NAVIGATIONS SERIES

GRADES 9–12

NAVIGATING *through* MATHEMATICAL CONNECTIONS

Introduction

Integrated Mathematics: Choices and Challenges (McGraw 2003) discusses many issues related to the development and implementation of integrated mathematics curricula. The introductory chapter, by Peggy House, appears on the CD-ROM that accompanies this volume.

All too often, mathematics educators think of integrated mathematics as just another curriculum option. However, in presenting and elaborating the Process Standards, *Principles and Standards for School Mathematics* (National Council for Teachers of Mathematics [NCTM] 2000) recommends that K–12 mathematics be taught and learned in an integrated, connected fashion, from prekindergarten through grade 12. This does not mean that *Principles and Standards* calls for teachers and schools to adopt textbooks titled "Integrated Mathematics," though such texts might have great potential for supporting the recommendations in the Standards. It does mean that teachers and schools should take advantage of the power of the Process Standards, which encompass connections, representation, problem solving, reasoning and proof, and communication. Teachers should use these Standards to link the mathematics curriculum's five essential content strands—number and operations, algebra, geometry, measurement, and data analysis and probability.

To become acquainted with the approach of this book and the topics that it covers, consider a well-known problem from the K–12 mathematics curriculum. Suppose that 500 mathematics teachers in an audience decide to shake hands. How much time will this activity take? The answer, of course, depends on how many handshakes the teachers give and what strategy they use to give them. Suppose that each of the 500 teachers shakes hands exactly once with every other teacher. How many handshakes will there be? An examination of several different strategies for solving this problem can illustrate the integrated nature of mathematics as well as highlight the themes of this book.

Combinatorial Strategy (S1)

A combinatorial strategy considers all the combinations of two teachers that we must make for each teacher to shake hands with every other teacher.

Enacting the problem

Once we have decided that every teacher shakes hands precisely once with every other teacher, we must determine an “enactment”—a way in which each teacher can shake hands with every other teacher, allowing us to make a count of the handshakes. To simplify the situation, we can start with a group of just five teachers and have every teacher shake hands once with every other teacher in the group while we count the handshakes.

In our simplified situation, the key is the same as in the original, more complex problem: we must make sure that every possible pair of teachers shakes hands. We could apply the same process to larger groups of teachers, but by *mathematizing* the process in the simplified case, we can discover how to use mathematics to find an answer to the original question.

Representing the problem mathematically

We can begin to treat the problem mathematically by applying a combinatorial strategy, which we will call S1, to our simplified situation. Our enactment of the problem suggests that a handshake corresponds to a single pairing of two teachers. We can think of each pair of teachers as a two-element subset of the set of teachers in the audience. The question then becomes, “How many two-element subsets are in a set with 500 elements?”

> *“The organized lists and tree diagrams that students will have used in the elementary and middle grades to count outcomes or compute probabilities can be used in high school to work on permutations and combinations.”*
> *(NCTM 2000, p. 293)*

Restructuring the mathematical representation

The entries in Pascal’s triangle tell us how many r-element subsets we can form from a set with n elements (see fig. 0.1). We call this number “n choose r” and denote it by

$$_{n}C_{r} \text{ or } \binom{n}{r},$$

since we are counting how many ways we can choose a subset of r elements from a set of n elements. To solve our simplified case of five teachers shaking hands with one another, we find the entry that corresponds to a subset of two elements chosen from a set with five elements. This entry, 10, shows the number of ways of making such subsets from a five-element set. This number tells us that if in a group of five teachers, every teacher shakes hands with every other teacher, then the teachers will exchange ten handshakes in all.

To find

$$\binom{500}{2},$$

	$_nC_r$	Number (r) of elements in a subset						
		0	1	2	3	4	5	....
Number (n) of elements in a set	0	1						
	1	1	1					
	2	1	2	1				
	3	1	3	3	1			
	4	1	4	6	4	1		
	5	1	5	10	10	5	1	
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Fig. **0.1.**

Pascal's triangle, presented as the entries in a table that gives the number of subsets with r elements, chosen from a set with n elements

or the number of handshakes that a group of 500 teachers would exchange if each teacher shook hands once with each other teacher, we would simply read from Pascal's triangle the number in the row labeled "500" and the column labeled "2." However, constructing Pascal's triangle for so many rows would be a prodigious task. Instead, we can use a combinatorial pattern that students in grades 9–12 frequently study:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Thus, we can easily compute $\binom{500}{2}$:

$$\binom{500}{2} = \frac{500!}{2!\,498!} = \frac{500 \times 499}{2} = 124,750.$$

Resolving the problem

Since there are 124,750 two-element subsets in a set with 500 elements, we expect the 500 teachers to exchange 124,750 handshakes.

Ordered-Pair Strategy (S2)

An ordered-pair strategy considers all the possible ordered pairs of teachers and handshakes that we would need to form for each teacher to shake hands with every other teacher.

Enacting the problem

Working with the same five-teacher simplification of the problem as in S1, we could vary the enactment by using a slightly different process and implementing it to ensure that every possible pair of teachers shakes hands once. We would start our new process by numbering the teachers from one to five. We would have teacher 1 shake hands with every other teacher and then sit down. Next, teacher 2 would shake hands with every other teacher still standing, and then teacher 2 would also sit down. We would continue this process, counting all the handshakes at each stage, until we had all the teachers seated. We would get a total of 4 + 3 + 2 + 1, or 10, handshakes for the group of five teachers.

As before, we could use our enactment in the case of the 500 teachers, but once again doing so would be a formidable task. However, we can mathematize the process, and as a result, we can again discover a way to use mathematics to answer the original question.

Representing the problem mathematically

Our new enactment suggests the idea of representing a teacher and a handshake with an ordered pair of natural numbers. In the larger, original problem, we would represent each teacher by a natural number from 1 to 500, and we would then use these numbers to form an ordered pair (m, n) representing a handshake between teachers m and n. Since we begin with teacher 1, and no teacher shakes hands with himself or herself, m is less than n.

For example, in the simplified case of five teachers, the ordered pairs that result from the enactment are (1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), and (4, 5). Thus, we can restate our original problem as a new question: “How many ordered pairs of natural numbers (using the numbers between 1 and 500, inclusive) can we form in which the first number is less than the second number?”

Sums of integers and their powers have been studied extensively in the history of mathematics. Formulas for these sums appear in many high school algebra texts and are even programmed into graphing calculators:

$$1+2+\ldots+k=\sum_{i=1}^{k} i=\frac{(k+1)k}{2},$$

$$1^2+2^2+\ldots+k^2=\sum_{i=1}^{k} i^2$$

$$=\frac{(2k+1)(k+1)k}{6},$$

and so forth.

Restructuring the mathematical representation

One way to answer this new question is to count the number of ordered pairs whose first coordinate is 1, add that to the number of ordered pairs whose first coordinate is 2, and so forth. In the simplified case of the five teachers, we get 4 + 3 + 2 + 1, or

$$\sum_{i=1}^{4} i.$$

In the case of 500 teachers, we get $\sum_{i=1}^{499} i.$

$$\sum_{i=1}^{499} i=\frac{500\times 499}{2},$$

or 124,750 handshakes. Thus, the number of ordered pairs (m, n) of natural numbers where $1 \le m < n \le 500$ is 124,750.

Another way to count these ordered pairs of natural numbers is to invoke their correspondence with lattice points in the Cartesian plane. In the simplified case of five teachers, the ordered pairs representing the handshakes correspond to the lattice points shown in figure 0.2a. We can count these lattice points easily by considering the rectangular array in figure 0.2b. The geometry of the array reveals that the numbers of dark lattice points and light lattice points are the same. Thus, there are (1/2)(5 × 4) dark lattice points. With 500 teachers, the resulting array would have 500 rows and 499 columns, yielding (1/2)(500 × 499), or 124,750 dark lattice points.

By studying the pattern of dark and light lattice points, we can see that the geometric strategy is equivalent to adding the numbers from 1 to 499 twice and then dividing by 2, a method attributed to Karl Friedrich Gauss for summing the integers from 1 to n (see fig. 0.3). We note that the columns of lattice points in figure 0.2b represent adding the terms from 1 to 4 in descending order (the dark lattice points: 4 + 3 + 2 + 1) to the terms from 1 to 4 in ascending order (the light lattice points: 1 + 2 + 3 + 4). The result is 4 columns, each with 5 lattice points, or a total of $(n - 1)(n)$, or (4)(5), lattice points.

Fig. **0.3.**

The sum of the integers from 1 to n – 1. Adding the terms in the first two expressions for the sum yields the expression in the third line for

$$2\sum_{i=1}^{n-1} i,$$

which simplifies to the expressions in lines 4 and 5.

$$\sum_{i=1}^{n-1} i = 1 + 2 + \ldots + (n-2) + (n-1)$$

$$\sum_{i=1}^{n-1} i = (n-1) + (n-2) + \ldots + 2 + 1$$

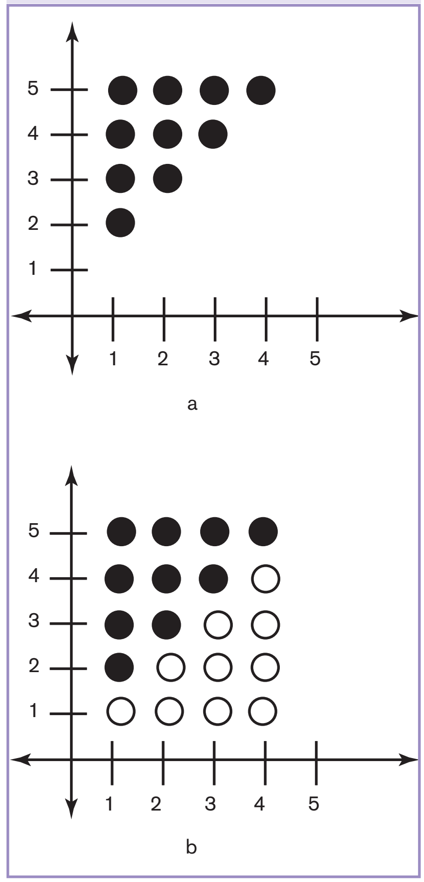
$$2\sum_{i=1}^{n-1} i = n + n + \ldots + n + n$$

$$2\sum_{i=1}^{n-1} i = (n-1)n$$

$$\therefore \sum_{i=1}^{n-1} i = \frac{(n-1)n}{2}$$

Fig. **0.2.**

Arrays of lattice points, showing (a) ordered pairs (m, n) representing teachers and handshakes in a group of five teachers, and (b) the rectangular array of which these ordered pairs are part



Resolving the problem

However we count the ordered pairs, we get the same result:

$$\frac{500 \times 499}{2},$$

or 124,750, ordered pairs. Therefore, we would expect 124,750 handshakes.

Graph-Theory Strategy (S3)

A graph-theory strategy considers all the segments that can connect two vertices representing any two teachers in a simple graph of the handshake situation.

Enacting the problem

We can represent yet another way of enacting the handshakes and counting them by drawing a picture of the situation. Figure 0.4 presents a drawing that shows each teacher in our simplified group of five as a dot and each handshake as a segment connecting two dots. By counting

“Three important areas of discrete mathematics are integrated within [the] Standards: combinatorics, iteration and recursion, and vertex-edge graphs. These ideas can be systematically developed from prekindergarten through grade 12.” (NCTM 2000, p. 31)

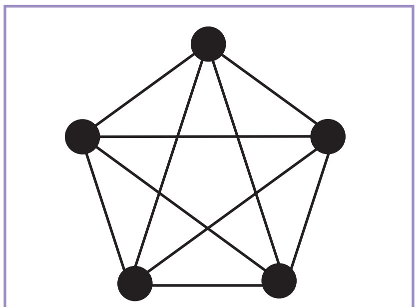


Fig. **0.4.**

A drawing to represent five teachers and the ten handshakes that occur when each teacher shakes hands with each other teacher

the segments, we arrive at the number of handshakes. We could use the same drawing process to handle the case of 500 teachers, but, once again, doing so would be impractical, and we choose instead to mathematize the process. By analyzing our mathematical model for the simple case, we again find a way to use mathematics to solve the original, larger problem.

Representing the problem mathematically

The drawing in figure 0.4 lets us think of the 500 teachers as vertices in a graph and a handshake as an edge containing exactly two of the vertices. With this representation, the question about how many handshakes the teachers exchange is transformed into a new question: “How many edges are in the graph?”

Restructuring the mathematical representation

From graph theory, we know that when every edge in a graph connects exactly two distinct vertices, the number of edges in the graph is one-half of the sum of the degrees of the vertices in the graph. The *degree* of a vertex is the number of edges containing the vertex. The graph in figure 0.4, which represents the simplified handshake problem, is called a *complete graph*, since exactly one edge connects each pair of vertices. (Every pair of teachers shakes hands exactly once.) In a complete graph with n vertices, each vertex must lie on $(n - 1)$ edges and therefore have degree $(n - 1)$. Thus, the sum of the degrees of the n vertices is $n(n - 1)$, and the number of edges is

$$\frac{n(n-1)}{2}.$$

Resolving the problem

Since there are

$$\frac{500 \times 499}{2},$$

or 124,750, edges in a complete graph with 500 vertices, then there should be 124,750 handshakes among the 500 teachers.

Function Strategy (S4)

A function strategy considers a rule that relates the number of teachers to the number of handshakes that all the pairs of teachers exchange.

Enacting the problem

To enact the handshake problem in still another way, we could start with one teacher in the room. With one teacher, we would have 0 handshakes. Then we could have a second teacher enter the room. With two teachers in the room, we would have 1 handshake when the teachers shook hands precisely once. We could then have a third teacher enter the room and shake hands once with each of the two teachers there, for 2 new handshakes. Thus, when there are three teachers in the room, there are 3 handshakes. We could record these handshakes in a table like table 0.1. We could then have a fourth teacher enter the room and shake hands with each of the three teachers there, while we added these 3 new handshakes to our total and recorded the result.

We could, of course, continue this process until we had all 500 teachers from the original problem in the room. However, we prefer to mathematize the process and apply our mathematics to the larger problem.

Table 0.1.
Handshakes between Pairs of Teachers in a Room

Number of Teachers in Room	Number of Handshakes
1	0
2	1
3	3
4	6
5	10
⋮	⋮
500	?

Representing the problem mathematically

This enactment of the problem leads us to treat the number of handshakes as a function of n, the number of teachers in the room. Table 0.1 presents a function table for the problem. With this enactment, the question changes again, now becoming, "What function $f(n)$ fits the data in the table, and what is $f(500)$?"

Restructuring the mathematical representation

From the results that we recorded for our latest enactment, we see that $f(1) = 0, f(2) = f(1) + 1, f(3) = f(2) + 2 = 3, f(4) = f(3) + 3 = 6$, and so on. The emerging recursive pattern, $f(n + 1) = f(n) + n$, or $f(n + 1) - f(n) = n$, makes sense, since the $(n + 1)$th person shakes n hands on entering the room. We add these n handshakes to the total of the handshakes, $f(n)$, which we counted when there were only n people in the room.

We can identify $f(n + 1) - f(n) = n$ as a difference equation. The theory of difference equations tells us that when the difference between consecutive terms in a sequence, or $f(n + 1) - f(n)$, is a linear function of n, the sequence $f(n)$ is a quadratic function of n. Thus, since $f(n + 1) - f(n) = n$, we know that $f(n)$ is a quadratic function; in other words, $f(n) = an^2 + bn + c$, for some real numbers a, b, and c.

We can use many methods to find a, b, and c. For example, we can use a quadratic regression on the ordered pairs in our function table. Or we can use the function table to set up and solve a system of three equations in three unknowns. A third possibility is to use methods from the theory of difference equations (see Cuoco's [2003] discussion of Newton's difference formula). All these methods lead to the same result:

"In making choices about what kinds of situations students will model, teachers should include examples in which models can be expressed in iterative, or recursive, form."
(NCTM 2000, p. 303)

Cuoco (2003; available on the CD-ROM) discusses using Newton's difference formula as a simple way to fit a polynomial to a table.

$$f(n) = \frac{1}{2}n^2 - \frac{1}{2}n.$$

Resolving the problem

Since

$$f(500) = \frac{1}{2}500^2 - \frac{1}{2}(500),$$

or 124,750, there are 124,750 handshakes when 500 teachers are in the room.

Integrating Mathematics

Our consideration of multiple approaches to a simple problem illustrates some important things about mathematics. The handshake problem involves teachers and a handshake relation. Solution strategies S1–S4 model this phenomenon in four different areas of mathematics. Indirectly, these areas are linked through their connection to the handshake problem.

These connections should not surprise us. Much of mathematics has blossomed from the efforts of men and women to model the real world. In fact, one can argue that the drive to model the physical world—a goal of a vast array of sciences—has been a primary force in the genesis of many mathematical ideas, their structures, and their interconnections. It is the rule rather than the exception that real-world problems can be modeled in several mathematical domains, and the resulting mathematical models, in turn, reveal interconnections among these areas.

Our multiple models for the handshake problem illustrate some fundamental principles of mathematical modeling and of problem solving. Mathematical modeling of a real-world phenomenon consists of a process of representation, in which a model in a mathematical domain replicates the key elements and relations in the phenomenon. The choice of a mathematical representation often depends on how someone conceptualizes the problem and its components in an enactment. This enactment is an integral part of the process of representing the phenomenon and allows for multiple modes: kinesthetic, visual, verbal, and symbolic.

"Students in grades 9–12 should develop an increased capacity to link mathematical ideas and a deeper understanding of how more than one approach to the same problem can lead to equivalent results, even though the approaches might look quite different." (NCTM 2000, p. 354)

Though the handshake problem makes the choice of a representation look relatively simple and straightforward, the process typically demands an involved analysis of the real-world situation. A cycle of assumptions and refinements customarily characterizes this analysis, and it is this process of rethinking, revising, and refining that gives mathematical modeling its justly deserved reputation for fostering interdisciplinary thinking.

The representational stage in the mathematical modeling process leads to a restructuring of the problem in the mathematical realm. This mathematicizing facilitates the deduction of new relationships among the key elements in the model. The restructuring often takes multiple paths, as in the ordered-pair strategy (S2) and the function strategy (S4).

Finally, the resolution phase translates the results of the model back to the original problem and validates them there. Here again, the handshake illustration makes this stage look easier than it ordinarily is. The process is usually involved; validating results depends on the adequacy of the assumptions underlying the representational stage and often leads to further refinements in the model.

The handshake problem also reveals that the modeling process can be useful in probing mathematical as well as real-world phenomena. The process can highlight connections in the mathematical world as well as between the mathematical world and the real world. For example, in S2, lattice points in a graph modeled the ordered pairs of teachers and handshakes. We then used the geometry of the graph to answer the question about the ordered pairs.

The practice of reserving the expression *mathematical modeling* for applications of mathematics to real-world phenomena is not uncommon, but this book is less restrictive in its use of the expression *modeling process.* The thought processes involved in modeling—whether one applies them to a real-world situation, like the handshake problem, or to a question about some mathematical phenomenon—are as intrinsic to mathematical thinking as assimilation and accommodation are to human learning.

The modeling process has exerted a powerful integrative and generative force on mathematics throughout history. For example, Descartes's modeling of Euclidean geometry's elements and relations within the domain of ordered pairs of numbers and their algebraic relations led to a historic restructuring of geometric questions in an algebraic framework. This restructuring ultimately resulted in the resolution of significant geometric problems. Descartes's groundbreaking work also led to a number line that included the negative numbers—a development that produced a momentous restructuring of algebra. Negative numbers had appeared in algebra before Descartes but were not widely accepted until after Descartes's innovation.

The history of imaginary numbers was similar. After emerging in algebra during the sixteenth century, these numbers did not gain wide acceptance among mathematicians until Wessel and Argand (ca. 1800) introduced the complex plane as a geometric model of the numbers and their operations.

The work of Beltrami, Klein, Riemann, Poincaré, and others can serve as a final example of the power of mathematical models to connect mathematical domains. These mathematicians created Euclidean models for non-Euclidean geometries. When the models established the independence of Euclid's parallel postulate from his other postulates, they greatly elevated the status of non-Euclidean geometry.

"Students can use insights gained in one context to prove or disprove conjectures generated in another, and by linking mathematical ideas, they can develop robust understandings of problems."
(NCTM 2000, p. 354)

An integrative, problem-solving habit of mind

In addition to illustrating the integrative power of the modeling process, the handshake problem highlights another vital, integrative force within mathematics: the habit of mind that the mathematician brings to the problem-solving process. Pólya (1973) described this habit of mind most impressively, observing that to the mathematician, "no problem whatever is completely exhausted" (p. 15). Connective

"Problem solving is an integral part of all mathematics learning, and so should not be an isolated part of the mathematics program. Problem solving in mathematics should involve all the five content areas described in [the] Standards…. Good problems will integrate multiple topics and will involve significant mathematics." (NCTM 2000, p. 52)

cognitive processes characterize the mathematical habit of mind. It uses deduction and proof to establish and organize interconnections among mathematical domains. It looks back at solutions, ever vigilant to find alternative, more effective ways to solve a problem and to see a solution at a glance. It extends and generalizes results to other problems within the same mathematical domain or, by analogy, to problems in other domains. It values conciseness, simplicity, and clarity in solutions—not only in notation but also in results.

It is this habit of mind that this book aims to cultivate, urging teachers to focus attention on the processes of representing and restructuring a problem—even in situations that do not involve real-world applications—while continuing to stress core, unifying concepts. Finally, the book emphasizes the importance of problem solving itself, which encourages reasoning and proof, exploring alternative routes, extending and generalizing results, and communicating results in simple, concise, and elegant ways.