

## CHAPTER 1

# Learning Trajectories and Professional Development

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When Ms. Williams, an elementary school teacher, is teaching her mathematics lessons, she knows her goals for every task she uses. She knows the *content* of the lesson and understands how this particular lesson fits within her mathematics objectives for the unit. She uses this knowledge to decide on probing questions to pose to her students as they work on the tasks and later discuss their work. Ms. Williams also knows her *students* and understands how students at different places in their learning engage with the tasks at hand. When planning her lesson, Ms. Williams anticipates what different students might need and prepares for a few different instructional scenarios, so that during the lesson she can be ready to work with students who need scaffolding to get started, as well as with others who need further challenges to remain engaged. As students work on the tasks, Ms. Williams pays attention to more and less mathematically advanced ways in which students approach tasks and supports each and every student in deepening their understanding of the topic. She then asks a few students to share their work with the whole group to engage the class in productive mathematics discussions. Ms. Williams considers the mistakes students are making, what these mistakes reveal about students' understanding of the topic, and how she will work with them as the lesson continues to unfold. She assigns follow-up tasks to different groups of students, and, for each of these groups, Ms. Williams has a solid understanding of what she wants them to learn next.

This portrait of Ms. Williams shows an elementary school teacher with a framework of student thinking at the center of her instruction. She uses this framework to interpret what students know and are yet to know and to guide her teaching, from task selection to student engagement to assessment. There was a time when Ms. Williams's hard work to understand the mathematics she teaches and develop the depth of her content knowledge (characterized in the first few sentences of the portrait) was enough to warrant what was considered good instruction. Though it is still the case that strong knowledge of the content is a requirement for good instruction, we now know that content

knowledge alone does not suffice! With a growing understanding about how students informally think about mathematics and develop mathematical ideas over time, great teachers like Ms. Williams seek out and use new knowledge about student learning (characterized in the remainder of the portrait). Good instruction rests upon knowledge of learners together with knowledge of content. Thus, excellent teachers know how their students learn, and use frameworks to organize this knowledge in ways that help them think about their teaching. These frameworks are often called learning trajectories (LTs). When teachers use them to make decisions about their teaching, as Ms. Williams does, they are engaged in what we call *learning trajectory based instruction* (Sztajn, Confrey, Wilson, & Edgington, 2012).

This book is about supporting teachers to strengthen their teaching, in keeping with our sketch of the teacher we describe as Ms. Williams. It focuses on understanding how teachers can learn to make LTs central to their teaching, engaging in learning trajectory based instruction. More specifically, the book shares the stories of four different professional development projects designed to promote teacher learning and the use of frameworks of student thinking—a research tool of growing interest to professional development designers (Sztajn, Borko, & Smith, 2017). The book represents an effort to look across projects that promote teacher learning of learning trajectory based instruction and discuss what these projects reveal about the design of such professional development (PD). It aims to inform professional development designers and teacher educators who want to improve their work in promoting teacher learning of LTs. It is also useful for teachers who want to use LTs to support their students' learning as well as for researchers interested in teacher education or PD. We begin with a brief introduction to LTs before discussing PD centered on these trajectories.

## WHAT ARE LEARNING TRAJECTORIES?

Analyzing his own teaching of mathematics, Simon (1995) captured the interest of mathematics education researchers when he coined the concept of a hypothetical learning trajectory. He suggested that mathematics instruction starts with a learning goal for the students, a hypothesis about what students currently understand and how their understanding evolves, and a set of learning activities to support students along that path. Together, the goal, the learning hypothesis, and the activities made up what, at the time, Simon (1995) called a hypothetical learning trajectory. In his words:

The teacher's learning goal provides a direction for a hypothetical learning trajectory. I use the term "hypothetical learning trajectory" to refer to the teacher's prediction as to the path by which learning might proceed. It is hypothetical because the actual learning

trajectory is not knowable in advance . . . A hypothetical learning trajectory provides the teacher with a rationale for choosing a particular instructional design; thus, I make my design decisions based on my best guess on how learning might proceed.

The choice of the word “trajectory” is meant to refer to a path, the nature of which can perhaps be clarified by the following analogy. Consider that you have decided to sail around the world in order to visit places that you have never seen. . . . You may initially plan the whole trip or only part of it. You set out sailing according to your plan. However, you must constantly adjust because of the conditions that you encounter. . . . You change your plans with respect to the order of your destination. You modify the length and nature of your visits as a result of your interactions with people along the way. You add destinations that prior to your trip were unknown to you. The path that you travel is your “trajectory.” The path that you anticipate at any point in time is your “hypothetical trajectory.” (p. 136)

The idea that teachers construct these hypothetical trajectories was profound. It spoke to teachers’ agency in the classroom as they engage with students and work to understand their thinking. It also showed that effective teachers do not approach a learning situation without knowledge about their students and an initial plan for students’ learning. This plan unfolds and changes into actualized learning through interactions with the students, building on and reorganizing the teachers’ initial predictions and the activities and goals that go with them.

Following Simon’s initial work, a new question arose: What if we substituted teachers’ best guesses for shared knowledge? The idea was that instead of teachers making instructional decisions “based on [their] best guess on how learning might proceed” (p. 136) they could make decisions based on shared knowledge about how learning might proceed. Teachers’ instructional decisions could be based on research on learning that is organized into trajectories, and these trajectories would be empirically developed and tested to give teachers a tool that represents how students’ learning might proceed in a particular topic. This tool would then be used in several classrooms with many students, year after year. The trajectories would still be hypothetical, as in Simon’s explanation of the sailing trip, functioning as starting points for teachers to work with students during instruction. However, teachers would not be recreating these trajectories alone or every time; instead they would use shared, research-based trajectories in the implementation and discussions of instructional decisions. This was an intriguing idea.

The concept of empirically developed, shared trajectories that mapped student learning captivated the imagination of educational researchers. Such trajectories could serve as a tool to guide curriculum development or provide a foundation for assessments to diagnose and monitor student learning. For teachers, trajectories could assist them in understanding their students’ thinking and provide a framework to guide their actions in planning, implementing, and reflecting on a lesson. Over time, the possibility of such a shared path

for student learning coalesced into the current attention to the concept of LTs and learning trajectory based instruction.

The concept of evidence-based mappings of students' learning, however, was not new. In their seminal work in the 1980s, Carpenter and Moser (1984) had created a framework of student thinking about addition and subtraction problems that later led to the work on Cognitively Guided Instruction (CGI) (Carpenter, Fennema, Franke, Levi, & Empson, 2015). This pioneering effort provided the foundation for teachers to use knowledge about the development of student thinking to inform their practice and, to date, Cognitively Guided Instruction remains one of the few approaches to teacher professional development that has demonstrated positive impacts on student learning (Wilson & Berne, 1999). Simon's work, nonetheless, brought forth the powerful idea of instruction that is guided by LTs, which, about 10 years later, emerged in a wave of attention to the topic.

By 2007, a National Research Council committee attending to the learning and teaching of science in schools released a report that called on researchers who examined student learning to map such learning into trajectories<sup>1</sup> that "describe the successively more sophisticated ways of thinking about a topic that can follow and build on one another as children learn about and investigate a topic over a broad span of time (e.g., 6 to 8 years)" (National Research Council, 2007, p. 213). The committee underscored that trajectories were "a promising direction" for organizing instruction and suggested that further research and development was needed to create these frameworks to support students' understanding. The National Research Council book was a catalyst to already existing interest among researchers about the idea of LTs (e.g., Battista, 2004; Clements & Sarama, 2004; Shapiro, 2004), and several conferences, publications, and policy briefs followed.

In addition to the broad definition used by the National Research Council, other definitions have been proposed for LTs. For example, Clements and Sarama (2004) defined LTs as "descriptions of children's thinking and learning in a specific mathematical domain, and a related conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking" (p. 83). Confrey, Maloney, Nguyen, Mojica, and Myers (2009) defined a LT as "a researcher-conjectured, empirically-supported description of the ordered network of constructs a student encounters through instruction (i.e., activities, tasks, tools, forms of interaction, and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time" (p. 347).

These definitions and others suggest variations in the object of learning, scale, and theoretical perspective considered in different LTs (Lobato & Walters, 2017). Yet these definitions also point to the idea that LTs provide

an initial mapping of student learning that can guide teachers' instructional decisions and exist in the context of teacher/learner interactions, when trajectories become "real" through instruction.

## **USING LEARNING TRAJECTORIES IN INSTRUCTION**

The potential of LTs in supporting instruction was recognized by the greater education community when researchers called for translating them into "usable tools for teachers" (Daro, Mosher, & Corcoran, 2011, p. 13). Still, much of the initial work on the use of LTs in mathematics education focused on policy (e.g., Common Core; Corcoran, Mosher, & Rogat, 2009), assessment (e.g., Battista, 2004; Gotwals & Songer, 2013; Confrey & Maloney, 2012), curriculum development (Clements, 2007), or the connections among them (e.g., Confrey, Maloney, & Corley, 2014; Daro, Mosher, & Corcoran, 2011; Duncan & Hmelo-Silver, 2009). This initial work was similar in science education, where Shapiro (2004) introduced the idea of learning progressions to offer curriculum developers insights into how students learn over long periods of time and how this learning is organized. As in mathematics education, most of this work was focused on the research and development of tools for teaching—and not on the actual teaching of mathematics or science.

How teachers interpret and use LTs as instructional tools is a much more recent object of research attention (Lobato & Walters, 2017). The authors of chapters in this book have made significant contributions to this line of research, and their efforts have provided insights about how teachers come to learn and use LTs in their classrooms. The paragraphs below highlight some of these insights.

Researchers in the Learning Trajectory Based Instruction project (Chapter 2) have shown that teachers can use LTs to select tasks and interact with students (Wilson, 2009), addressing students' misconceptions (Edgington, 2012). Teachers' own content knowledge, however, matters when they interact with and learn about the trajectories (Wilson, Sztajn, Edgington, & Confrey, 2014). In the classroom, teachers can use LTs to anticipate students' strategies, monitor small-group work, and sequence large-group discussions (Wilson, Sztajn, Edgington, & Myers, 2015). Teachers can also use LTs to learn to talk differently about their students. LTs, nonetheless, do not challenge existing talk that includes deficit language about mathematical abilities (Wilson, Sztajn, Edgington, Webb, & Myers, 2017).

Researchers in the Children's Measurement Project (Chapter 3) have found that LTs can support teachers' focus on the conceptual building blocks of mathematical content (Wickstrom, Baek, Barrett, Tobias, & Cullen, 2012). Teachers who learn about trajectories can develop more specific language and lesson designs. They can also use the levels in the trajectories

to mark students' partial progress instead of expecting an "all or nothing" understanding (Wickstrom, 2014). Teachers benefit from LTs as prompts to explore conceptual aspects of mathematical content, building their own content knowledge for teaching (e.g., Wickstrom & Jurczak, 2016; Wickstrom, Nelson, & Chumbley, 2015).

Focusing on instructional practices that build on students' thinking, researchers in the Responsive Teaching in Elementary Mathematics Project (Chapter 4) have revealed the critical role of noticing a student's mathematical thinking (Jacobs, Lamb, & Philipp, 2010). Focusing on teachers' responses, researchers have identified categories of questioning that build on students' thinking (Jacobs & Ambrose, 2008; Jacobs & Empson, 2016). They have highlighted the purpose of each category. These categories are important because questioning that is grounded in the specific details of students' thinking provides increased opportunities for students to advance their understanding (Jacobs, Franke, Carpenter, Levi, & Battey, 2007; Steinberg, Empson, & Carpenter, 2004).

Research in the Building Blocks and TRIAD Projects (Chapter 5) showed that when teachers participate in PD on LTs, their practices improve and there are increases in children's achievement in mathematics and also in other domains, such as language. Teachers demonstrate increasing levels of fidelity in implementing ideas from the professional development on LTs both 2 years and 6 years after the professional development (Sarama, Lange, Clements, & Wolfe, 2012). For these researchers, LTs that include mathematics goals, developmental progression of levels of children's thinking, and instructional tasks and strategies linked to specific levels provide the conditions for promoting sustainability in teacher practices. Research suggests these trajectories can address a climate of low expectations in some urban schools as teachers increase their understanding of the capacities of all children to learn mathematics.

In the chapters that follow, the researchers who contributed to these and other findings discuss what they have learned through their experiences in designing and providing professional development around LTs. More important, they go beyond their research findings to discuss their practice as professional development designers and share their assumptions and experiences in developing and providing these professional development programs. They attend to their work in conceptualizing, creating, leading, or scaling mathematics professional development for elementary school teachers and present lessons learned.

## **MATHEMATICS TEACHING AND PROFESSIONAL DEVELOPMENT**

It is important to consider LTs in the context of what constitutes high-quality instruction. This type of instruction is ambitious in its goal of advancing the mathematical understanding of each and every student (Lampert, Beasley,

Ghousseini, Kazemi, & Franke, 2010), not only promoting instruction guided by trajectories but also fostering equitable learning trajectory based instruction (Myers, Sztajn, Wilson, & Edgington, 2015). The vision for mathematics teaching that is key for the design of many professional development programs in mathematics is also central for the programs presented in this book. This vision includes “instruction that builds on rich mathematical tasks, attends to student thinking, values interactions as a learning mechanism, and is considered ‘ambitious’” (Sztajn, Borko, & Smith, 2017, p. 796).

Jacobs and Spangler (2017) proposed that this type of teaching attends to student thinking and is contingent on students’ comments, questions, and strategies. They highlighted that such teaching benefits teachers and students: “When teachers explore students’ ways of reasoning, they benefit by gaining a window into students’ reasoning, which can be mathematically powerful but often differs from teachers’ reasoning. Students benefit because they not only have opportunities to articulate and reflect on their reasoning but also learn to value their own and peers’ sense making” (p. 767).

This vision for high-quality mathematics instruction aligns with the idea of learning trajectory based instruction in which LTs are central and shape teachers’ mathematical knowledge for teaching, task selection, orchestration of discourse, formative assessment, and equitable practices. LTs can also be placed at the center of practices considered core for mathematics instruction, such as teacher noticing and leading discussions (Jacobs & Spangler, 2017). Given the central role of instruction in learning, as well as recent progress in better articulating important features of high-quality mathematics instruction and equitable teaching, it is important to continue to articulate the vision for instruction that guides the work of those designing professional development focused on LTs and learning trajectory based instruction.

## **DESIGN OF PROFESSIONAL DEVELOPMENT**

Of particular importance to this book is attention to how to best design programs that address the key ideas about LTs while also attending to what is known about effective professional development. Thus, we want to highlight some shared features across the professional development programs discussed in this book. All programs build on current consensus features of quality professional development, such as implementation over a significant number of hours, spanning many months, having a content focus on mathematics teaching and learning, and using pedagogies that promote teachers’ active learning. However, as Sztajn, Borko, and Smith (2017) suggested, the programs understand these features as necessary features of effective professional development but far from sufficient to explain what makes professional development successful. In the accompanying chapters, each of the professional development programs attends to the specific design features of its program.

The authors discuss what makes their professional development effective and the ways in which frameworks of student thinking are central to their projects. Perhaps more interesting, they discuss what they learned in the process of designing and implementing their professional development.

When designing learning opportunities for teachers, professional developers attend to teachers' learning goals and interests and to how teachers help shape the design of their own learning. It is also important to consider that teachers learn in the context of their classrooms, within their schools, and in interaction with their colleagues. Thus, practice plays a key role in the design of professional development, and the programs presented make use of important artifacts of practice such as video clips or student work. Understanding how these various tools were used in different programs, and the lessons learned from such use, can improve the use of such tools when designing other professional development programs for student thinking frameworks in different contexts. Those seeking to take ideas from the programs presented here into their own practice can benefit from the detailed descriptions the chapters provide concerning program design, what teachers learned, and what researchers learned.

### **CAUTION POINTS ABOUT LEARNING TRAJECTORIES**

Before discussing how LTs can serve as starting points to guide instruction and the design of professional development, we acknowledge that the research on LTs has not come without criticism and potential pitfalls. We also recognize that skeptical criticism is a fundamental part of the knowledge development process. Thus, in this section we briefly present concerns that have emerged regarding LT research.

An important idea when considering LTs is that researchers' perspectives on learning shape the development of the trajectories. Lobato and Walters (2017) reported that different perspectives on learning have been used in the development of different trajectories. Perspectives that attend to learning as a social, rather than an individual, phenomenon are underrepresented in LT research. This implies that, in many cases, the learning described in LTs is conceived as paths for individuals to gain knowledge, when, for many researchers, learning is not the isolated experience of single individuals. For several of the projects in this book, learning is a social phenomenon, or should at least be considered from a combination of outlooks that includes both the individual and collective views of learning. From a more social perspective on learning, trajectories are negotiated in the larger context of classroom interactions as well as in professional development settings.

The idea of trajectories as paths for individuals to gain knowledge can lead to a problematic interpretation that suggests every student follows the steps specified in LTs as they learn. Empson (2011) showed that, even in topics

such as students' strategies for solving addition and subtraction problems, where there is robust evidence for a progression of learning, students' use of these strategies varies between classrooms and children. This means that the development of student learning is highly context-dependent and should not be equated with the following of predetermined steps from a trajectory. It is important to understand that LTs represent common patterns in aspects of learning, rather than a determined path. In her early work, Confrey (2006) suggested that trajectories provide conceptual corridors with landmarks as well as constraints and obstacles. Teachers and students navigate these corridors, finding their paths from prior knowledge to learned ideas.

There are also concerns about equity regarding the research used to generate LTs, particularly about the lack of attention to the variety of cultural practices learners bring to learning situations (Anderson et al., 2012). Myers (2014) problematized the paucity of questions about the reproduction of systems of class, race, and gender in LT research. She suggested that particular attention be given to understanding how cultural artifacts can limit learning opportunities for certain students, particularly students of color. Myers also called for further attention to the students participating in the empirical research used to develop and validate LTs. She suggested that when such research does not attend to the diversity of its sample, it can privilege the knowledge of particular groups in the development of LTs while affirming the resulting trajectories as universal. Myers suggested that we carefully examine the concept of universality in research results such as LTs.

For the authors in this book, attention to potential pitfalls in research on LTs is of utmost importance and needs to guide those supporting teacher learning and use of these trajectories. In particular, equity in mathematics instruction is a central tenet for mathematics education and needs to be at the forefront of research on how teachers come to use LTs in diverse classrooms. Still, the agreement that research can address emerging concerns and work toward the development of shared, empirically tested, continuously improved frameworks about student learning remains key to the work of authors in this book. Further, these frameworks are useful starting points for teachers across contexts and groups of students.

## **OVERVIEW OF THE BOOK**

In October 2016, the researchers working on the four projects presented in this book met to discuss what they were learning from their various projects centered on teacher learning.<sup>2</sup> The goal of the 2-day meeting was to share, accumulate, and synthesize evidence of what teachers learn and how they use frameworks of student thinking across the four projects. Descriptions of what happens inside each professional development and how professional learning tasks are designed to foster teacher learning were at the center of the

conversations. The meeting also highlighted similarities and differences across projects. Attention to issues of equity emerged as a fundamental point in the discussion among researchers.

The conversations that took place during the meeting strengthened the collaboration across projects and led to this book. The meeting generated recommendations for professional development focused on LTs—which are embedded throughout the chapters in the book and further highlighted in Chapter 6. Chapters 2–5 are organized around the four projects participating in the meeting.

In Chapter 2, Sztajn, Edgington, Wilson, Webb, and Myers describe the Learning Trajectory Based Instruction project. Attending to the issue of how to bring a research-developed tool such as LTs to teachers in ways that promote productive research and practice connections, they discuss their design of a professional development program that was respectful of teachers' knowledge. For them, professional development is a space in which teachers and researchers come together to share knowledge and learn from each other. Thus, they examine how they adapted LTs—created by researchers—to be useful to teachers. They present the principles they used to create professional learning tasks around these trajectories.

In Chapter 3, Barrett, Wickstrom, Tobias, Cullen, Cullen, and Baek examine the Children's Measurement Project: Sharing Trajectories with Teachers. Here they integrate LT research into their professional development to help teachers look more carefully at students' thinking, modify or develop tasks, and use lesson study as a structure to improve the teaching of geometric measurement. Specifically, they want teachers to use the trajectories for formative assessment in the classroom. The authors discuss how the project supported teachers' task design as well as their noticing and responding to students' mathematics.

In Chapter 4, Jacobs, Empson, Pynes, Hewitt, Jessup, and Krause discuss the Responsive Teaching in Elementary Mathematics (RTEM) project. Their focus is on professional development that helps teachers develop instruction that is responsive to children's fraction thinking. Responsive teaching requires expertise in eliciting and attending to the details of children's thinking to make decisions in the midst of instruction. To support teachers in developing this expertise, these authors work with frameworks that encapsulate research on children's thinking and on the instructional practices of noticing children's thinking and questioning to build on children's thinking. They identify design principles that create opportunities for teachers to integrate these frameworks into their teaching.

In Chapter 5, Sarama and Clements examine the design, scale-up, and evaluation of a professional development based on LTs in the context of the Building Blocks and TRIAD projects. With attention to the development of evidence-based curriculum materials and a focus on equity, the chapter shows how these projects built on the 10 phases from Clements's (2007)

Curriculum Reform Framework. The authors discuss how they designed a professional development focused on teachers learning to engage young children in mathematics that is both appropriate and challenging, and they touch on the fidelity of teachers' implementation of what they learned. They examine the challenges of designing a program that can be delivered at scale and discuss principles that can support both implementation and sustainability long after the original professional development ends.

All the authors present principles that guided the projects' professional development design, provide rationales for professional development features, describe key decisions that shaped the resulting programs, and share what the designers learned in the process of developing and enacting their professional development. Such attention to design provides a rare opportunity to identify and examine key factors that influence both the implementation of the professional development and teacher learning. Therefore, in Chapter 6, we examine issues regarding professional development design discussed in Chapters 2–5. We present important design features for professional development in general as well as for professional development that focuses on frameworks of student thinking. In particular, we discuss issues related to the ways in which trajectories are presented to teachers as well as connections between trajectories and instruction. We also turn to the issues of context and equity and offer a set of recommendations for designing professional development for LTs. This summary chapter is followed by Chapter 7, in which Hilda Borko adds her perspective on the book. She looks at LTs as conceptual tools for professional development design and connects the ideas in this book to larger discussions on professional development.

## NOTES

1. In science education, the term *learning progression* is used more often than *trajectory*. For this book, we use the word *trajectory* instead of *progression* whenever possible, in alignment with the discursive practice and perspective of the mathematics education community.

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