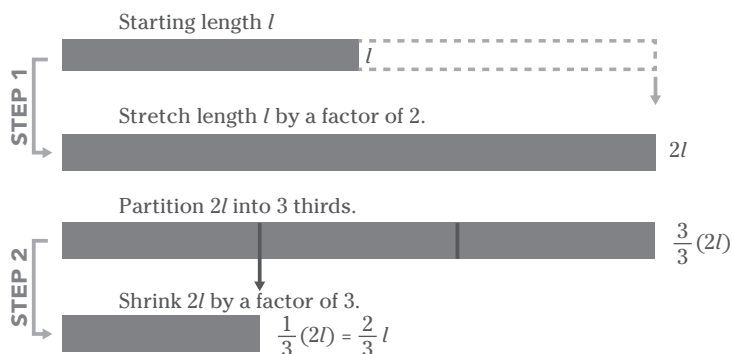
The Multiplicative Operator Meaning of $\frac{a}{b}$

The final interpretation of a fraction in the intermediary grades is a *multiplicative operator.* In this sense, $\frac{a}{b}$ "changes or transforms another number or quantity by magnifying, shrinking, enlarging, reducing, expanding, or contracting it" (Barnett-Clarke et al., 2010, p. 27), depending on the nature of the quantity it's acting on. The action of change here is multiplication. Consider first a measurable quantity, such as a strip of tape of length l, and the fraction $\frac{2}{3}$. What does $\frac{2}{3}l$ mean from the operator perspective?

We can think of $\frac{2}{3}$ as having a *stretching-shrinking* effect on length l. First, we stretch l by a factor of 2 (the numerator as *stretcher*), and then we shrink the resulting length, $2l$, by a factor of 3 (the denominator as *shrinker*). The result is one-third of $2l$, or $\frac{2}{3}l$ (Figure 1.7).

FIGURE 1.7

The Stretching-Shrinking Effect of $\frac{2}{3}$ on l

STEP 1

Starting length l

l

Stretch length l by a factor of 2.

$2l$

STEP 2

Partition $2l$ into 3 thirds.

$\frac{3}{3}(2l)$

Shrink $2l$ by a factor of 3.

$\frac{1}{3}(2l) = \frac{2}{3}l$

As students' understanding of fractions matures and they are able to see fractions as numbers and to quickly assess their magnitude with respect to the whole-number or fraction benchmarks they know well, instead of decomposing the operator effect of $\frac{2}{3}$ on l into two steps ($\times$ 2 followed by $\div$ 3), they simply shrink l in one step to the resulting $\frac{2}{3}l$. Similarly, they would stretch or shrink l in one step to get $\frac{7}{3}l$, $\frac{3}{2}l$, or $\frac{3}{4}l$ (Figure 1.8). Just as they would stretch or shrink l to get $2l$, $3l$, or $\frac{1}{2}l$ (Figure 1.8).

FIGURE 1.8

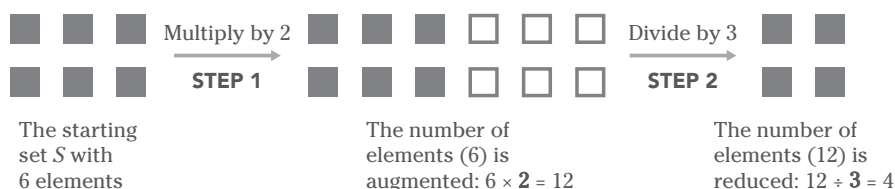**Whole and Fractional Multiplicative Operators Acting on Length l**
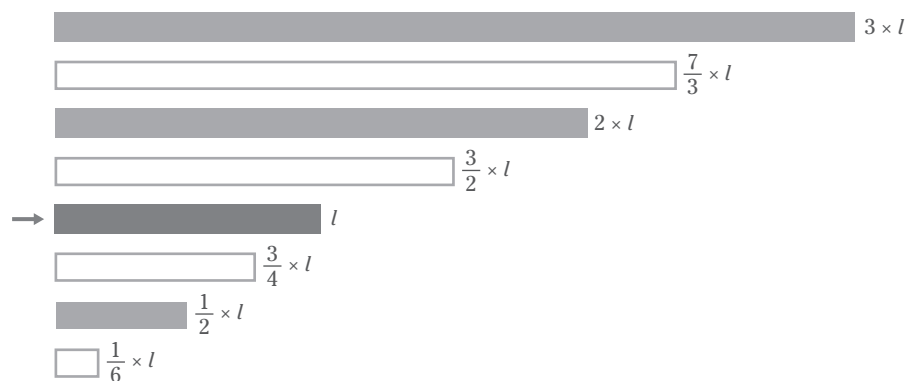
This stretching-shrinking metaphor of multiplication is more powerful than repeated addition and serves students better as they move from whole-number multiplication to fraction multiplication. Indeed, interpreting $3 \times n$ as $n + n + n$ breaks down when we move to $\frac{3}{4} \times n$, whereas stretching (e.g., $3 \times n$) or shrinking $\left(\text{e.g., } \frac{3}{4} \times n\right)$ works for any multiplier.

Next, consider a countable quantity, such as a set S containing six elements. Interpreting $\frac{2}{3}$ as an operator in the expression $\frac{2}{3}S$ means that $\frac{2}{3}$ has a *multiplier-divider* effect on the number of elements comprising S, which is six. First, we multiply 6 by 2 (the numerator or multiplier), and then

we divide the resulting number, 12, by 3 (the denominator or divisor) to obtain 4, the number of elements in the set $\frac{2}{3}S$ (Figure 1.9).

FIGURE 1.9

Multiplicative Operator $\frac{2}{3}$ Acting on Set S

The starting set S with 6 elements	The number of elements (6) is augmented: $6 \times \mathbf{2} = 12$	The number of elements (12) is reduced: $12 \div \mathbf{3} = 4$

Again, as students move through the grades, their increasing familiarity with fraction values allows them to quickly see $\frac{2}{3}$ of 6 and therefore proceed from 6 to 4 in one step. That said, it is nevertheless important to understand the decomposition of multiplying a number or quantity by $\frac{a}{b}$ into two steps (multiply by a, then divide by b), as this process is not only applicable to all fractions but also especially useful with less familiar fractions.

Finally, since multiplication and division have equal status in order of operations, students will discover (with your prodding and questioning) that multiplying first and then dividing is equivalent to dividing first and then multiplying. Similarly, stretching first and then shrinking is equivalent to shrinking first and then stretching. As students become more comfortable with symbolism, they can translate these equivalent processes as follows:

- $n \rightarrow 2 \times n \rightarrow \frac{2 \times n}{3} \rightarrow \frac{2}{3}n$, which can also be thought of as
  $n \rightarrow 2 \times n \rightarrow \frac{1}{3} \times (2 \times n) \rightarrow \frac{2}{3}n$

- $n \rightarrow \frac{n}{3} \rightarrow 2 \times \left(\frac{n}{3}\right) \rightarrow \frac{2}{3}n$, which can also be thought of as
  $n \rightarrow \left(\frac{1}{3} \times n\right) \rightarrow 2 \times \left(\frac{1}{3} \times n\right) \rightarrow \frac{2}{3}n$

Teaching Tip: Math Talk

The mental processes employed by students engaged in fraction tasks are often quite different from the instructional procedures they are taught. Through quality mathematical discourse—lately referred to as *math talk*—we can access what students are really thinking and how they are producing their answers and then consider how those findings might modify our instruction. Regarding the interpretation of a fraction as a multiplicative operator $\left(\text{say, } \frac{2}{3}\right)$ operating on any number n, different views usually emerge within the same class, depending on students' prior experiences, knowledge, and instruction:

> **Alex:** I first multiplied the number by 2, and then I divided by 3.
>
> **Talia:** I put a 1 under the number $\left(\frac{n}{1}\right)$, and then I multiplied the numerator by 2 and multiplied the denominator by 3.

Alex's approach, the one explored in this chapter, uses whole-number multiplication and division. Note that if $2n$ is not divisible by 3, the answer remains in the fraction form $\frac{2n}{3}$. This approach is taught and learned well before the standard algorithm for fraction multiplication.

By contrast, Talia has clearly been taught the standard algorithm for multiplying a fraction by a number n and has applied it correctly—though it is unclear whether she fully understands it. The important pedagogical point here is to reconcile both approaches so all students see they are equivalent, even though they may hear Alex saying "*divided* by 3" and Talia saying "*multiplied* by 3."

A teacher's intervention at this point is crucial.

The Rational Number Meaning of $\frac{a}{b}$ Embodied by the Number Line

The rational-number concept is the outgrowth of extensive work on fractions over many years—the beautiful confluence of the part-whole, measure, division/quotient, multiplicative operator, and even ratio concepts. When conceptualizing a number in grades 3–8, be it whole, integer, or rational, students visualize its physical embodiment as a single point on a number line—a line with a selected point called the *origin*, *O*, composed of units of 1. ("Number" as an abstract concept occurs much later.) In locating or placing a fraction $\left(\text{say, } \frac{4}{5}\right)$ on the number line, we find aspects of all of the connected concepts we've discussed.
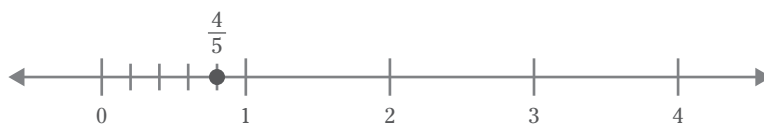
- **Part-whole.** The whole (the unit from 0 to 1) is sliced into five parts called *fifths*. We count four copies (iterations) of $\frac{1}{5}$, starting from the origin, and mark the point $\frac{4}{5}$.

- **Measure.** If we consider the unit of measure to be the line segment from 0 to 1, then the length of the line segment from 0 to $\frac{4}{5}$ represents the measure $\frac{4}{5}$ of the unit.

- **Division/quotient.** Partitioning the line segment from 0 to 4 into five equal "shares" yields five smaller line segments of length $\frac{4}{5}$. Try it!

- **Multiplicative operator.** There are two ways to consider this:

  1. Stretch the unit by a factor of 4 (i.e., land on 4) and then shrink the four-unit segment by a factor of 5 to land on $\frac{4}{5}$.

2. Shrink the unit by a factor of 5 $\left(\text{i.e., land on } \frac{1}{5}\right)$ and then stretch the $\frac{1}{5}$ segment by a factor of four to land on $\frac{4}{5}$.

- **Ratio.** This is more complex. We can form many ratios of lengths, such as $1{:}\frac{4}{5}$ and $4{:}\frac{4}{5}$. The ratio of 1 to $\frac{4}{5}$ is $\frac{5}{4}$ because 1 is $\frac{5}{4}$ $\left(\text{or } 1\frac{1}{4}\right)$ times longer than $\frac{4}{5}$. The ratio 4 to $\frac{4}{5}$ is 5 because 4 is 5 times longer than $\frac{4}{5}$. Take a moment to verify this!

Targeting Misconceptions with Challenging Problems

Even though there are ample fraction questions and problems available online and in books, many of them are rote problems that students carry out mechanically without deepening their understanding of fractions. We must formulate more thought-provoking problems that force students to change gears from number numbness to productive struggle. Seven such problems are suggested in the following list.

Problem 1: Multiple meanings. Pick a fraction $\frac{a}{b}$ of your choice. For each of the following cases, describe a real-world situation, and then formulate a question to which your fraction is the answer:

- $\frac{a}{b}$ is a part of a whole.
- $\frac{a}{b}$ is a measure.
- $\frac{a}{b}$ is a quotient (the result of a division).
- $\frac{a}{b}$ is a multiplicative operator.
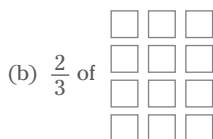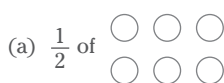- $\frac{a}{b}$ is a ratio.

💡 **Problem 2: Part-whole meaning.** We know that a fraction can represent a part of a region or a collection.

- Draw, shade, or otherwise represent the fractional part of each figure.

    (a) $\frac{1}{3}$ of △

    (c) $\frac{2}{5}$ of ⬠

    (b) $\frac{3}{4}$ of ▭

    (d) $\frac{7}{6}$ of ⬡
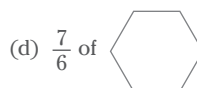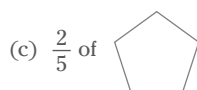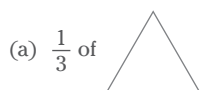
- Draw, shade, or otherwise represent the fractional part of each given set.

    (a) $\frac{1}{2}$ of ○○○ ○○○

    (c) $\frac{5}{3}$ of ▭▭ □□□□ □□□

    (b) $\frac{2}{3}$ of □□□ □□□ □□□ □□□

    (d) $\frac{7}{4}$ of △△ △△△△ △△

💡 **Problem 3: Number and linear measure meanings.** We know that a fraction can represent a number or a measure.

- Place a point on the number line that represents the number $\frac{4}{5}$.

- This piece of packing tape is 5 units long (using the unit of measure in the previous number line). Draw a piece of tape below this one that is $\frac{4}{5}$ as long.

- In what ways are your number line and tape models similar? Explain.
- In what ways are your number line and tape models different? Explain.

Problem 4: Multiplicative operator meaning. Remember that a fraction can represent a multiplicative operator. Imagine that the figures below are flexible bands of fabric and that each piece can be stretched or shrunk to a desired length. Draw the result of the stretching or shrinking effect of each fraction on the given length of fabric.

$\frac{2}{3}$ of

$\frac{5}{2}$ of

$\frac{7}{4}$ of

$\frac{2}{5}$ of

Problem 5: Connecting part-whole, quotient, and ratio meanings. Use everything you have learned so far about fractions to complete the following activities.

- Draw a rectangular chocolate bar, and shade $\frac{5}{6}$ of the bar, representing the part of the whole that you ate (i.e., the *part-whole meaning* of the fraction $\frac{5}{6}$).
- Draw five bars of chocolate that are the same size. Then model with a drawing or diagram how you might share the five bars

equally among six people $\Big($i.e., the *quotient meaning* of the

fraction $\frac{5}{6}\Big)$.

- In what ways are the fractional meanings in the first two activities related? Are there other ways that they could be related? Explain.

- Drawing a rectangle for a chocolate bar once again, explain what a $\frac{5}{6}$ part-to-whole ratio might represent. Do the same for a $\frac{5}{6}$ part-to-part ratio.

💡 **Problem 6: Rational number meaning.** You have learned that a fraction can represent a rational number, which is shown by a point on the number line.

- Looking at this number line, cross out the fractions that clearly could not be represented by point *A*:

$$\frac{1}{2} \quad \frac{3}{5} \quad \frac{7}{6} \quad \frac{7}{10} \quad \frac{3}{2} \quad \frac{4}{5} \quad \frac{5}{4} \quad \frac{5}{7} \quad \frac{3}{4} \quad \frac{4}{3}$$

- Circle the fractions that could be represented by point *B*:

$$\frac{8}{5} \quad \frac{2}{3} \quad \frac{5}{4} \quad \frac{7}{10} \quad \frac{10}{7} \quad \frac{3}{2} \quad \frac{4}{3} \quad \frac{9}{7} \quad \frac{3}{4} \quad \frac{5}{3}$$

- Place point *C* on the number line to represent the number $\frac{7}{4}$. Explain how you used the values of the numerator and denominator of the fraction to find its place on the line.

💡 **Problem 7: Whole numbers and rational numbers on the number line.** Use everything you have learned so far about fractions to solve the following problems.

N1

```
←——+——————————————+——————————————+——→
    1              2              3
```

- The whole numbers 0, 2, and 3 segment number line N1 into four intervals. Place a fraction, as accurately as you can, in each of the four intervals.

N2

```
←——+——————+———————————————————————+——→
   1/2    5/4                     11/3
```

- The fractions $\frac{1}{2}, \frac{5}{4}$, and $\frac{11}{3}$ segment number line N2 into four intervals. Place a whole number, as accurately as you can, in each of the four intervals.

🖥 **What's the App for That?**

By playing **Fraction Action**, students come to appreciate various aspects of fraction representation, including visual, geometric, numeric, and artistic.