into practice

Chapter 1
Prenumeric Ideas

Big Idea 1

Number is an extension of more basic ideas about relationships between quantities.

Essential Understanding 1*a*

Quantities can be compared without assigning numerical values to them.

Essential Understanding 1*b*

Physical objects are not in themselves quantities. All quantitative comparisons involve selecting particular attributes of objects or materials to compare.

Essential Understanding 1*c*

The relation between one quantity and another quantity can be an equality or inequality relation.

Essential Understanding 1*d*

Two important properties of equality and order relations are conservation and transitivity.

In *Developing Essential Understanding of Number and Numeration for Teaching Mathematics in Prekindergarten–Grade 2* (2010), Dougherty and colleagues noted that understanding the concept of number is much more than learning how to rote count or how to count a discrete number of objects in a set. The concept of number must be first understood by recognizing that counting provides a measure of a quantity or a set of objects. Counting is based on the idea that some identified unit is used to determine how many of that unit is present in the quantity or set being counted (Dougherty and Venenciano 2007).

However, this process of counting assumes that there is an understanding of quantities and relationships that can be present within and across quantities. This understanding is the conceptual aspect of interpreting number and forms the foundation of number development from its inception before prekindergarten and beyond.

Foundational Ideas about Number and Numeration

Using prenumeric comparisons to understand quantities and relationships

The foundational aspects of number begin by first identifying what attributes of objects can be compared and, in essence, measured. These attributes may be the length, area, volume, or mass of an object (Dougherty and Venenciano 2007). By directly comparing the attributes of objects, students can determine equality or inequality. The explicit identification of attributes that can be measured and compared helps students to differentiate among the attributes that are more difficult to compare and measure and may not be mathematical in nature at all, such as color. If you've never thought deeply how young children develop ideas of equivalence, Reflect 1.1 invites you to do so now.

Reflect 1.1

Look at figure 1.1. What ideas about equivalence can come from student observations of the direct comparison?

How might the attributes of volume, area, or mass be used to demonstrate equivalence relationships?

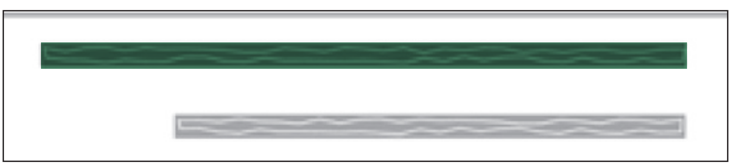

Fig. 1.1. Direct comparison of the length of two straws (Dougherty et al. 2010, p. 10)

For example, in figure 1.1, the lengths of two straws are compared by aligning one end of each straw with the other. By placing them in this manner, students can see that the length of one straw is greater than the length of the other straw OR the length of one straw is less than the length of the other straw.

There are many benefits in having students compare lengths, areas, volumes, and masses as part of the foundational development of number. For young children to determine which number is greater (or lesser or equal) when presented with 3 and 5, they need a mental picture of what these amounts might look like when two quantities are not equal. The mental picture of the quantitative relationship is developed through prior consistent use of physical materials that can be manipulated in multiple ways to model the relationship and then make an interpretation of what is being modeled. This picture is called a *mental residue* (Dougherty 2008), and it provides a way for students to access these comparisons more abstractly when they might only be represented with numerical quantities.

These initial experiences in number can also provide a way to illustrate the importance of the precision of language. Words like *bigger, littler,* and *huger* are often used by young children but are not necessarily indicative of the attribute or quality that was being compared.

Comparing unlike objects

Twenty-four kindergarteners and thirty first-graders were shown the two objects displayed in figure 1.2 and asked to decide which was larger. Consider in Reflect 1.2 how children might react when encountering an “apple to oranges” comparison situation.

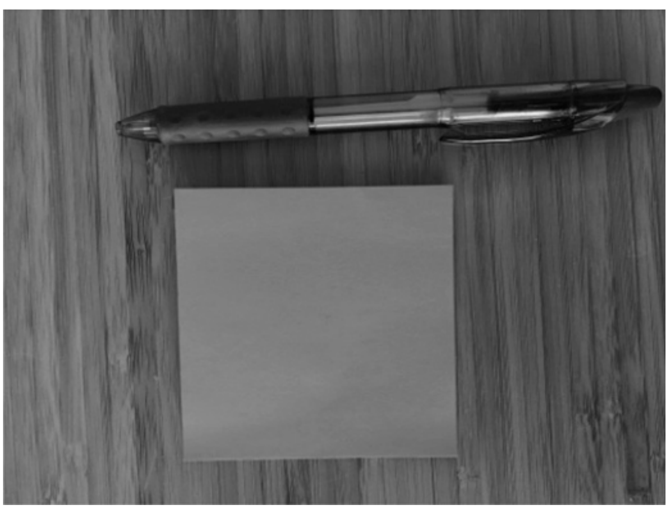

Fig. 1.2. Comparison of size of two objects

Reflect 1.2

Consider the comparison task given above. How might young children respond to the question?

On what attribute(s) might they focus? Why?

All of the kindergartners and 86.6 percent (26 out of 30) of the first graders responded that the blue paper was "bigger." When asked why it was larger or greater than the pen, students had a difficult time explaining why they chose it. Corey, a kindergartener, said, "Well, it just looks bigger 'cause the pen looks littler." Micah, a first grader, said, "Everybody can see it's bigger. Can't you see it?" Their responses are typical of all the students who selected the blue paper.

Conversely, the four first graders who did not select the blue paper had other ideas. June said, "Hmm, it's hard for me." When asked why, she said, "'Cause the pen is longer but the paper is bigger this way [*motioning to the width of the paper*]. So, I don't know." Naveah had a similar explanation: "I don't know what larger means when they [objects] are different. Maybe the pen? But maybe the paper?"

From their responses, you can see that the concepts of *greater than* or *less than* are relevant to the attribute of the quantities being compared and students are not sure how to use those attributes. When we ask questions such as, "Which one is larger?" students may be confused about what they are actually comparing.

Notions of equality

While the modeling of the relationships of greater than, less than, equal, and unequal are important, there are even more powerful ideas that stem from this component of prenumber development. Without using numbers, students can explore some properties of equality at a very early age. Consider the following task adapted from the Measure Up project (Dougherty and Venenciano 2007). Before presenting the task to your students, take time to consider Reflect 1.3. Even without a formal understanding of volume, pre-K and kindergarten children have a sense of equal and unequal amounts and how to make them "the same."

Task: Two Bottles of Water

Sammi had two bottles of water. She wanted to have the same amount of water in both bottles, but she cannot fill the bottles to the top. What could she do?

Figure 1.3. Sammi's water bottle comparison

Reflect 1.3

Consider the task presented above. What properties of equality might students use? What background knowledge might they have that could support ideas they could share?

First-grade students brainstormed ideas about what Sammi could do. To facilitate their discussion, the teacher labeled the water on the left as volume *C* and the water on the right as volume *M*. She identified the volumes of water in this way to help students better communicate their proposed actions and to provide a way to document what they said and the result of their actions. Note that it is the *volume* of water in the bottle that is labeled, and not the bottle itself.

Larry proposed that Sammi could pour other water into volume C until the water level was the same as volume M. Tori thought that another way would be to pour water out from volume M until the height of the water was the same as volume C. The teacher asked how the amount of water added to volume C and the amount of water taken out of volume M were related. The students all agreed that the volume of water removed and the volume of water added would be the same. The teacher then identified the volume removed or added as volume H. She called this quantity of water the *difference*.

This discussion resulted in the students and teacher creating the following equations:

$$C + H = M$$

$$M - H = C$$

Including the symbolic representation (equations) gives students a meaningful way to document the process they used and attaches a language to the physical action.

This type of task is powerful because it encourages students to think about relationships that cannot be easily modeled and understood at such an early age. The big idea that emerges here for children is that when two quantities are unequal, they can be made equal by adding or subtracting the same amount from the respective quantity. This idea is an important underpinning for future number work because it illustrates *difference* in a meaningful way.

Further tasks should focus on maintaining equality, given two quantities that are equal. For example, if two masses on a balance scale are equal, what can be done to add or take away mass on both sides to maintain equality? Students notice that if the same amount is added or taken away to both sides of the balance scale, the masses will remain equal. This important concept about number and equality will support student understanding as children move forward in their development of algebraic thinking.

Also, in this prenumeric stage of development, students can begin to apply the reflexive, symmetric, and transitive properties of equality, which are rarely discussed in the early grades. These properties are directly related to the first-grade standard in the Common Core State Standards for Mathematics (CCSSM; National Governors Association Center for Best Practices and Council of Chief State School Officers 2010).

Common Core State Standards for Mathematics

Related to the Big Idea and Essential Understandings for Chapter 1

Grade 1 (1.OA.7)

7. Understand the meaning of the equal sign, and determine if equations involving addition and subtraction are true or false. *For example, which of the following equations are true and which are false? 6 = 6, 7 = 8 – 1, 5 + 2 = 2 + 5, 4 + 1 = 5 + 2.*

(National Governors Association Center for Best Practices and Council of Chief State School Officers [NGA Center and CCSSO] 2010.)

This standard is often interpreted as students being able to demonstrate that 3 + 2 = 5 but the meaning of the equal sign is much more than a symbol in an equation. In the pre-numeric stage, students recognize that the equal sign indicates that two quantities represent the same amount. The equal sign is not a signal that the "answer" comes next as it is often interpreted.

The three properties of equality (reflexive, symmetric, and transitive) can all be modeled by using physical materials. For example, it is clear that an area, say the area of a tabletop, must be equal to itself. Hence, if the area of a tabletop is represented by area K, then we can symbolize that relationship as $K = K$ (reflexive property).

The symmetric property is easily modeled by having two congruent areas, area F and area P. Students can directly compare the two areas by laying one area on top of the other and seeing whether there are any overlaps. As students describe what they notice about the areas being the same, some will naturally say area F is equal to (the same as) area P while others will say that area P is equal to area F. This relationship should be recorded as both $F = P$ and $P = F$.

The transitive property of equality is initiated by having students compare, for example, three areas such as, area B, area L, and area W. By direct comparison, they determine that $L > B$. The teacher asks students to find out how area B compares to area W without moving the region that represents area B. Students suggest that instead, they can compare area L to area W, and upon doing that, they discover that $L = W$. Because those two areas are equal, students can now surmise that $B < W$ (or $W > B$).

The transitive property of equality is more complex than the reflexive and symmetric properties, but it, too, is accessible by very young students when physical materials are used. Number development without using numbers seems contradictory, but this approach provides opportunities for students to consider the quantitative relationships that are aligned with significant number concepts and use them with greater understanding.

Summary: Learners, Curriculum, Instruction, and Assessment

To effectively teach the mathematical ideas presented in this chapter, teachers must have knowledge of the four components—learners, curriculum, instructional strategies, and assessment—presented in the Introduction. The following sections summarize some key ideas for each of these elements.

Knowledge of learners

Young students often do not have adequate opportunity to learn the foundational aspects of number, particularly those concepts associated with quantitative relationships. They have not yet gained the sophistication in their thinking to solely work with symbolic representations. And, the exclusive use of discrete sets of objects to count and compare quantities does not provide a context that can be generalized to other number systems, such as rational numbers. The use of physical objects combined with a symbolic recording of the relationships between the objects fits with young children's developmental level and affords them the opportunity to work with more complex mathematical ideas and truly focus on conceptual understanding.

Knowledge of curriculum

The understanding of the meaning of the equal sign and its related properties is critical in constructing a strong foundation for students' number knowledge. Building on experiences from pre-K and the early grades, students need to acquire the realization that the equal sign indicates that the two quantities being compared represent the same amount; otherwise, they consider the equal sign as indicating that the answer comes next. The explicitness of instruction that focuses student attention on quantitative reasoning provides more opportunities for students to truly see the meaning of this most important symbol. Additionally, the properties of equality (or inequality) lead to other number understandings, but these properties are often assumed and not overtly taught.

Knowledge of instructional strategies

The careful use of language is a critical consideration when helping students develop understanding of number. Careful and explicit identification of the attributes of objects being compared or measured focuses students' attention and allows them to construct more precise descriptions of the relationships they notice. This attention to units leads, even in the earlier grades, to substantive mathematical discussions where students can justify their reasoning and confront misconceptions.

Knowledge of assessment

Assessments, such as formative assessments generated by the tasks presented, should provide a window into students' thinking and understanding. Children can be asked to interpret physical representations and justify their thinking about a relationship between these representations. Conversely, students can be given a relationship and asked to provide a model that shows it. We can initially be vague in our description of a relationship, such as the book is bigger than the sheet of paper, and then ask students to find better ways of describing and conveying the relationship. Tasks should be varied to promote student flexibility in considering and interpreting quantitative relationships.

Conclusion

In order to help children develop a deep understanding of number, teachers must provide their students with opportunities to interpret and model relationships using physical objects along with symbolic representations. Focusing on precise language within the instructional tasks is particularly important in developing ways of communicating about relationships as well as supporting the structures represented in the models. The deep understanding of these concepts will form the basis for further work with number by developing the concept of a unit and how it supports the comparison of numerical quantities, the focus of chapter 2.