into practice

Chapter 1
From Whole Numbers to Fractions

Essential Understanding 1*c*
The rational numbers allow us to solve problems that are not possible to solve with just whole numbers or integers.

Big Idea 1 for rational numbers, as stated in *Developing Essential Understanding of Rational Numbers for Teaching Mathematics in Grades 3–5* (Barnett-Clarke et al. 2010), captures the foundational notion that extending from whole numbers to rational numbers creates a more powerful and complicated number system. This system allows students to solve problems that they cannot solve with just whole numbers or integers. This is the important insight that students gain when they grasp Essential Understanding 1*c*, the idea on which this chapter focuses.

Working toward Essential Understanding 1*c*

Designing, adapting, or selecting worthwhile assessment tasks, interpreting the responses of individual students, and making instructional decisions based on the results requires specialized knowledge. For example, Reflect 1.1 asks you to consider the mathematical assessment tasks shown in figures 1.1 and 1.2. As you examine these tasks, which were used with students entering grades 3–5, think about the questions for reflection.

Reflect 1.1

What aspects of students' understanding of fractions are being assessed in figures 1.1 and 1.2?

Do you predict that students in grades 3–5 would perform better on the task in figure 1.1 or on the task in figure 1.2? Why?

Do the brownies below have $\frac{1}{2}$ of the brownie shaded? Explain your thinking.

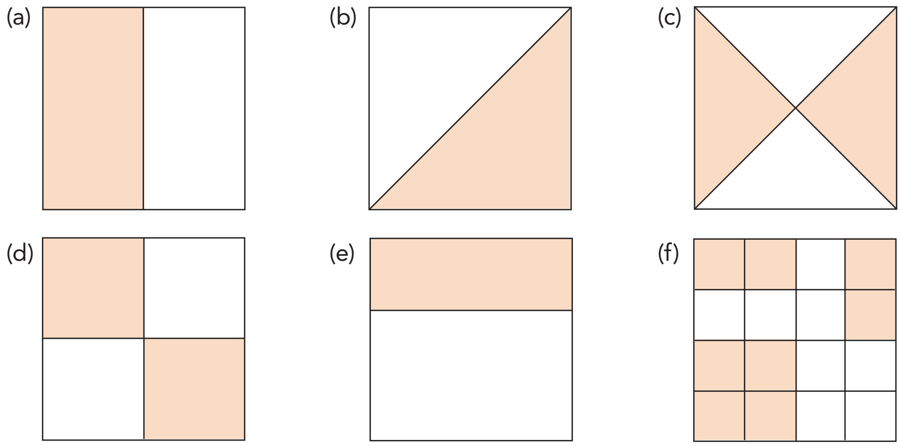


Fig. 1.1. A fraction task exploring the meaning of one-half

Each picture below shows a brownie that is shared. You receive the shaded amount, and your friend receives the unshaded amount.

Which of these brownies make fair shares? Explain your thinking.

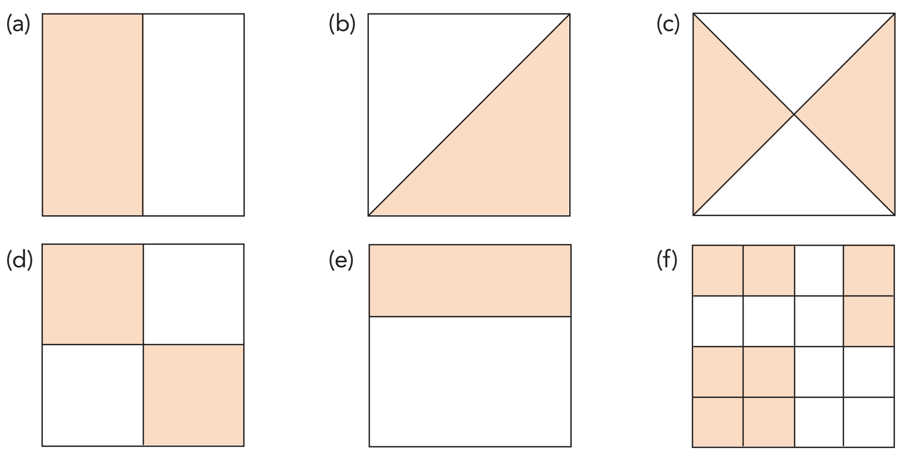


Fig. 1.2. A fraction task exploring the sharing of brownies

The mathematical tasks in figures 1.1 and 1.2 are closely connected, but they also have an important distinction. The task in figure 1.1 requires students to determine whether the shading represents one-half, whereas the task in figure 1.2 involves determining whether the diagrams are equally partitioned, or "fair shared." Battista (2012) argues that a student may understand partitioning but have no concept of the meaning of fractions. To some extent, these two tasks assess similar ideas about the meaning that students give to one-half. In particular, for both tasks, students must consider whether the shaded and nonshaded regions are the same size.

Many times, your students' work can provide you with further insight into their understanding of the mathematical ideas that emerge in a task, allowing you greater insight into their understandings and misconceptions. Consider the work of two students on these two tasks. Figures 1.3 and 1.4 present the work of Miriam, a student who is entering grade 4. Figures 1.5 and 1.6 show the work of Jaden, a student who is entering grade 5. As you look at Miriam's and Jaden's work, consider what it reveals about their understanding or misunderstanding of fractions. Miriam and Jaden completed the tasks in figures 1.1 and 1.2 during the same class period. Moreover, their work is representative of the way in which the majority of children to whom we provided these items responded in grades 3–5.

For the task shown in figure 1.1, Miriam decided that only parts (a) and (b) have $\frac{1}{2}$ of the brownie shaded. For part (c), she stated, "No. Because half is 2 peices and this is 4 peices" (spelling as in original). She provided a similar argument for parts (d) and (f). She claimed that part (e) does not have $\frac{1}{2}$ shaded, "because half is 2 equals peices and these are not equals." Jaden provided similar responses for this task. On part (f), he wrote, "No. Because there are 16 in all and 8 shaded there has to be 1 and 2." He also provided an example that looked like the picture in (a).

Both Miriam and Jaden appear to have what Tall and Vinner (1981) refer to as a "concept image" of $\frac{1}{2}$ as a whole divided into two equal pieces, with one of the parts shaded. A concept image includes the mental images and associated properties and actions. Tall and Vinner explain that a concept image builds over years and changes as students mature and encounter conflicting ideas. Miriam and Jaden have a mental image of $\frac{1}{2}$ that is likely to have developed in their out-of-school experiences with the word "half"—experiences in which "half" is often used to mean a partition into two equal parts. When they think about the representation of the fraction $\frac{1}{2}$, they suppose that the unit must be divided into two, and only two, congruent pieces.

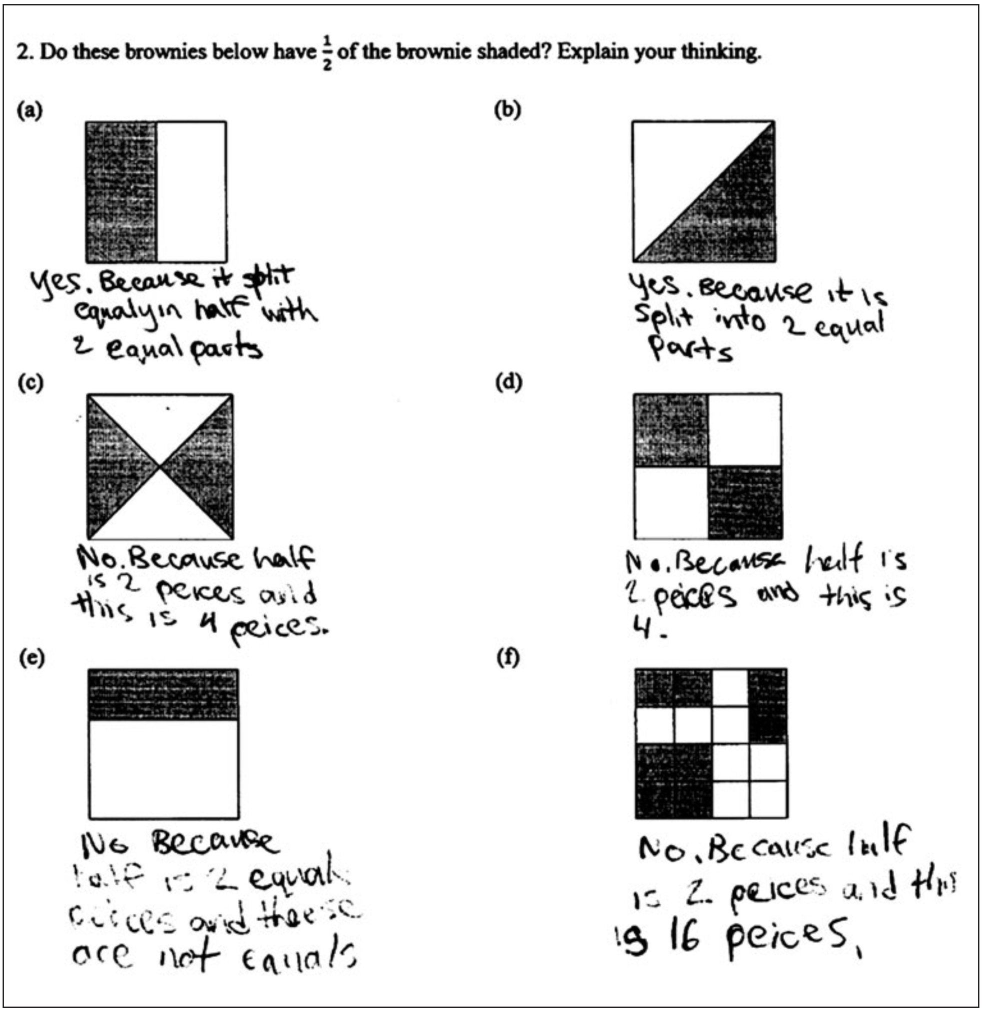


Fig. 1.3. Miriam's responses to the task in figure 1.1

For the task in figure 1.2, Miriam stated that all of the brownies were divided into fair shares except for part (e). For example, when evaluating the partitioning of the brownie in part (c), she noted, "This can [be fair shares] because it has 4 equal

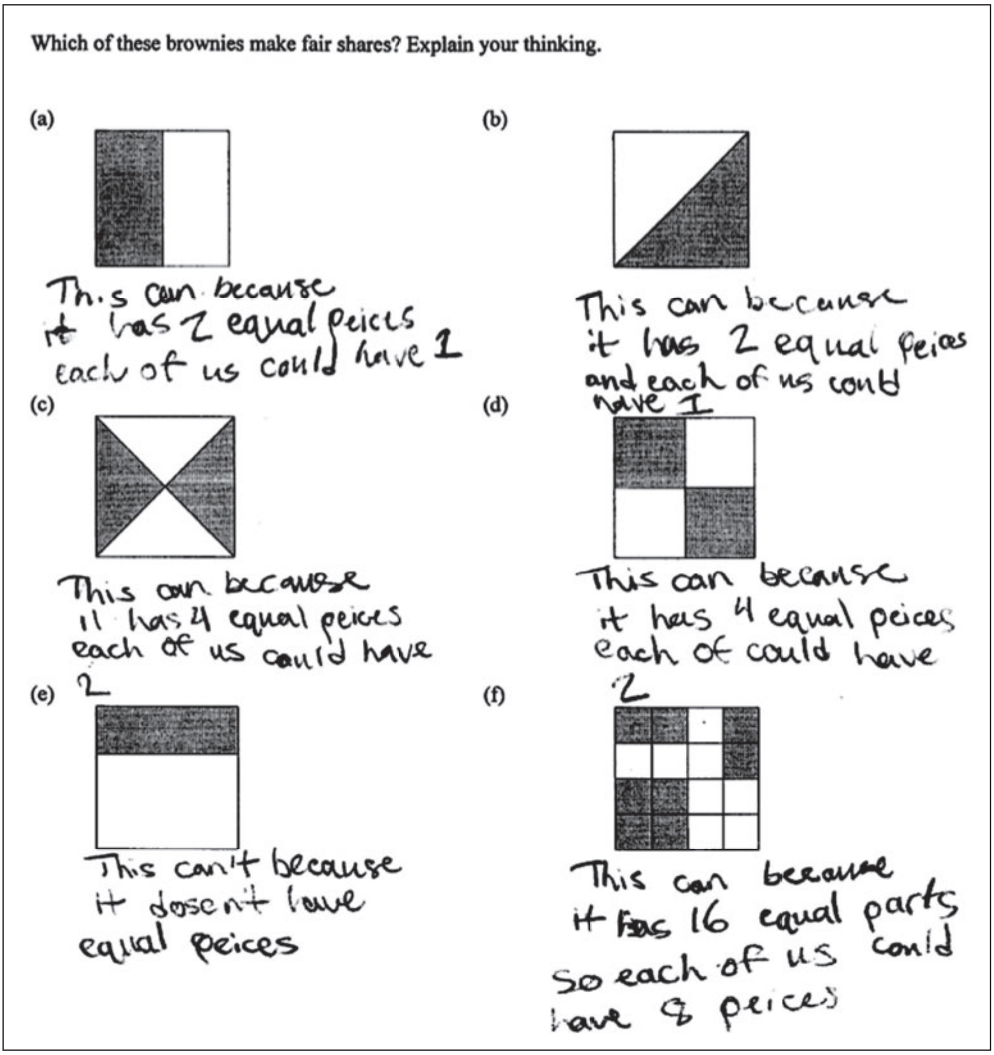


Fig. 1.4. Miriam's responses to the task in figure 1.2

peices each of us could have 2." In the same way, Jaden wrote for part (f), "Fair. We each get 8 pieces." Compare Miriam's and Jaden's responses and then respond to the questions in Reflect 1.2.

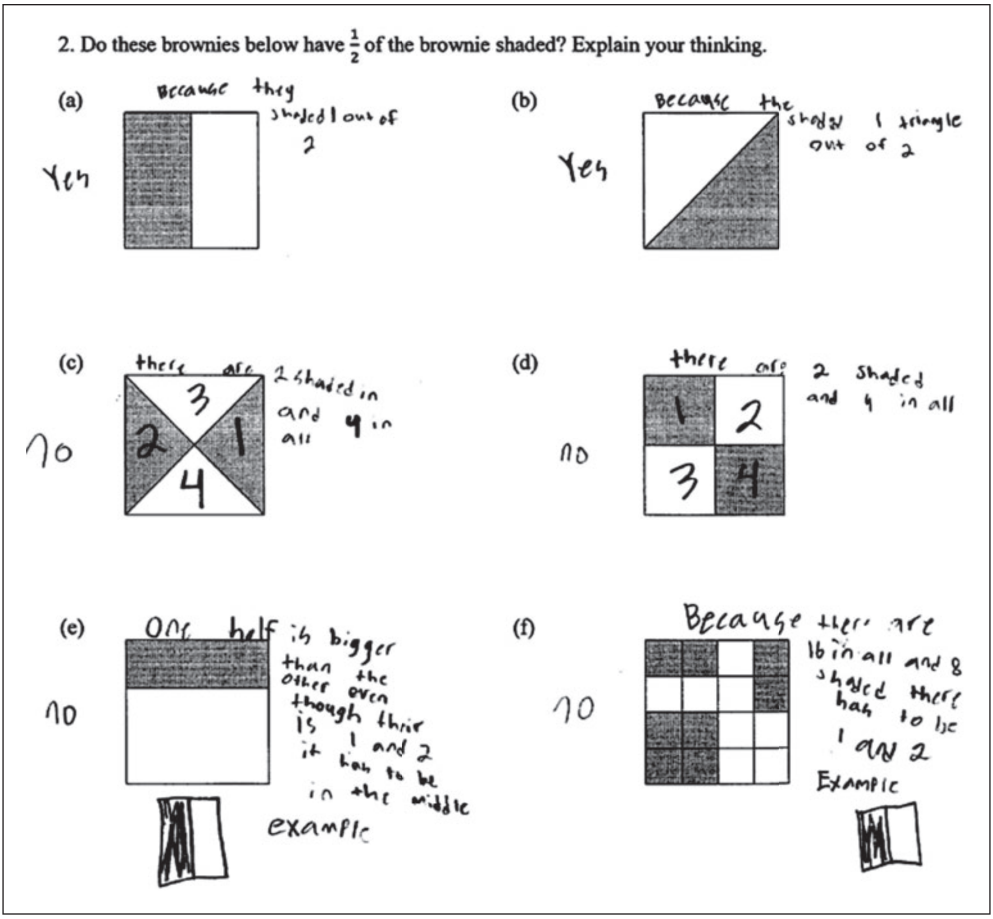


Fig. 1.5. Jaden's responses to the task in figure 1.1

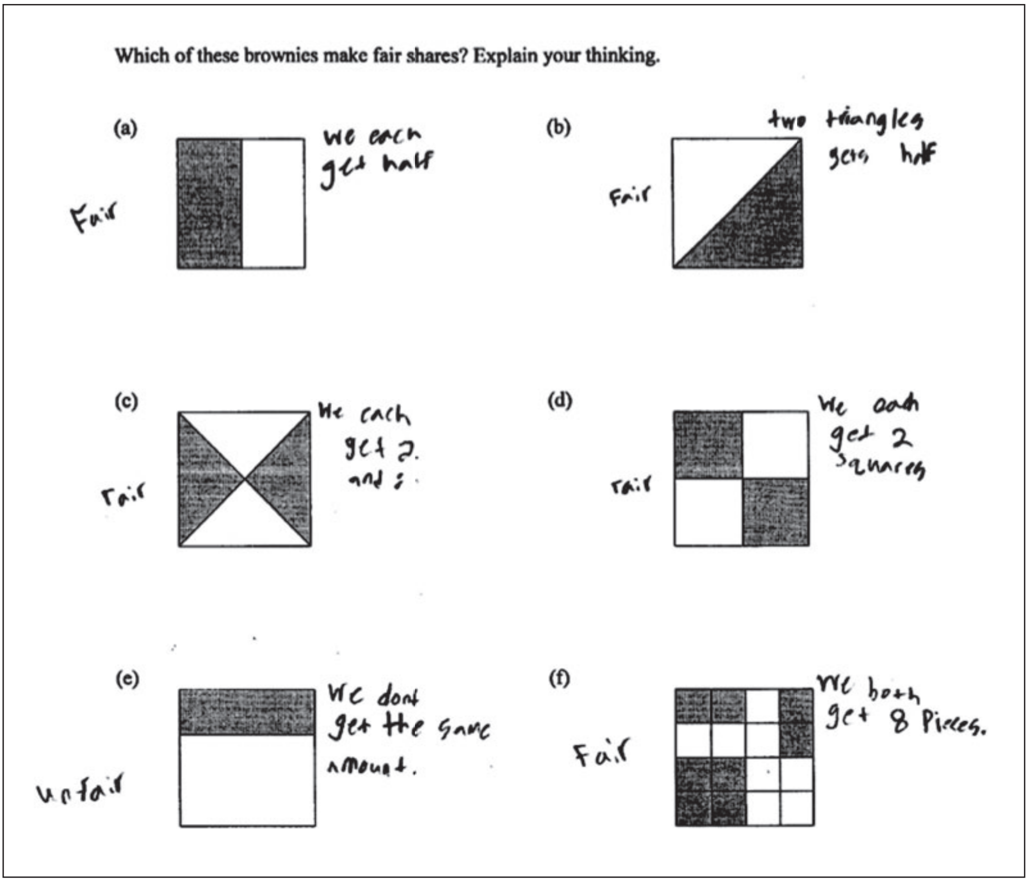


Fig. 1.6. Jaden’s responses to the task in figure 1.2

Reflect 1.2

What do Miriam and Jaden appear to understand about fair shares for two people?

How do their responses on the various parts of the task in figure 1.2 conflict with their responses on the parallel parts of the task in figure 1.1?

Both Miriam and Jaden recognize that making "fair shares" for two people does not require that the whole be split into only two pieces. They demonstrate that they have a strong understanding of "fair sharing" between two people that is disconnected from their understanding of the formal mathematical meaning of $\frac{1}{2}$.

You may wonder whether Miriam and Jaden recognize that the fractions $\frac{1}{2}$, $\frac{2}{4}$, and $\frac{8}{16}$ are equivalent. Another task on the same assessment asked students to list all the fractions they could that are equivalent to $\frac{1}{2}$. Miriam listed $\frac{2}{4}$, $\frac{4}{8}$, $\frac{8}{16}$, $\frac{16}{32}$, and $\frac{32}{64}$, and Jaden listed $\frac{2}{4}$, $\frac{1}{2}$, $\frac{3}{6}$, $\frac{4}{8}$, $\frac{5}{10}$, $\frac{7}{14}$, $\frac{16}{32}$, and $\frac{118}{236}$. It is clear that both students are familiar with a procedure for writing equivalent fractions, but their prior knowledge and previously developed meaning for $\frac{1}{2}$ interfered with their ability to make connections between the symbols and diagrams that represent $\frac{1}{2}$.

It is critical not only to consider students' mathematical conceptions or misconceptions, but also to provide them with possibilities for developing their understanding further. Consider how you might extend the understanding of $\frac{1}{2}$ that Miriam and Jaden currently demonstrate. It appears that they both understand the idea of fair sharing between two people, and they also have an idea of representations of equivalent fractions. One strategy might be to pose one or more of the questions shown in figure 1.7.

Examining how Miriam and Jaden responded to the tasks in figure 1.7 would allow further assessment of their mathematical understanding and suggest ways of extending it. You might anticipate that Miriam and Jaden would respond that the first two brownies have $\frac{2}{4}$ shaded, and the third has $\frac{8}{16}$ shaded. If they used their knowledge that $\frac{2}{4}$ and $\frac{8}{16}$ are equivalent to $\frac{1}{2}$, they might realize that the three models do have $\frac{1}{2}$ shaded. Furthermore, part (b) of the task in figure 1.7 could

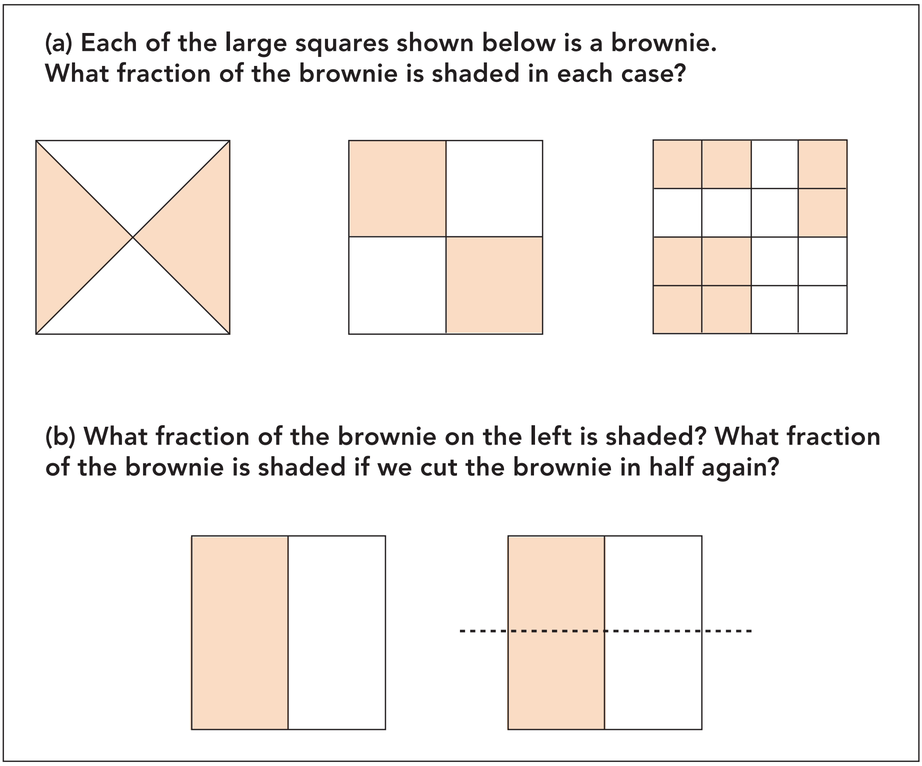


Fig. 1.7. Questions to extend Miriam and Jaden's understanding of $\frac{1}{2}$

formalize this insight for them. They should recognize that the brownie on the left has $\frac{1}{2}$ shaded. By introducing the idea of partitioning the brownie again, this part of the task could help Miriam and Jaden recognize that the size of the shaded region did not change as a result.

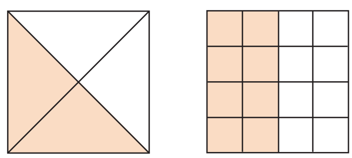
Further, Miriam and Jaden appear to need assistance in connecting their informal ideas about sharing with the meaning of $\frac{1}{2}$. Discussing the students' various responses to the tasks in figures 1.1 and 1.2 in a whole-class session might allow you to make the idea explicit that when shapes are divided into two equal-sized shares, each share represents $\frac{1}{2}$, although the two shares do not have to be the same shape, adjacent to each other, or "split" with just one cut, as is the brownie on the left in part (b) of the task in figure 1.7.

Next, consider the work of Henri, a student who has just completed the fourth grade. In his response to the task in figure 1.1, Henri stated that only (a) and (b) show brownies with $\frac{1}{2}$ of the brownie shaded. He wrote, "The answer is a or b because $\frac{1}{2}$ of a something is like you have candy and you give one to yourself and to your friend. That's $\frac{1}{2}$." He said that the shaded portions of the brownies in part (c) and part (d) did not represent $\frac{1}{2}$, offering the following explanation: "No, it is not a $\frac{1}{2}$ because they are separated. But look at a and b. They have to be together like a or b." In evaluating the shaded portions of the brownie in part (f), he wrote, "They're not $\frac{1}{2}$ because look at f. It doesn't even look close to be a $\frac{1}{2}$." In completing the task in figure 1.2, Henri stated that all the shares of brownies were fair except for those shown in part (e).

Questions to gather information from Henri

1. Why do the shaded pieces have to touch?

2. Is $\frac{1}{2}$ of the brownie shaded in the following diagrams? Why or why not?



Questions to help move Henri forward

3. If you rearranged the shaded pieces in the figures above, would they still represent $\frac{1}{2}$? Why or why not?

4. John looked at the brownie on the left and said, "This has $\frac{1}{2}$ shaded. If you flip one of the shaded pieces over, it adds to half." Do you agree with John? Use John's strategy to determine if the other brownies have $\frac{1}{2}$ shaded.

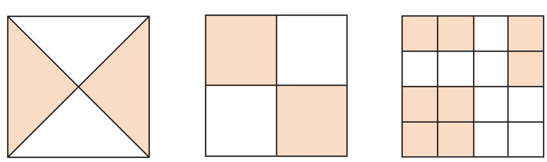


Fig. 1.8. Questions for Henri

What does Henri appear to understand about the meaning of $\frac{1}{2}$? Clearly, something is missing, given his claim that the pieces have to be "together." Using a questioning strategy may be helpful. You can use questions to gather information about what your students understand. Moreover, effective questioning can help move them forward. Think of some questions that would give you more information about what Henri understands about $\frac{1}{2}$, along with some questions that would help him gain a better understanding of $\frac{1}{2}$. Figure 1.8 presents several such questions.

John, who has completed the fourth grade, stated that all the diagrams in figure 1.1 show $\frac{1}{2}$ except for part (e). In response to part (c), he wrote, "Yes, if you flip one of the shaded pieces over it adds to half." He used similar reasoning on part (f) and indicated that the two shaded squares in the upper right corner could fill in the gap on the left.

It may be that the concept images that Henri and John hold for $\frac{1}{2}$ allow a figure to be partitioned into more than two pieces, but Henri's requires that the shaded pieces be touching. John's, by contrast, permits mental rearrangement of the pieces and accepts that the shaded pieces may be separated from one another.

From what John has stated, it is not clear whether or not he believes that the pieces need to be the same shape. That is to say that as his teacher, you would not necessarily know how he might complete the task in figure 1.9, which shows a diagram in which the shaded pieces do not fit conveniently together.

The following picture shows a brownie that is shaded. Is $\frac{1}{2}$ of the brownie shaded? Explain your thinking.

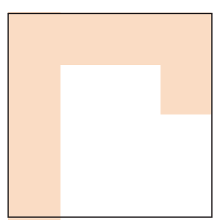


Fig. 1.9. An extension of task in figure 1.1

Figures 1.10 and 1.11 show assessment tasks for students entering grades 4–6. Reflect 1.3 poses questions for you to consider as you examine and compare these tasks with the previous ones in figures 1.1 and 1.2.

Reflect 1.3

Examine the tasks for students entering grades 4–6 shown in figures 1.10 and 1.11. What aspects of students' understanding of fractions are being assessed in these two tasks?

Do you think that students who have completed grades 3–5 are likely to perform better on the task in figure 1.10 or on that in figure 1.11? Why?

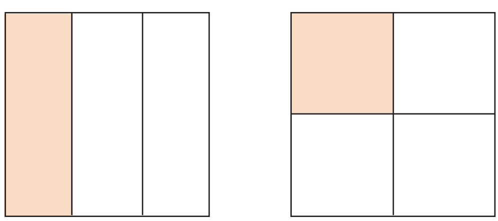
Suppose that a student in your class said the following:

$\frac{1}{4}$ is more than $\frac{1}{3}$ because 4 is more than 3.

Do you agree with this student? Explain your thinking.

Fig. 1.10. A fraction task about $\frac{1}{4}$ and $\frac{1}{3}$

Look at the shaded part of the two brownies below. Circle the brownie that has a larger amount of a brownie shaded.



Explain why you think this is a larger amount of brownie.

Fig. 1.11. A fraction task about a larger amount of brownie

The mathematical tasks in figures 1.10 and 1.11 are closely connected, but they also have an important distinction. The first task asks students to evaluate another student's reasoning that is based on a common misconception. The second task, in contrast, requires students to determine which diagram displays a larger shaded portion of the whole. To an extent, these two tasks assess similar ideas about comparing fractions. Whereas the first task focuses students on the symbolic forms $\frac{1}{4}$ and $\frac{1}{3}$, the second uses diagrams of $\frac{1}{4}$ and $\frac{1}{3}$.

We provided these tasks to students entering grades 4–6 during the same class period. On one hand, in response to the task in figure 1.10, some students agreed while others disagreed, for various reasons. On the other hand, almost every student circled the brownie on the left in the task in figure 1.11, signifying that a larger amount was shaded. However, the reasons that students gave for this choice differed. Consider, for example, work on these tasks by Ryan (entering fourth grade) and Amber (entering sixth grade) and what it reveals about their understanding or misunderstanding of fractions. Ryan's work is shown in figures 1.12 and 1.13.

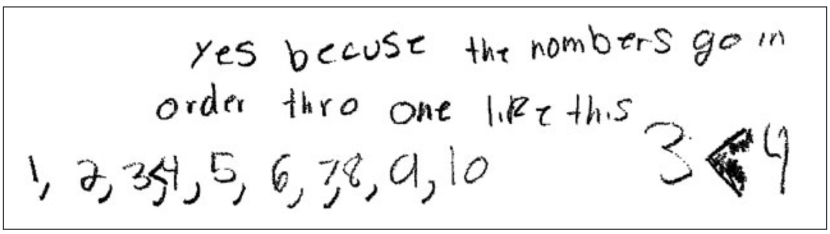


Fig. 1.12. Ryan's response to the task in figure 1.10

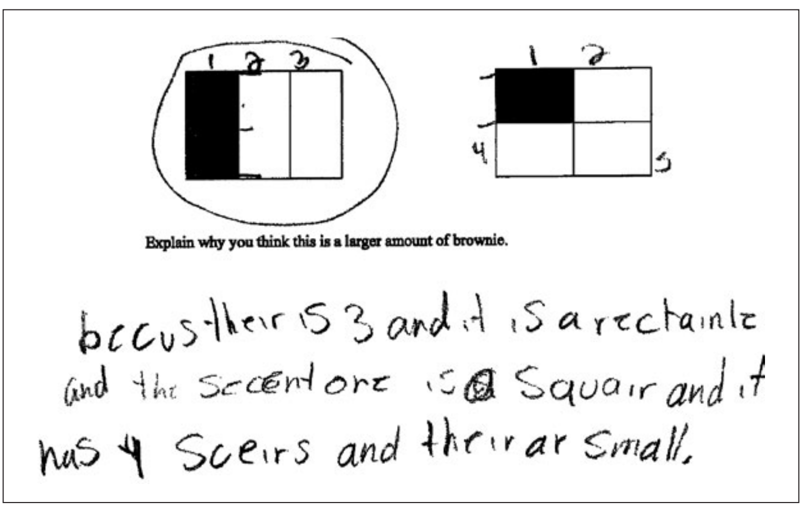


Fig. 1.13. Ryan's response to the task in figure 1.11

As Ryan's teacher, you could ask Ryan to write a fraction for the shaded pieces of brownies shown in figure 1.11. This could help him connect the symbols $\frac{1}{4}$ and $\frac{1}{3}$ to the pictures used in the tasks, as well as highlight the role of the denominator in a fraction. The connection between pictorial and symbolic representation could also emphasize that the ordering for whole numbers does not directly apply to the case presented in figure 1.10.

Amber's responses to both of these tasks are shown in figures 1.14 and 1.15. Use the questions in Reflect 1.4 to guide you in assessing her understanding and misunderstanding.

Reflect 1.4

What does Amber appear to understand about the fractions $\frac{1}{3}$ and $\frac{1}{4}$?

How does her understanding compare with Ryan's?

What questions could you ask Amber that might help her clarify or strengthen her understanding of the meaning of $\frac{1}{3}$ and $\frac{1}{4}$?

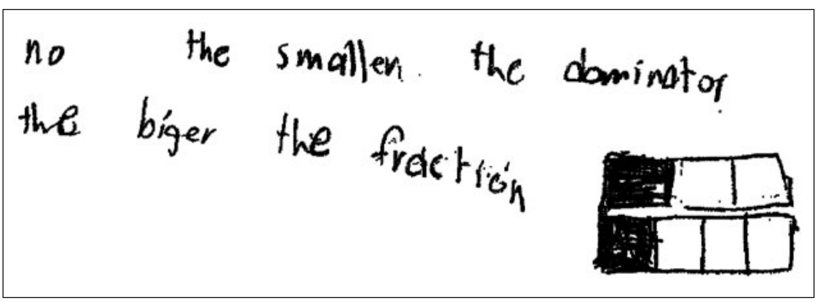


Fig. 1.14. Amber's responses to the task in figure 1.10

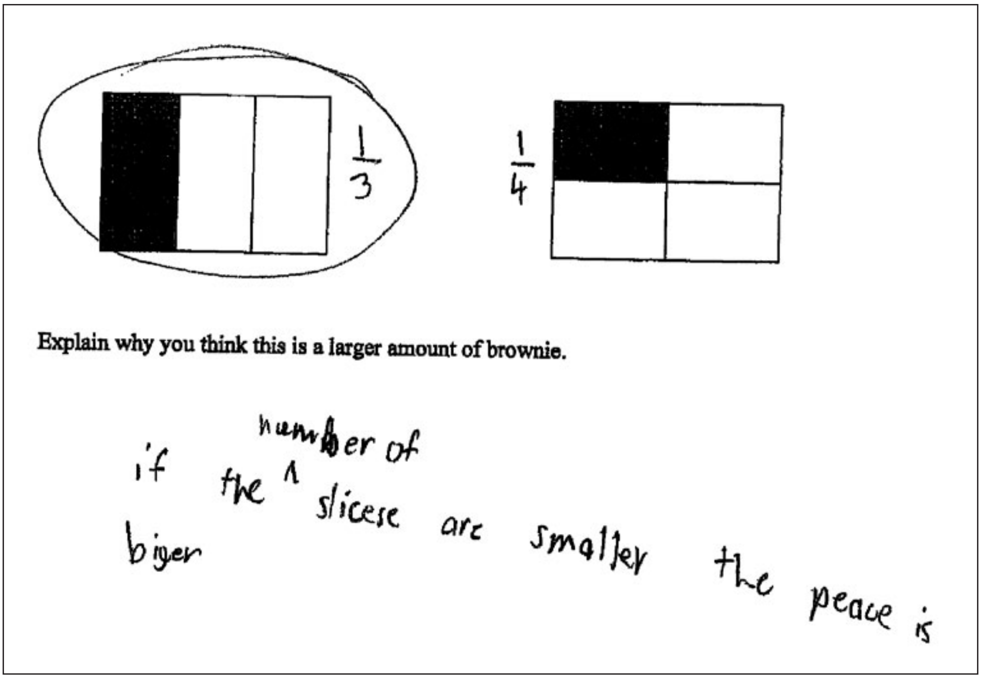


Fig. 1.15. Amber's responses to the task in figure 1.11

Amber's work indicates that Amber understands the meaning of both the pictorial and symbolic forms of $\frac{1}{3}$ and $\frac{1}{4}$ and demonstrates that she has made connections between the two forms. Without prompting, in her response to the task in figure 1.10, she draws a pictorial model of $\frac{1}{4}$ and $\frac{1}{3}$ to explain why $\frac{1}{4}$ is smaller than $\frac{1}{3}$ (see fig. 1.14). In her response to the task in figure 1.11, she correctly labels the models (see fig. 1.15).

In response to the task in figure 1.10, Amber wrote, "the smaller the d[en]ominator, the big[g]er the fraction." This reasoning works well here, because the numerators are the same. But what would you expect if you asked Amber to compare fractions with different numerators, such as $\frac{2}{3}$ and $\frac{3}{4}$, or different denominators, like $\frac{5}{8}$ and $\frac{3}{5}$? Amber would probably realize that $\frac{5}{8}$ was made of smaller pieces, but her response to the task in figure 1.10 does not make clear how she would use the numerator to answer this question.

Summarizing Pedagogical Content Knowledge to Support Essential Understanding 1c

Teaching the mathematical ideas in this chapter requires specialized knowledge related to the four components presented in the Introduction: learners, curriculum, instructional strategies, and assessment. The four sections that follow summarize some examples of these specialized knowledge bases in relation to Essential Understanding 1c. Although we separate them to highlight their importance, we also recognize that they are connected and support one another.

Knowledge of learners

Battista (2012, p. 1) states an important idea that has been illustrated by the students' work included in this chapter:

> Before students can understand fractions, they must understand partitioning. To partition a whole is to divide it into equal portions, like dividing a pizza equally among four people. Being able to partition, however, does not mean that one understands fractions.

Battista goes on to outline eight levels of sophistication in students' reasoning about fractions. At level 0, students may understand equal partitioning but not yet have an understanding of fractions. At level 1, students recognize only familiar diagrams of fractions, whereas at level 2, they understand fractions as numbers that involve counting all parts and shaded parts. As evidenced by their work on the task in figure 1.2 (see figs. 1.4 and 1.6), Miriam and Jaden appear to have a strong understanding of partitioning or fair shares, and, as their work on the later assessment item demonstrates, they can identify some examples of fractions that are equivalent to $\frac{1}{2}$. They also demonstrated their ability to recognize common images of $\frac{1}{2}$ in their responses to parts (a) and (b) in figure 1.1. Yet, they share a misconception that $\frac{1}{2}$ can be depicted only in diagrams that include two pieces that are congruent. It is critical that as a teacher you develop knowledge specific to learners' mathematical understandings and misconceptions to design instruction, assessment, and curricular tasks that will enhance their understanding.

Knowledge of curriculum

Barnett-Clarke and colleagues (2010, p. 19) discuss the importance of recognizing three features of rational numbers to understand them:

> Rational numbers can be used in many different contexts, their interpretation can change depending on the context, and defining the unit is key to the interpretation (Behr et al. 1983; Carraher 1992, 1996; Kieren 1992; Lamon 2007).

These three ideas are crucial for students to know and understand, and consequently they should influence the selection of curricular tasks that use different contexts (for example, brownies, people, string), different language (the tasks in figs. 1.1 and 1.2, for instance, use the same diagrams but introduce the task in different words), and different units. As you think about which contexts, language, and units to use to explore and extend your students' mathematical ideas, it is also important to consider which ones are effective for which specific purposes. Fosnot and Dolk (2002, p. 28) argue persuasively for the use of meaningful contexts:

> When the context is a good one, the children talk about the situation. When a problem is camouflaged school mathematics, children talk about numbers abstractly; they lose sight of the problem as they try to figure out what the teacher wants.

You also need to carefully select tasks that will not perpetuate misconceptions, such as any faulty concept images that your students may have. For example, you may have students who assume that parts must be congruent (that is, the same size and shape) on the basis of the examples that they have experienced. Students should recognize that parts may be the same size without being the same shape.

Knowledge of instructional strategies

Teachers have a multitude of instructional strategies to draw on in developing students' understanding of fractions. The preceding discussion has suggested a few examples, which this section highlights.

Empson (1995) writes, "Most young children have experienced equal sharing and have quite a bit of intuitive knowledge of those situations" (p. 110). Capitalizing on those prior experiences and knowledge is an important instructional strategy. Empson also argues that the context of equal sharing can help children understand that pieces need to be the same size. The cases of Miriam and Jaden, whose performance on the brownie sharing task in figure 1.2 was stronger than on the more abstract task in figure 1.1, illustrate the truth of these observations.

A second strategy is to ask students to interpret diagrams such as those in the task in figure 1.1, but, more important, to create tasks that will encourage comparisons and connections. Fosnot and Dolk (2002) suggest that students should wonder, notice patterns, and ask questions such as “Why?” and “What if?” Facilitating a whole-class discussion in which your students compare and contrast tasks like those in figures 1.1 and 1.2 will generate interesting ideas and questions from them. This strategy will help students connect their knowledge about sharing with the meanings of fractions.

Another instructional strategy illustrated in this chapter’s examples is to engage students in analyzing responses from fictitious students to a particular task. The task in figure 1.10 illustrates this approach: students must agree or disagree with the conjecture from a fictitious student in their class. Again, this strategy can lead to interesting whole-class discussions. In this case, the fictitious student’s conjecture involves an erroneous connection between whole numbers and fractions, implying that comparing fractions is similar to comparing whole numbers. This is a misconception that teachers in grades 3–5 frequently encounter among students. Rather than introducing this misconception through the work of an actual student in the classroom, the “fictitious student” strategy permits highlighting this idea and initiating a discussion through the use of the work of “other students.” Students can then justify their reasoning for their agreement or disagreement.

Knowledge of assessment

As noted in the Introduction, Wiliam (2007) emphasizes the importance of selecting tasks that provide opportunities for students to demonstrate their thinking and for teachers to make instructional decisions. The work that students generated in response to the tasks presented in this chapter provides evidence about their understanding and misunderstanding of fractions. The tasks ask students to explain their thinking or justify a response to give their teachers more information to use in interpreting their thinking and making determinations about future instruction.

Another assessment strategy that the examples in this chapter have illustrated is the questioning that can and should follow the initial assessment. Miriam’s and Jaden’s teacher made some decisions about what questions and tasks to pose next. This strategy may serve the purpose of enabling the teacher to collect additional assessment information, positioning the teacher to move the students’ understanding forward or to challenge an identified misconception.

Conclusion

The children’s work highlighted in this chapter has illustrated knowledge that teachers need to design, adapt, and select worthwhile curricular and assessment tasks related to fractions; interpret responses from specific students; and make instructional decisions based on the results. The chapter has discussed some essential understandings that children need to develop—as well as misconceptions that teachers need to challenge—to enable them to be successful with fractions. As teachers, we need to assess and build on children's understanding of partitioning, fair shares, and the meaning of fractions. Further, we need to determine whether our students have any faulty concept images of fractions. This work requires not only careful selection of tasks but also effective questions, as discussed in the Introduction. This effort will help students lay a foundation for developing the critical understanding of unit that we discuss in the next chapter.