CHAPTER 1

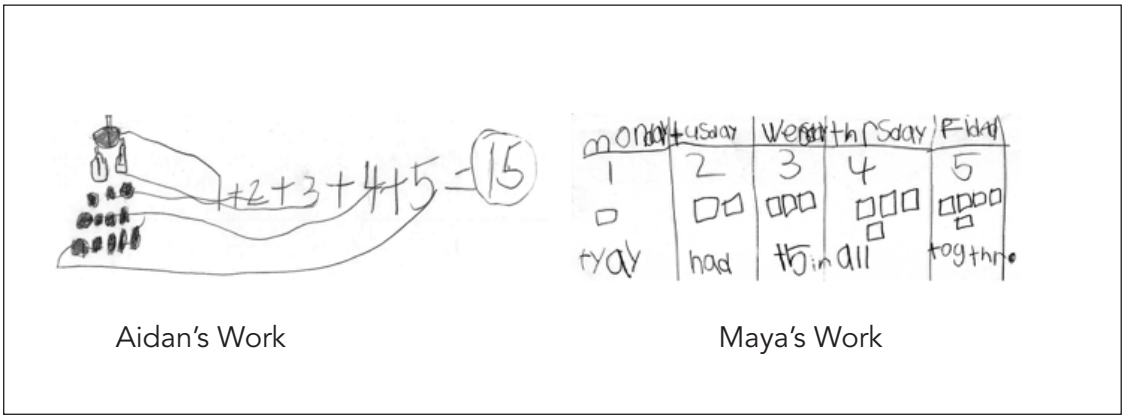
Setting the Stage

Imagine walking into a first-grade classroom where students are sitting on a rug and their teacher, Ms. Bouchard, is reading *The Very Hungry Caterpillar* by Eric Carle. After discussing the story, she comments, "I wonder how many pieces of fruit were eaten through by that caterpillar? He sure was hungry." This question prompts the task shown in figure 1.1 and launches her students into a mathematical investigation.

On Monday, the hungry caterpillar ate through one apple, but he was still hungry. On Tuesday he ate through two pears, but he was still hungry. On Wednesday he ate through three plums. On Thursday he ate through four strawberries. On Friday he ate through five oranges. How many pieces of fruit did the hungry caterpillar eat during the week?

Fig. 1.1. The Caterpillar task

Some students use cubes, while other students start drawing on paper. Regardless of how the students solve the problem, the teacher expects all students to "put their thinking on paper." As the students work, Ms. Bouchard makes her way around the classroom asking questions, "So, tell me about your picture" or "How does the picture show your thinking?" or "How does the picture show the story?" The teacher also makes note of their approaches so she can decide which students she wants to present their work, and in which order, later during the whole-class discussion. The work of two students, Aidan and Maya, are shown in figure 1.2.



Aidan's Work

Maya's Work

Fig. 1.2. Samples of student work for the Caterpillar task

Before holding a class discussion, the teacher asks students to describe their picture to a partner. This gives all students a chance to share their own thinking, and begins orienting them to the thinking of other students, certainly not an easy task for first-grade students. Next the class gathers on the rug to present and compare their work. Throughout the lively discussion, Ms. Bouchard focuses her students on the mathematics learning goals for the lesson—to see and model the mathematical elements in a contextual problem, to understand addition as putting together sets of objects, and to understand that counting can be used to answer "how many" questions. To close the lesson, Ms. Bouchard asks each student to look at their picture and to revise it so that it better shows their thinking using ideas they got by looking at and talking about the work of their classmates.

A Vision for Students as Mathematics Learners and Doers

The lesson portrayed in the opening scenario exemplifies the vision of school mathematics that the National Council of Teachers of Mathematics (NCTM) has been advocating in a series of policy documents over the last twenty-five years (NCTM 1989, 2000, 2006). In this vision, as in the scenario, students are active learners, constructing their knowledge of mathematics through exploration, discussion, and reflection. The tasks in which students engage are both challenging and interesting and cannot be answered quickly by applying a known rule or procedure. Students must reason about and make sense of a situation and persevere when a pathway is not immediately evident. Students use a range of tools to support their thinking and collaborate with their peers to test and refine their ideas. A whole-class discussion provides a

forum for students to share ideas and clarify understandings, develop convincing arguments, and learn to see things from other students' perspectives.

In the Caterpillar scenario, students were encouraged to use their imagination to envision the caterpillar eating his way through the apple, the pears, the plums, the strawberries, and the oranges. Through this context, they were presented with mathematical work to find the total number of pieces of fruit. Even though combining five quantities was a challenge for these young learners, all students could enter the problem by using their choice of materials. The context interested the students and they persevered in making sense of the situation and keeping track of all those quantities. Putting their thinking on paper, while not new to the students, was difficult for them. Presenting their work to a partner had multiple purposes. It allowed some students to get help from a peer in revising their drawing while allowing all students the opportunity to communicate their own mathematical thinking and to consider the reasoning of other students. During the whole-class discussion, students were pressed to not only analyze the work of individual students but to compare approaches and discuss how they were similar and different. In this way, the teacher introduced students to new ways to represent and organize their thinking on paper, such as with realistic pictures, math drawings, or equations (e.g., $1 + 2 + 3 + 4 + 5 = 15$). While most students counted by ones, other students shared how they had grouped the numbers such as making fives, $(1 + 4) + (2 + 3) + 5$, or how they discovered that adding $1 + 2 + 3 + 4$ made 10. During the closing minutes of the lesson students were allowed time to revise their drawings, reinforcing that mathematical work occurs within a community of practice in which we develop shared understanding and learn from each other.

The vision for student learning advocated by NCTM, and represented in our opening scenario, has gained growing support over the past decade as states and provinces have put into place world-class standards (e.g., National Governors Association Center for Best Practices and Council of Chief State School Officers 2010). These standards focus on developing conceptual understanding of important mathematical ideas, flexible use of representations, strategies, and procedures, and the ability to engage in a set of mathematical practices that include reasoning, problem solving, and communicating mathematically.

A Vision for Teachers as Facilitators of Student Learning

Meeting the demands of world-class standards for student learning requires teachers to engage in what has been referred to as "ambitious teaching." Ambitious teaching stands in sharp contrast to what many teachers experienced themselves as learners of mathematics. On one hand, many of us remember memorizing facts and procedures with little emphasis on understanding, problem solving, and application. Ambitious teaching, on the other hand,

views students as capable of making sense of mathematical ideas and being able to use their understanding to solve authentic problems (Lampert, Boerst, and Graziani 2011) and values students' thinking, including emergent understanding and errors, and attends to student thinking in an equitable and responsive manner (Anthony et al. 2015).

In ambitious teaching, the teacher engages students in challenging tasks and collaborative inquiry, and then observes and listens as students work so that she or he can provide an appropriate level of support to diverse learners. The goal is to ensure that each and every student succeeds in doing meaningful, high-quality work, not simply executing procedures with speed and accuracy. In our opening scenario with Ms. Bouchard, we see a teacher who is engaging her students in mathematics reasoning and problem solving. She presents an authentic task for her students to explore that emerges naturally from reading the children's literature book. She provides resources to support their work, such cubes, paper, and crayons, as well as partners with whom to exchange ideas, and monitors their actions providing support as needed. During the whole-class discussion, the teacher orchestrates discourse that builds on students' ways of thinking about the task while making connections to important mathematical ideas. It is this central focus on student sense making, thinking, and reasoning that exemplifies ambitious teaching. As Anthony and colleagues (Anthony et al. 2015, p. 46) underscore:

> Ambitious mathematics teaching involves skilled ways of eliciting and responding to each and every student in the class so that they learn worthwhile mathematics and come to view themselves as competent mathematicians.

This book is intended to support teachers in meeting the challenge and complexities of ambitious teaching by describing and illustrating a set of teaching practices that facilitate the type of "responsive and discipline-connected instruction" (Lampert et al. 2010, p. 130) that is at the heart of ambitious teaching.

Support for Ambitious Teaching

Principles to Actions: Ensuring Mathematical Success for All (NCTM 2014) provides guidance on what it will take to make ambitious teaching, and rigorous content standards, a reality in classrooms, schools, and districts in order to support mathematical success for each and every student. At the heart of this book is a set of eight teaching practices that provide a framework for strengthening the teaching and learning of mathematics (see fig. 1.3). These teaching practices describe intentional and purposeful actions taken by teachers to support the engagement and learning of every student. These teaching practices, based on knowledge of mathematics teaching and learning accumulated over more than two decades, represents "a core set of high-leverage practices and essential teaching skills necessary to promote deep learning of mathematics" (NCTM 2014, p. 9). Each of these teaching practices is examined in more depth through illustrations and discussions in the subsequent chapters of this book.

Establish mathematics goals to focus learning. Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.
Implement tasks that promote reasoning and problem solving. Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.
Use and connect mathematical representations. Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.
Facilitate meaningful mathematical discourse. Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.
Pose purposeful questions. Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.
Build procedural fluency from conceptual understanding. Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.
Support productive struggle in learning mathematics. Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.
Elicit and use evidence of student thinking. Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.

Fig. 1.3. Eight effective teaching practices for mathematics

Ambitious mathematics teaching must be equitable. Driscoll and his colleagues (Driscoll, Nikula, and DePiper 2016, pp. ix-x) acknowledge that defining equity can be elusive, but argue that equity is really about fairness in terms of providing access to "each learner with alternative ways to achieve, no matter the obstacles they face" and believing in each student's potential "to do challenging mathematical reasoning and problem solving." Hence teachers need to pay attention to the instructional opportunities that are provided to students, particularly to

historically underserved or marginalized youth (i.e., students who are Black, Latina/Latino, American Indian, low income) (Gutierrez 2013, p. 7). Every student has the right to participate substantially in all phases of a mathematics lesson and be challenged and supported in developing deep understanding and proficiency in mathematics (Jackson and Cobb 2010).

Toward this end, throughout this book we relate the eight effective teaching practices to specific equity-based practices (see fig. 1.4) shown to strengthen mathematical learning and cultivate positive student mathematical identities (Aguirre, Mayfield-Ingram, and Martin 2013). Equitable mathematics classrooms provide every student with access to meaningful mathematics by leveraging students' strengths (mathematical competencies), drawing on students as resources of knowledge, and challenging spaces of marginality. These classroom communities of collaboration and coherent discourse position each and every student to make sense of mathematics and develop positive mathematics identities.

Go deep with mathematics. Develop students' conceptual understanding, procedural fluency, and problem solving and reasoning.
Leverage multiple mathematical competencies. Use students' different mathematical strengths as a resource for learning.
Affirm mathematics learners' identities. Promote student participation and value different ways of contributing.
Challenge spaces of marginality. Embrace student competencies, diminish status, value multiple mathematical contributions.
Draw on multiple resources of knowledge (math, language, culture, family). Tap students' knowledge and experiences as resources for mathematics learning.

Fig. 1.4. Five equity-based practices to support mathematics learning

Central to ambitious teaching, and at the core of the five equity-based practices, is helping each and every student develop an identity as a capable member of a mathematical community of practice. Aguirre and her colleagues (Aguirre et al. 2013, p. 14) define mathematical identities as

> the dispositions and deeply held beliefs that students develop about their ability to participate and perform effectively in mathematical contexts and to use mathematics in powerful ways across the contexts of their lives.

Students enter elementary school eager to engage in collaborative, mathematical problem solving. Yet as students progress through these early years of their education, beliefs about

their mathematical abilities sometimes shift. It is sad when such young learners begin to see themselves as "not good at math" and question their own sense-making abilities, as well as wonder whether they have a place in mathematical communities. When students begin to approach mathematics with fear, anxiety, and lack of confidence, it is too often the result of unambitious instructional practices based on the belief that not everyone can do serious mathematics (Delpit 2012). Allen and Schnell (2016, p. 398) argue, "teachers have a unique opportunity to steer their students' mathematical development in a more positive direction." The effective teaching practices discussed and illustrated in this book are intended to help in this regard.

Contents of this Book

This book is written for teachers and teacher educators who are committed to ambitious teaching that provides their students with increased opportunities to experience mathematics as meaningful, challenging, and worthwhile. It is likely, however, that any education professional working with teachers would benefit from the illustrations and discussions of the effective teaching practices for mathematics.

This book can be used in several ways. Teachers can read through the book on their own, stopping to engage in the activities suggested and trying out the ideas in their own classrooms. Alternatively, and perhaps more powerfully, teachers can work their way through the book with colleagues in professional learning communities, grade-level meetings, or staff development sessions. Working together to discuss and examine teaching practices brings considerable added value to our professional learning. Teacher educators or professional developers might use this book in college or university education courses for prospective teachers or in professional development workshops during the summer or school year. The book would be a good choice for a face-to-face or online book study for any group of mathematics teachers interested in improving their instructional practices.

In this book we provide a rationale for and discussion of each of the eight effective mathematics teaching practices, and when appropriate, connect them to the equity-based practices for mathematics classrooms. We provide examples and activities intended to help elementary teachers develop their understanding of each teaching practice, how it can be enacted in the classroom, and how it can promote equity in student learning. Towards this end, we invite the reader to actively engage in two types of activities that are presented throughout the book: Analyzing Teaching and Learning (ATL) and Taking Action in Your Classroom. The Analyzing Teaching and Learning activities invite the reader to engage with specific artifacts of classroom practice (e.g., mathematics tasks, narrative cases of classroom instruction, teacher-student dialogues, video clips, and student work samples). Taking Action in Your Classroom provides specific suggestions for a teacher to begin exploring specific teaching practices in her or his own classroom. The ATLs are drawn, in part, from activities found in the Principles

to Actions Professional Learning Toolkit (http://www.nctm.org/PtAToolkit/). Additional activities, beyond what can be found in the Toolkit, have been added in order to provide a more extensive investigation of each of the eight effective mathematics teaching practices.

The video clips, featured in several Analyzing Teaching and Learning activities, show teachers endeavoring to engage in ambitious instruction in their urban classrooms and show students preserving in solving mathematical tasks that require reasoning and problem solving. The videos, made available by the Institute for Learning at the University of Pittsburgh, provide images of effective mathematics teaching. As such, they are examples to be analyzed rather than models to be copied. You can access and download the videos and their transcripts by visiting NCTM's More4U website (nctm.org/more4u). The access code can be found on the title page of this book

As you read this book and engage with both types of activities, we encourage you to keep a journal or notebook in which you record your responses to questions that are posed, as well as make note of issues and new ideas that emerge. These written records can serve as the basis for your own personal reflections, informal conversations with other teachers, or for planned discussions with colleagues.

Each of the next eight chapters focuses explicitly on one of the eight effective teaching practices for mathematics. We have arranged the chapters in an order that makes it possible to highlight the ways in which the teaching practices are interrelated. (Note that this order differs from the one shown in figure 1.2 and in *Principles to Actions* [NCTM 2014]).

Chapter 2: Establish Mathematics Goals to Focus Learning

Chapter 3: Implement Tasks that Promote Reasoning and Problem Solving

Chapter 4: Build Procedural Fluency from Conceptual Understanding

Chapter 5: Pose Purposeful Questions

Chapter 6: Use and Connect Mathematical Representations

Chapter 7: Facilitate Meaningful Mathematical Discourse

Chapter 8: Elicit and Use Evidence of Student Thinking

Chapter 9: Support Productive Struggle in Learning Mathematics

Each of these chapters will follow a similar structure. We begin a chapter with a short discussion of the focal teaching practice. Then we ask the reader to engage in a series of Analyzing Teaching and Learning (ATL) activities that highlight key features of the teaching practice for teachers and students through artifacts of classroom practice. Each chapter also includes a summary of research findings related to the focal teaching practice, and describes how the teaching practice promotes equity in mathematics classrooms. We end each chapter with suggestions for Taking Action in Your Classroom. This includes a summary of the key messages regarding the focal teaching practice and an activity in which the reader is encouraged to apply aspects of the teaching practice in her or his own classroom instruction.

In the final chapter of the book (Chapter 10: Pulling It All Together), we consider how the set of eight effective mathematics teaching practices are related and how they work in concert to support student learning of mathematics. In chapter 10 we also consider the importance of thoughtful and thorough planning in advance of a lesson and evidence-based reflection following a lesson as critical components of the teaching cycle and necessary for successful use of the effective teaching practices.

An Exploration of Teaching and Learning

We close this chapter with the first Analyzing Teaching and Learning activity, which takes you into the classroom of Mr. Harris, where his third-grade students are exploring representations for multiplication. The case presents a situation from his classroom in which his students are first exploring and solving the Band Concert task and then engaging in a whole-class discussion to analyze and compare the various strategies they used.

When new teaching practices are introduced in chapters 2 through 9, we occasionally refer the new teaching practice to some aspect of the Case of Mr. Harris and the Band Concert task. In so doing, we are using the case as a touchstone throughout the book to relate the teaching practices. Hence the case provides a unifying thread that brings coherence to the book and makes salient the synergy of the effective mathematics teaching practices (i.e., the combined effect of the effective teaching practices is greater than the impact of any individual teaching practice).

Analyzing Teaching and Learning 1.1

Investigating Teaching and Learning in a Third-Grade Classroom

As you read the case of Mr. Robert Harris, consider the follow questions and record your observations in your journal or notebook so that you can revisit them when we refer to Mr. Harris and the Band Concert task in subsequent chapters:

- What does Mr. Harris do during the lesson to support his students' engagement in and learning of mathematics?
- What aspects of Mr. Harris's teaching are similar to or different from your own teaching of mathematics?
- Which aspects of his teaching might you want to incorporate into your own teaching of mathematics?
- In what ways does the case illustrate the eight effective teaching practices in support of ambitious teaching of mathematics?

Exploring Representations for Multiplication
The Case of Robert Harris and the Band Concert Task

Robert Harris wanted his third-grade students to understand the structure of multiplication and decided to develop a task that would allow students to explore multiplication as equal groups through a familiar context—the upcoming spring band concert. He thought that the Band Concert task (see next page) would prompt students to make or draw arrays and provide an opportunity to build conceptual understanding toward fluency in multiplying one-digit whole numbers by multiples of 10 using strategies based on place value and properties of operations—all key aspects of the standards for third-grade students. He felt that the task aligned well with his math goals for the lesson and supported progress along math learning progressions, had multiple entry points, would provide opportunities for mathematical discourse, and would challenge his students. As students worked on the task he would be looking for evidence that his students could identify the number of equal groups and the size of each group within visual or physical representations, such as collections or arrays, and connect these representations to multiplication equations.

The Band Concert Task

The third-grade class is responsible for setting up the chairs for the spring band concert. In preparation, the class needs to determine the total number of chairs that will be needed and ask the school's engineer to retrieve that many chairs from the central storage area. The class needs to set up 7 rows of chairs with 20 chairs in each row, leaving space for a center aisle. How many chairs does the school's engineer need to retrieve from the central storage area?

Mr. Harris began the lesson by asking students to consider how they might represent the problem. "Before you begin working on the task, think about a representation you might want to use and why, and then turn and share your ideas with a partner." The class held a short conversation sharing their suggestions, such as using cubes or drawing a picture. Then the students began working individually on the task.

As Mr. Harris made his way around the classroom, he noticed many students drawing pictures. Some students struggled to organize the information, particularly those who tried to represent each individual chair. He prompted these students to pause and review their work by asking, "So, tell me about your picture. How does it show the set up of the chairs for the band concert?" Other students used symbolic approaches, such as repeated addition or partial products, and a few students chose to use cubes or grid paper. He made note of the various approaches so he could decide which students he wanted to present their work, and in which order, later during the whole-class discussion.

In planning for the lesson, Mr. Harris prepared key questions that he could use to press students to consider critical features of their representations related to the structure of multiplication. As the students worked, he often asked: "How does your drawing show the seven rows?" "How does your drawing show that there are 20 chairs in each row?" "Why are you adding all those twenties?" "How many twenties are you adding and why?"

He also noticed a few students changed representations as they worked. Dominique started to draw tally marks, but switched to using a table. When Mr. Harris asked her why, she explained she got tired of making all those marks. Similarly, Jamal started to build an array with cubes, but then switched to drawing an array. Their initial attempts were valuable, if not essential, in helping each of these students make sense of the situation.

Before holding a whole-class discussion, Mr. Harris asked the students to find a classmate who had used a different representation and directed them to take turns explaining and comparing their work as well as their solutions. He encouraged them to also consider how their representations were similar and different. For example, Jasmine who had drawn a diagram compared her work with Kenneth who had used equations

(see fig. 1.5). Jasmine noted that they had gotten the same answer and Kenneth said they both had the number 20 written down seven times. Molly, in particular, was a student who benefited from this sharing process because she was able to acknowledge how confused she had gotten in drawing all those squares (see fig. 1.5) and had lost track of her counting. Her partner helped her mark off the chairs in each row in groups of ten and recount them. The teacher repeated this process once more as students found another classmate and held another sharing and comparing session.

During the whole-class discussion, Mr. Harris asked the presenting students to explain what they had done and why and to answer questions posed by their peers. He asked Jasmine to present first since her diagram accurately modeled the situation, and it would likely be accessible to all students. Kenneth went next as his approach was similar to Jasmine's but without the diagram. Both clearly showed the number 20 written seven times. Then Teresa presented. Her approach allowed the class to discuss how skip counting by twenties was related to the task and to multiplication, a connection not apparent for many students. Below is an excerpt from this discussion.

Mr. H: So, Teresa skipped counted by twenties. How does this relate to the Band Concert situation?

Connor: She counted seven times like she wrote on her paper.

Mr. H: I'm not sure I understand. Can someone add on to what Connor was saying?

Grace: Well each time she counted it was like adding 20 more chairs, just like what Kenneth did.

Mr. H: Do others agree with what Grace is saying? Can anyone explain it in their own words?

Mason: Yeah, the numbers on top are like the 7 rows and the numbers on the bottom are the total number of chairs for that many rows.

Mr. H: This is interesting. So what does the number 100 mean under the 5?

Mason: It means that altogether five rows have 100 total chairs, because there are 20 chairs in each row.

Mr. H: Then what does the 140 mean?

Mason: It means that seven rows would have a total of 140 chairs.

Mr. H: *[Mr. Harris paused to write this equation on the board: $7 \times 20 = 140$.]* Some of you wrote this equation on your papers. How does this equation relate to each of the strategies that we have discussed so far? Turn and talk to a partner about this equation.

[After a few minutes, the whole-class discussion continued and Grace shared what she talked about with her partner.]

Grace: Well, we talked about how the 7 means seven rows like Jasmine showed in her picture and how Teresa showed. And the 20 is the number of chairs that go in each row like Jasmine showed, and like how Kenneth wrote down. Teresa didn't write down all those twenties but we know she counted by twenty.

Toward the end of the lesson, Mr. Harris had Tyrell and Ananda present their representations (see fig. 1.5) because they considered the aisle and worked with tens rather than with twenties. After giving the students a chance to turn and talk with a partner, he asked them to respond in writing whether it was okay to represent and solve the task using either of these approaches and to justify their answers. He knew this informal experience with the distributive property would be important in subsequent lessons, and the student writing would provide him with some insight into whether or not his students understood that quantities could be decomposed as a strategy in solving multiplication problems.

The case was written by DeAnn Huinker (University of Wisconsin-Milwaukee), drawing on her professional experiences with teachers and students in the Milwaukee area.

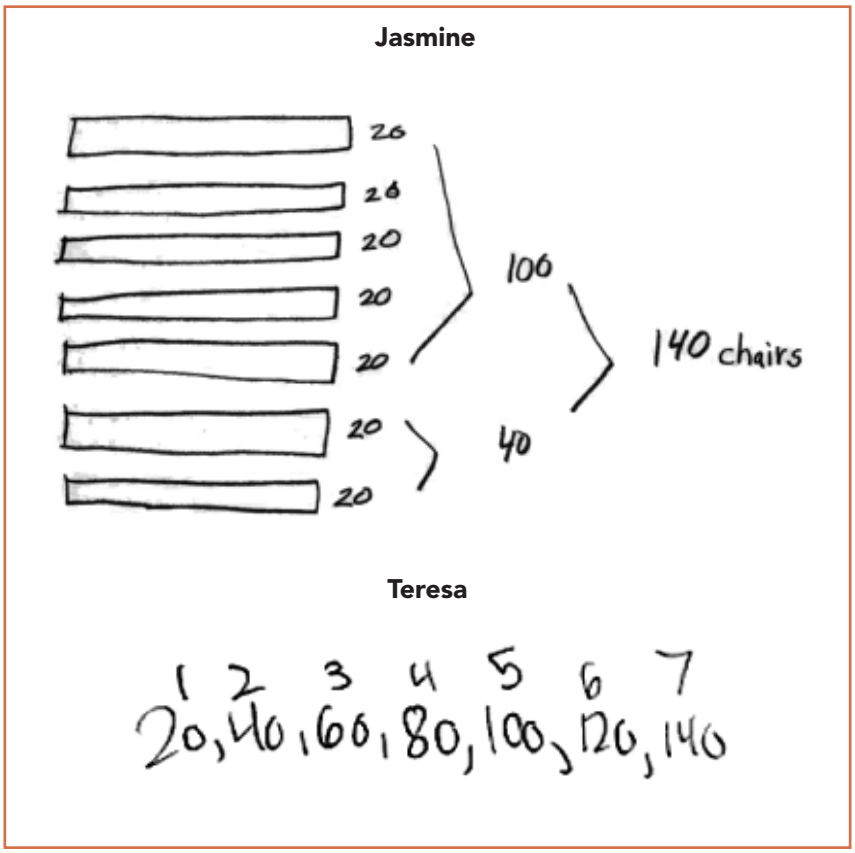


Fig. 1.5. Student work for the Band Concert task
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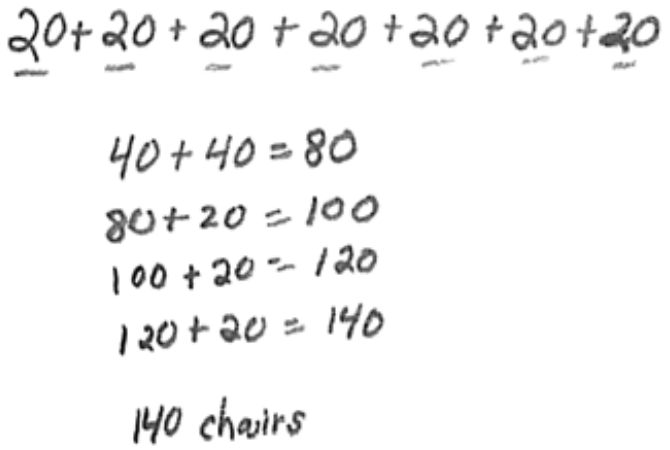
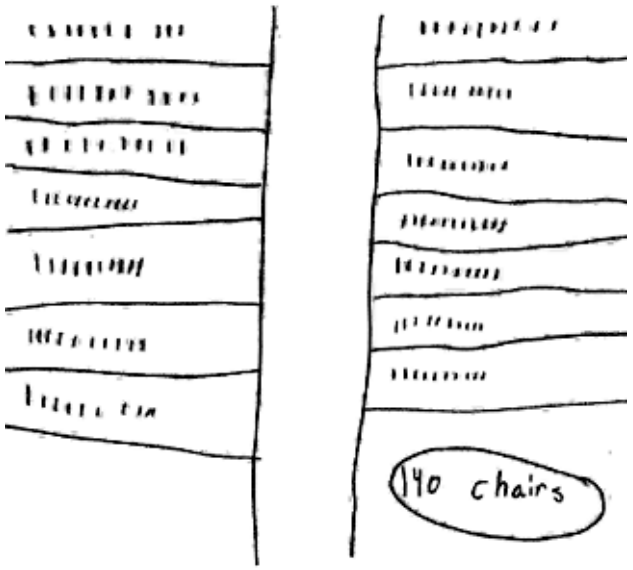
Kenneth



Molly

Fig. 1.5. Student work for the Band Concert task
(*continued on next page*)

Tyrell



Ananda

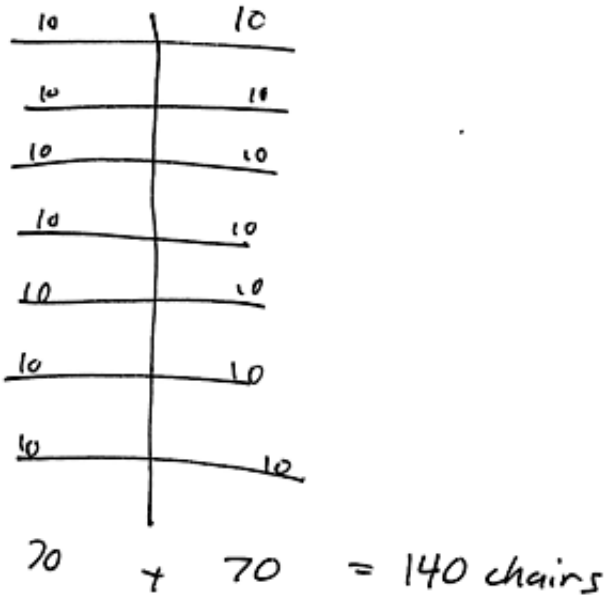


Fig. 1.5. Student work for the Band Concert task

Moving Forward

The case of Robert Harris presents many noteworthy aspects of mathematics instruction and examples of his use of the effective mathematics teaching practices. However, we are not going to provide an analysis of this case here. Rather, as you work your way through chapters 2 through 9 you will revisit the case of Robert Harris and consider the extent to which he engaged in the effective mathematics teaching practices and the impact it appeared to have on student learning and engagement. As you progress through the chapters, you may want to return to the observations you made during your initial reading of the case and consider the extent to which you are now seeing aspects of the case through a new lens.

As you read the chapters that follow, we encourage you to continue to reflect on your own instruction and how the effective teaching practices can help you in improving your teaching of mathematics. The Taking Action in Your Classroom activity at the end of each chapter is intended to support you in this process. Cultivating a habit of systematic and deliberate reflection holds the key to improving one's teaching as well as sustaining lifelong professional learning.