NAVIGATIONS SERIES

GRADES 3–5

NAVIGATING *through* NUMBER *and* OPERATIONS

Introduction

What could be more fundamental in mathematics than numbers and the operations that we perform with them? Thus, it is no surprise that Number and Operations heads the list of the five Content Standards in *Principles and Standards for School Mathematics* (NCTM 2000). Yet, numbers and arithmetic are so familiar to most of us that we run the risk of underestimating the deep, rich knowledge and proficiency that this Standard encompasses.

Fundamentals of an Understanding of Number and Operations

In elaborating the Number and Operations Standard, *Principles and Standards* recommends that instructional programs from prekindergarten through grade 12 enable all students to—

- understand numbers, ways of representing numbers, relationships among numbers, and number systems;
- understand meanings of operations and how they relate to one another;
- compute fluently and make reasonable estimates.

The vision that *Principles and Standards* outlines in the description of this Standard gives Number and Operations centrality across the entire mathematics curriculum. The *Navigating through Number and Operations* volumes flesh out that vision and make it concrete in activities for students in four grade bands: prekindergarten through grade 2, grades 3–5, grades 6–8, and grades 9–12.

Understanding numbers, ways of representing numbers, relationships among numbers, and number systems

Young children begin to develop primitive ideas of number even before they enter school, and they arrive in the classroom with a range of informal understanding. They have probably learned to extend the appropriate number of fingers when someone asks, "How old are you?" and their vocabulary almost certainly includes some number words. They are likely to be able to associate these words correctly with small collections of objects, and they probably have been encouraged to count things, although they may not yet have mastered the essential one-to-one matching of objects to number names. During the years from prekindergarten through grade 2, their concepts and skills related to numbers and numeration, counting, representing and comparing quantities, and the operations of adding and subtracting will grow enormously as these ideas become the focus of the mathematics curriculum.

The most important accomplishments of the primary years include the refinement of children's understanding of counting and their initial development of number sense. Multiple classroom contexts offer numerous opportunities for students to count a myriad of things, from how many children are in their reading group, to how many cartons of milk their class needs for lunch, to how many steps they must take from the chalkboard to the classroom door. With experience, they learn to establish a one-to-one matching of objects counted with number words or numerals, and in time they recognize that the last number named is also the total number of objects in the collection. They also discover that the result of the counting process is not affected by the order in which they enumerate the objects. Eventually, they learn to count by twos or fives or tens or other forms of "skip counting," which requires that quantities be grouped in certain ways.

Though children initially encounter numbers by counting collections of physical objects, they go on to develop number concepts and the ability to think about numbers without needing the actual objects before them. They realize, for example, that five is one more than four and six is one more than five, and that, in general, the next counting number is one more than the number just named, whether or not actual objects are present for them to count. Through repeated experience, they also discover some important relationships, such as the connection between a number and its double, and they explore multiple ways of representing numbers, such as modeling six as six ones, or two threes, or three twos, or one more than five, or two plus four.

Young children are capable of developing number concepts that are more sophisticated than adults sometimes expect. Consider the prekindergarten child who explained her discovery that some numbers, like 2 and 4 and 6, are "fair numbers," or "sharing numbers," because she could divide these numbers of cookies equally with a friend, but numbers like 3 or 5 or 7 are not “fair numbers,” because they do not have this property.

As children work with numbers, they discover ways of thinking about the relationships among them. They learn to compare two numbers to determine which is greater. If they are comparing 17 and 20, for example, they might match objects in two collections to see that 3 objects are “left over” in the set of 20 after they have “used up” the set of 17, or they might count on from 17 and find that they have to count three more numbers to get to 20. By exploring “How many more?” and “How many less?” young children lay the foundations for addition and subtraction.

Continual work with numbers in the primary grades contributes to students’ development of an essential, firm understanding of place-value concepts and the base-ten numeration system. This understanding often emerges from work with concrete models, such as base-ten blocks or linking cubes, which engage students in the process of grouping and ungrouping units and tens. They must also learn to interpret, explain, and model the meaning of two- and three-digit numbers written symbolically. By the end of second grade, *Principles and Standards* expects students to be able to count into the hundreds, discover patterns in the numeration system related to place value, and compose (create through different combinations) and decompose (break apart in different ways) two- and three-digit numbers.

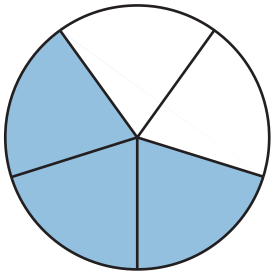
In addition, students in grade 2 should begin to extend their understanding of whole numbers to include early ideas about fractions. Initial experiences with fractions should introduce simple concepts, such as the idea that halves or fourths signify divisions of things into two or four equal parts, respectively.

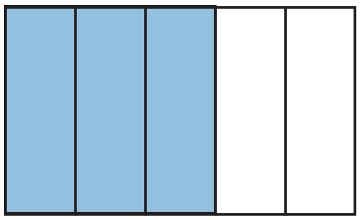
As students move into grades 3–5, their study of numbers expands to include larger whole numbers as well as fractions, decimals, and negative numbers. Now the emphasis shifts from addition and subtraction to multiplication and division, and the study of numbers focuses more directly on the multiplicative structure of the base-ten numeration system. Students should understand a number like 435 as representing $(4 \times 100) + (3 \times 10) + (5 \times 1)$, and they should explore what happens to numbers when they multiply or divide them by powers of 10.

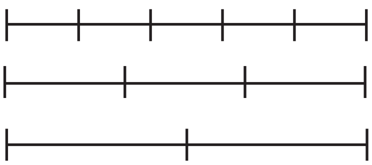
The number line now becomes an important model for representing the positions of numbers in relation to benchmarks like 1/2, 1, 10, 100, 500, and so on. It also provides a useful tool at this stage for representing fractions, decimals, and negative integers as well as whole numbers.

Concepts of fractions that the curriculum treated informally in the primary grades gain new meaning in grades 3–5 as students learn to interpret fractions both as parts of a whole and as divisions of numbers. Various models contribute to students’ developing understanding. For example, an area model in which a circle or a rectangle is divided into equal parts, some of which are shaded, helps students visualize fractions as parts of a unit whole or determine equivalent fractions.

Number-line models are again helpful, allowing students to compare fractions to useful benchmarks. For instance, they can decide that 3/5 is greater than 1/3 because 3/5 is more than 1/2 but 1/3 is less than 1/2,







or they can recognize that 9/10 is greater than 7/8 because 9/10 is closer to 1 than 7/8 is. Parallel number lines, such as one marked in multiples of 1/3 and another in multiples of 1/6, can help students identify equivalences.

During these upper elementary years, students also encounter the concept of percent as another model for a part of a whole. Their work should help them begin to develop benchmarks for common percentages, such as 25 percent, $33\frac{1}{3}$ percent, or 50 percent.

In grades 6–8, students expand their understanding of numbers to include the integers, and now they learn how to add, subtract, multiply, and divide with negative as well as positive numbers. Developing a deeper understanding of rational numbers is another very important goal for these students, who must increase their facility in working with rational numbers represented by fractions, decimals, and percents.

At this level, the curriculum places particular emphasis on developing proportional reasoning, which requires students to understand and use ratios, proportions, and rates to model and solve problems. Fraction strips, circles, number lines, area models, hundreds grids, and other physical models provide concrete representations from which students can draw conceptual meaning as they hone their understanding of rational numbers. Exposure to these models develops students' abilities to translate fluently from one representation to another, to compare and order rational numbers, and to attach meaning to rational numbers expressed in different but equivalent forms.

The concept of proportionality, which is a central component of the middle-school curriculum, serves to connect many aspects of mathematics, such as the slope of the linear function $y = mx$ in algebra, the scale factor in measurements on maps or scale drawings, the ratio of the circumference to the diameter of a circle (π) in geometry, or the relative frequency of a statistic in a set of data. Thus, students have numerous opportunities to develop and use number concepts in multiple contexts and applications. In some of those contexts, students encounter very large or very small numbers, which necessitate scientific notation and a sense of orders of magnitude of numbers.

Finally, students in grades 6–8 are able to focus more directly on properties of numbers than they were at earlier stages of development. They can investigate such key ideas as the notions of factor and multiple, prime and composite numbers, factor trees, divisibility tests, special sets (like the triangular and square numbers), and many interesting number patterns and relationships, including an introduction to some irrational numbers, such as $\sqrt{2}$.

When students move on to grades 9–12, their understanding of number should continue to grow and mature. In these grades, students customarily encounter many problems, both in mathematics and in related disciplines like science or economics, where very large and very small numbers are commonplace. In working such problems, students can use technology that displays large and small numbers in several ways, such as 1.219 E17 for 1.219 (10^{17}), and they need to become fluent in expressing and interpreting such quantities.

High school students also have many opportunities to work with irrational numbers, and these experiences should lead them to an understanding of the real number system—and, beyond that, to an understanding of number systems themselves. Moreover, students in grades 9–12 should develop an awareness of the relationship of those systems to various types of equations. For example, they should understand that the equation $A + 5 = 10$ has a whole-number solution, but the equation $A + 10 = 5$ does not, though it does have an integer solution. They should recognize that the equation $10 \cdot A = 5$ requires the rational numbers for its solution, and the equation $A^2 = 5$ has a real-number solution, but the equation $A^2 + 10 = 5$ is solved in the complex numbers.

Students should also understand the one-to-one correspondence between real numbers and points on the number line. They should recognize important properties of real numbers, such as that between any two real numbers there is always another real number, or that irrational numbers can be only approximated, but never represented exactly, by fractions or repeating decimals.

In grades 9–12, students also encounter new systems, such as vectors and matrices, which they should explore and compare to the more familiar number systems. Such study will involve them in explicit examination of the associative, commutative, and distributive properties and will expand their horizons to include a system (matrices) in which multiplication is not commutative. Using matrices, students can represent and solve a variety of problems in other areas of mathematics. They can find solutions to systems of linear equations, for instance, or describe a transformation of a geometric figure in the plane. Using algebraic symbols and reasoning, students also can explore interesting number properties and relationships, determining, for example, that the sum of two consecutive triangular numbers is always a square number and that the sum of the first N consecutive odd integers is equal to N^2.

Understanding meanings of operations and how operations relate to one another

As young children in prekindergarten through grade 2 learn to count and develop number sense, they simultaneously build their understanding of addition and subtraction. This occurs naturally as children compare numbers to see who collected more stickers or as they solve problems like the following: "When Tim and his dad went fishing, they caught seven fish. Tim caught four of the fish. How many did his dad catch?" Often, children use concrete materials, such as cubes or chips, to model "joining" or "take-away" problems, and they develop "counting on" or "counting back" strategies to solve problems about "how many altogether?" and "how many more?" and similar relationships.

Even at this early stage, teachers who present problems in everyday contexts can represent the problem symbolically. For example, teachers can represent the problem "How many more books does Emily need to read if she has already read 13 books and wants to read 20 books before the end of the school year?" as $13 + \square = 20$ or as $20 - \square = 13$. Such expressions help students to see the relationship between addition and subtraction.

Young children also build an understanding of the operations when they explain the thinking behind their solutions. For example, a child who had just celebrated his sixth birthday wondered, "How much is

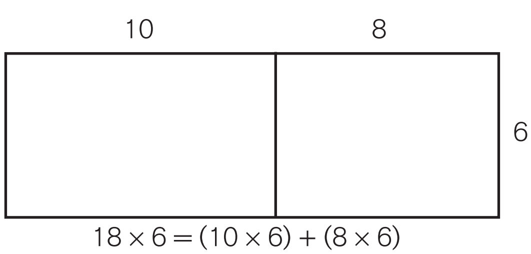
6 and 7?" After thinking about the problem for a moment, he decided that 6 + 7 = 13, and then he explained how he knew: "Well, I just had a birthday, and for my birthday I got two 'five dollars,' and my $5 and $5 are $10, so 6 and 6 should be 12, and then 6 and 7 must be 13."

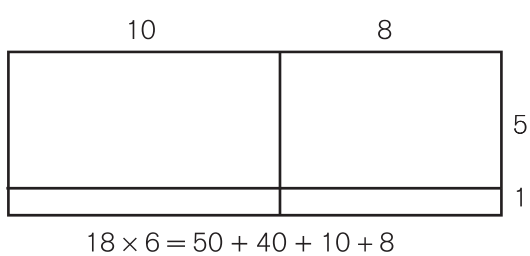
As young students work with addition and subtraction, they should also be introduced to the associative and commutative properties of the operations. They should learn that when they are doing addition, they can use the numbers in any order, but they should discover that this fact is not true for subtraction. Further, they should use the commutative property to develop effective strategies for computation. For example, they might rearrange the problem 3 + 5 + 7 to 3 + 7 + 5 = (3 + 7) + 5 = 10 + 5 = 15.

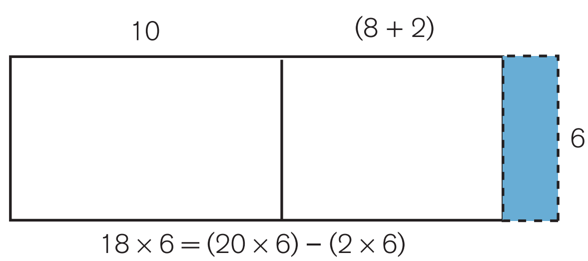
Early work with addition and subtraction also lays the conceptual groundwork for later study of operations. Multiplication and division are all but evident when students repeatedly add the same number—for example, in skip-counting by twos or fives—or when they solve problems requiring that a collection of objects be shared equally among several friends. The strategies that young children use to solve such problems, either repeatedly adding the same number or partitioning a set into equal-sized subsets, later mature into computational strategies for multiplication and division.

The operations of multiplication and division, and the relationships between them, receive particular emphasis in grades 3–5. Diagrams, pictures, and concrete manipulatives play important roles as students deepen their understanding of these operations and develop their facility in performing them.

For example, if an area model calls for students to arrange 18 square tiles into as many different rectangles as they can, the students can relate the three possible solutions (1 by 18, 2 by 9, and 3 by 6) to the factors of 18. Similar problems will show that some numbers, like 36 or 64, have many possible rectangular arrangements and hence many factors, while other numbers, like 37 or 41, yield only one solution and thus have only two factors. By comparing pairs of rectangular arrangements, such as 3 by 6 and 6 by 3, students can explore the commutative property for multiplication. As illustrated in the three examples below,





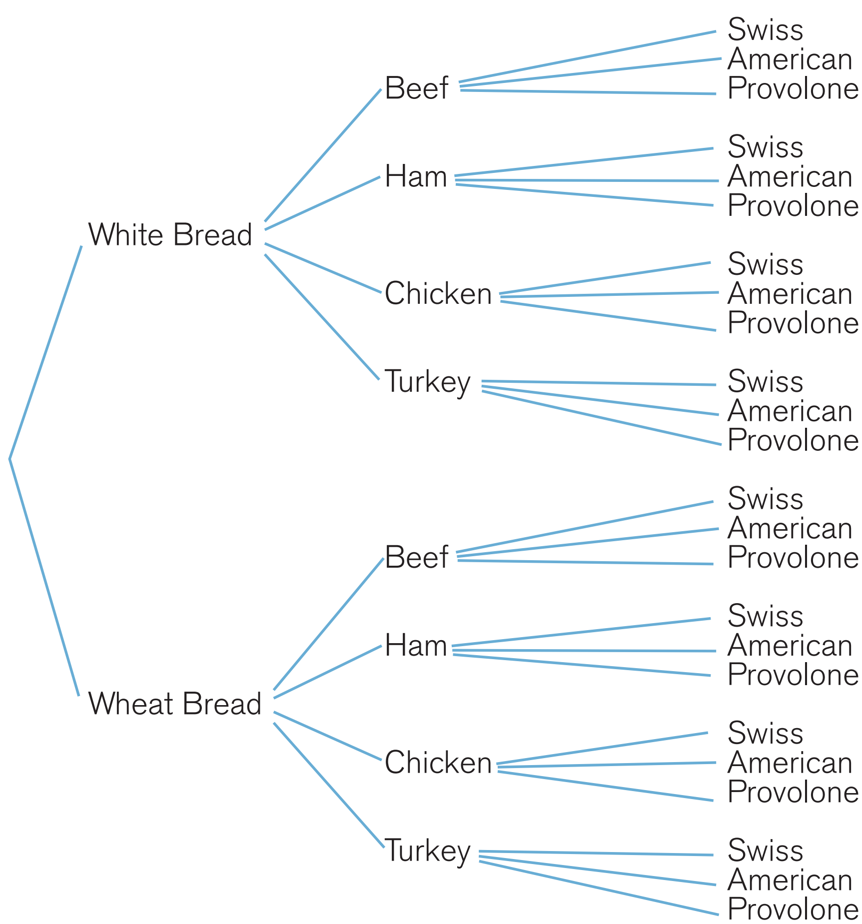


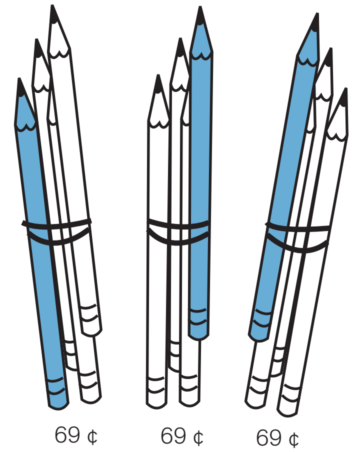
by decomposing an 18-by-6 area model, students can develop an understanding of the distributive property.

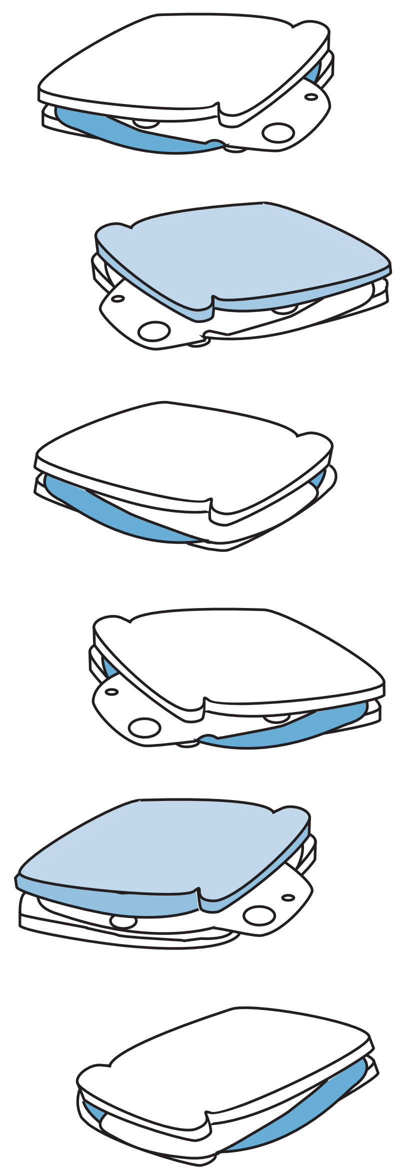
Other models for multiplication might involve rates or combinations. In grades 3–5, a typical problem involving a rate might be, “If 4 pencils cost 69¢, how much will a dozen pencils cost?” Problems involving combinations at this level are often similar to the following: “How many different kinds of meat-and-cheese sandwiches can we make if we have 2 kinds of bread (white and wheat), 4 kinds of meat (beef, ham, chicken, and turkey), and 3 kinds of cheese (Swiss, American, and provolone)?” (See the tree diagram below.)

To develop students’ understanding of division, teachers should engage them in working with two different models—a partitioning model (“If you have 36 marbles and want to share them equally among 4 people, how many marbles should each person receive?”) and a repeated-subtraction model (“If you have 36 marbles and need to place 4 marbles into each cup in a game, how many cups will you fill?”). Students should be able to represent both models with manipulatives and diagrams.

In exploring division, students in grades 3–5 will inevitably encounter situations that produce a remainder, and they should examine what the remainder means, how large it can be for a given divisor, and how to interpret it in different contexts. For example, arithmetically,







$28 \div 5 = 5\frac{3}{5}$, but consider the solutions to each of the following problems:

- “Compact disks are on sale for \$28 for 5 disks. How much should one disk cost?” (\$5.60)
- “Muffins are packaged 5 to a box for the bake sale. How many boxes can you make up if you bake 28 muffins?” (5).
- “Parents will be transporting children in minivans for the class field trip. Each van can take 5 children. The class has 28 children. How many vans will parents need to drive for the trip?” (6).

The understanding of all four operations that students build with whole numbers in the upper elementary grades broadens during grades 6–8, when they apply those operations to fractions, decimals, percents, and integers. Moreover, as students operate with rational numbers and integers, they encounter new contexts that may challenge their conceptual foundations. For example, when students are multiplying or dividing with fractions or decimals between 0 and 1, they see results that expose as misconceptions the commonly held beliefs that “multiplication makes larger” and “division makes smaller.”

Other challenges that middle-grades students must confront include understanding when the result of a computation with integers is positive and when it is negative, knowing how to align decimals in computations with decimal fractions, and recognizing where in an answer to place a decimal point. Operating with fractions has proven difficult for many students. Lacking conceptual understanding, many have tried to get by with rote application of procedures that they don’t understand. In the middle grades, therefore, it is important that students develop an understanding of the meaning of such concepts as numerator, denominator, and equivalent fractions and their roles in adding and subtracting fractions.

Middle school students need to model and compare expressions that are frequently subject to confusion, such as “divide by 2,” “multiply by 1/2,” and “divide by 1/2,” and they must see that different models of division are sometimes required to give meaning to such ideas. For example, “divide by 2” can be modeled by a partitioning model (“separate the amount into two equal quantities”), but “divide by 1/2” is more appropriately represented by a repeated-subtraction model:

> “You made $2\frac{3}{4}$ gallons of lemonade. How many $\frac{1}{2}$-gallon bottles can you fill?”
>
> $$\left(2\frac{3}{4} \div \frac{1}{2} = 5, \text{ with a remainder of } \frac{1}{4} \text{ gallon}\right)$$

Encouraging students to estimate and evaluate the reasonableness of the results of their computations is important in helping them expand their number sense.

As students’ algebraic concepts grow during grades 6–8, they will also frequently face computations involving variables, and they will need to extend their understanding of the operations and their properties to encompass simplification of and operations with algebraic expressions. Understanding the inverse relationship between addition and subtraction, between multiplication and division, and between “square” and “square root” will be important in such tasks.

In grades 9–12, students should go beyond producing the results of specific computations to generalize about operations and their properties and to relate them to functions and their graphs. For example, they should describe and compare the behavior of functions such as $f(x) = 2x$, $g(x) = x + 2$, $h(x) = x^2$, or $j(x) = \sqrt{x}$. They should reason about number relations, describing, for instance, the value of $a \cdot b$ where a and b are positive numbers and $a + b = 50$. They should understand and correctly apply the results of operating with positive or negative numbers when they are working with both equations and inequalities.

In addition, high school students should learn to perform operations in other systems. They should find vector sums in the plane, add and multiply matrices, or use multiplicative reasoning to represent counting problems and combinatorics.

Computing fluently and making reasonable estimates

Although an understanding of numbers and the meanings of the various operations is essential, it is insufficient unless it is accompanied by the development of computational proficiency and a sense of the reasonableness of computational results. Computational skills emerge in the prekindergarten and early elementary years in conjunction with students' developing understanding of whole numbers and counting.

Young children's earliest computational strategies usually involve counting. As they think about number problems involving addition or subtraction, young students devise different solution schemes, and teachers should listen carefully to their students' explanations of these thinking strategies. Encouraging children to explain their methods and discussing different students' strategies with the class helps students deepen their understanding of numbers and operations and refine their computational abilities.

At first, young children rely heavily on physical objects to represent numerical situations and relationships, and they use such objects to model their addition and subtraction results. Over time, they learn to represent the same problems symbolically, and eventually they carry out the computations mentally or with paper and pencil, without needing the actual physical objects. Students should have enough experience and practice to master the basic one-digit addition and subtraction combinations, and they should combine that knowledge with their understanding of base-ten numeration so that, by the end of grade 2, they can add and subtract with two-digit numbers.

As students become more proficient with addition and subtraction, teachers can help them examine the efficiency and generalizability of their invented strategies and can lead them to an understanding of standard computational algorithms. When students understand the procedures that they are employing, they are able to carry them out with accuracy and efficiency.

In grades 3–5, students should extend their knowledge of basic number combinations to include single-digit multiplication and division facts, and by the end of the upper elementary years they should be able to compute fluently with whole numbers. As students develop their computational proficiency, teachers should guide them in examining and explaining their various approaches and in understanding algorithms for addition, subtraction, multiplication, and division and employing them effectively. In turn, teachers must understand that there is more than one algorithm for each of the operations, and they should recognize that the algorithms that are meaningful to students may not be the ones that have traditionally been taught or that some people have come to assume offer “the right way” to solve a problem.

In grades 3–5, students are beginning to work with larger numbers, and it is important for them to develop a strong sense of the reasonableness of a computational result and a facility in estimating results before computing. It will often be appropriate for students to use calculators when they are working with larger numbers. At other times, paper and pencil may be appropriate, or it may be reasonable for teachers to expect mental computation. Teachers and students should discuss various situations to assist students in developing good judgment about when to use mental arithmetic, paper and pencil, or technology for whole-number computation.

Other aspects of computational fluency in the 3–5 grade band involve understanding the associative, commutative, and distributive properties and seeing how those properties can be used to simplify a computation. Students at this level will also encounter problems that require the introduction of order-of-operations conventions.

While students in grades 3–5 are honing their skills with whole-number computation, they also will be spending a great deal of time developing an understanding of fractions and decimals. However, computing with rational numbers should not be the focus of their attention yet. Rather, students should apply their understanding of fractions and decimals and the properties of the operations to problems that include fractions or decimals. For example, “How many sheets of construction paper will Jackie need to make 16 Halloween decorations if each decoration uses $2\frac{1}{4}$ sheets of paper?” General procedures for calculating with rational numbers and integers will be the focus of instruction in the next grade band.

In grades 6–8, students learn methods for computing with fractions and decimals as extensions of their understanding of rational numbers and their facility in computing with whole numbers. As with whole-number computation, students develop an understanding of computing with fractions, decimals, and integers by considering problems in context, making estimates of reasonable expectations for the results, devising and explaining methods that make sense to them, and comparing their strategies with those of others as well as with standard algorithms. When calculating with fractions and decimals, students must learn to assess situations and decide whether an exact answer is required or whether an estimate is appropriate. They should also develop useful benchmarks to help them assess the reasonableness of results when they are calculating with rational numbers, integers, and percents. Computational fluency at the middle grades also includes a facility in reasoning about and solving problems involving proportions and rates.

In grades 9–12, students should extend their computational proficiency to real numbers and should confidently choose among mental mathematics, paper-and-pencil calculations, and computations with technology to obtain results that offer an appropriate degree of precision. They should perform complex calculations involving powers and roots, vectors, and matrices, as well as real numbers, and they should exhibit a well-developed number sense in judging the reasonableness of calculations, including calculations performed with the aid of technology.

Numbers and Operations in the Mathematics Curriculum

Without numbers and operations there would be no mathematics. Accordingly, the mathematics curriculum must foster the development of both number sense and computational fluency across the entire pre-K–12 continuum. The Number and Operations Standard describes the core of understanding and proficiency that students are expected to attain, and a curriculum that leads to the outcomes envisioned in this Standard must be coherent, developmental, focused, and well articulated across the grades. At all levels, students should develop a true understanding of numbers and operations that will undergird their development of computational fluency.

The *Navigating through Number and Operations* books provide insight into the ways in which the fundamental ideas of number and operations can develop over the pre-K–12 years. These Navigations volumes, however, do not—and cannot—undertake to describe a complete curriculum for number and operations. The concepts described in the Number and Operations Standard regularly apply in other mathematical contexts related to the Algebra, Geometry, Measurement, and Data Analysis and Probability Standards. Activities such as those described in the four *Navigating through Number and Operations* books reinforce and enhance understanding of the other mathematics strands, just as those other strands lend context and meaning to number sense and computation.

The development of mathematical literacy relies on deep understanding of numbers and operations as set forth in *Principles and Standards for School Mathematics.* These *Navigations* volumes are presented as a guide to help educators set a course for the successful implementation of this essential Standard.