Mathematical Tasks as a Framework for Reflection: From Research to Practice

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According to the *Professional Standards for Teaching Mathematics* (NCTM 1991), a primary factor in teachers' professional growth is the extent to which they "reflect on learning and teaching individually and with colleagues" (p. 168). Reflecting on their classroom experiences is a way to make teachers aware of how they teach (Hart et al. 1992) and how their students are thriving within the learning environment that has been provided. Although all teachers think informally about their classroom experiences, cultivating a habit of systematic and deliberate reflection may hold the key to improving one's teaching as well as to sustaining lifelong professional development.

One of the most difficult aspects of reflection is figuring out on what to focus (Hart et al. 1992). In our five years of experience with middle school teachers in the QUASAR Project (see Silver and Stein [1996]), we have seen how focusing on mathematical tasks and their phases of classroom use can assist teachers in the reflection process. QUASAR (Quantitative Understanding: Amplifying Student Achievement and Reasoning) is a national reform project aimed at fostering and studying the development and implementation of enhanced mathematics instructional programs in six urban middle schools. It is housed at the Learning Research and Development Center at the University of Pittsburgh and is directed by Edward A. Silver. In this article, we describe a framework for reflection based on the mathematical tasks used during classroom instruction and the ways in which it has been used by teachers. In the framework, a task is defined as a segment of classroom activity that is devoted to the development of a particular mathematical idea. A task can involve several related problems or extended work, up to an entire class period, on a single complex problem. Defined in this way, most tasks are from twenty to thirty minutes long.

Focusing on Mathematical Tasks

Our focus on mathematical tasks is built on the idea that the tasks used in the classroom form the basis for students' learning (Doyle 1988). Tasks that ask students to perform a memorized procedure in a routine manner lead to one type of opportunity for student thinking; tasks that require students to think conceptually and that stimulate students to make connections lead to a different set of opportunities for student thinking. The day-in and day-out cumulative effect of classroom-based tasks leads to the development of students' implicit ideas about the nature of mathematics—about whether mathematics is something about which they can personally make sense and about how long and how hard they should have to work to do so.

The example shown in **figure 1** illustrates four ways in which the task of determining the relationships between fractions and their decimal and percent equivalents can be approached, each of which places a different kind of cognitive demand on students. As shown on the left side of the figure, lower-level approaches to the task consist of memorizing the equivalent forms of specific fractional quantities, for example, 1/2 = 0.5 = 50%, or performing conversions of fractions to percents or decimals with standard conversion algorithms in the absence of additional context or meaning, for example, converting the fraction 3/8 to the decimal 0.375 by dividing the numerator by the denominator or changing 0.375 to a percent by moving the decimal point two places to the right. When these lower-level approaches are used, students typically work many similar problems, twenty or more, within a given task.

A different approach to this same task—one that presents higher-level demands—might also use procedures, but in a way that *builds connections to the mathematical meanings* of fractions, decimals, and percents. One way to build such connections is to encourage students to grapple with the underlying concept of part-whole relationships by working with a 10×10 grid. As shown on the upper-right-hand side of **figure 1**, students might be asked to use the grid to illustrate how 0.6 represents the same quantity as the fraction 3/5,

Lower-Level Demands

Memorization

What are the decimal and percent equivalents for the fractions 1/2 and 1/4?

Expected student response:

$$\frac{1}{2} = 0.5 = 50\% \qquad \frac{1}{4} = 0.25 = 25\%$$

Procedures without connections

Convert the fraction 3/8 to a decimal and a percent.

Expected student response:

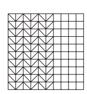
Fraction	Decimal	Percent
$\frac{3}{8}$	$\begin{array}{r} 0.375 \\ 8\overline{)3.000} \\ \underline{24} \\ 60 \\ \underline{56} \\ 40 \\ \underline{40} \end{array}$	0.375 = 37.5%

Higher-Level Demands

Procedures with connections

Using a 10 × 10 grid, identify the decimal and percent equivalents of 3/5.

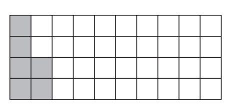
Expected student response:

Pictorial	Fraction	Decimal	Percent
	$\frac{60}{100} = \frac{3}{5}$	$\frac{60}{100} = 0.60$	0.60 = 60%

Doing mathematics

Shade 6 small squares in a 4 × 10 rectangle. Using the rectangle, explain how to determine each of the following: (a) the percent of area that is shaded, (b) the decimal part of area that is shaded, and (c) the fractional part of area that is shaded.

One possible student response:



(a) One column will be 10%, since there are 10 columns. So four squares is 10%. Then 2 squares is half a column and half of 10%, which is 5%. So the 6 shaded blocks equal 10% plus 5%, or 15%.

(b) One column will be 0.10, since there are 10 columns. The second column has only 2 squares shaded, so that would be one-half of 0.10, which is 0.05. So the 6 shaded blocks equal 0.1 plus 0.05, which equals 0.15.

(c) Six shaded squares out of 40 squares is 6/40, which reduces to 3/20.

Fig. 1. Lower-level versus higher-level approaches to the task of determining the relationships among different representations of fractional quantities

or 60 percent. Students might also be asked to record their results in a chart containing the decimal, fraction, percent, and pictorial representations, thereby allowing them to make connections among the various representations and to attach meaning to their work by referring to the pictorial representation of the quantity every step of the way.

Another high-level approach to the task—a *doing mathematics* approach—could entail asking students to explore the relationships among the various ways of representing fractional quantities. Students would not, at least initially, be given the conventional conversion procedures. They might once again use grids; but this time, grids of varying sizes, not just 10 × 10, would be used. For example, students could be asked to shade six squares of a 4×10 rectangle and, after doing so, might be asked to represent the shaded area as a percent, a decimal, and a fraction. When students use the visual diagram to solve this problem, they are challenged to apply their understandings of the fraction, decimal, and percent concepts in novel ways. For example, once a student has shaded the six squares, he or she must determine how the six squares relate to the total number of squares in the rectangle. In **figure 1**, we see an example of a student's response, which illustrates the kind of mathematical reasoning used to come up with an answer that makes sense and that can be justified. In contrast to the lower-level approaches discussed earlier, when "procedures with connections" or "doing mathematics" approaches are used, students typically perform far fewer problems, sometimes as few as two or three within a given task.

Focusing on Task Phases

As shown in **figure 2**, the Mathematics Task Framework distinguishes three phases through which tasks pass: first, as they appear in curricular or instructional materials on the printed pages of textbooks, ancillary materials, and so on; next, as they are set up or announced by the teacher; and finally, as they are actually implemented by students in the classroom—in other words, the way in which students actually go about working on the task. All these, but especially the implementation phase, are viewed as important influences on what students actually learn, illustrated by the triangle in **figure 2**.

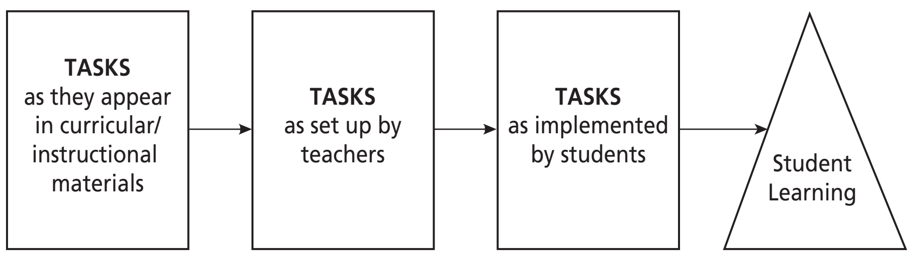


Fig. 2. The Mathematics Tasks Framework

The nature of tasks often changes as they pass from one phase to another. In other words, the task that appears in the curricular or instructional materials is not always identical to the task that is set up by the teacher; in turn, it is not always exactly the same task that the students actually do. The evolution of tasks as they pass from the *setup* to the *implementation* phase has been closely examined in QUASAR classrooms (see Stein, Grover, and Henningsen [1996]). High-level tasks were sometimes found to be implemented in such a way that students thought and reasoned in complex and meaningful ways. Sometimes, however, tasks that were set up to place high levels of cognitive demand on students' thinking changed dramatically in terms of how students actually went about working on them. Recognizing this phenomenon can be a fruitful focus for reflection.

Applying the framework: The case of Ms. Bradford

In our work, we have seen how the Mathematics Tasks Framework can give teachers insight into the evolution of their own lessons. After teachers learned about the framework, they began to use it as a lens for reflecting on their own instruction and as a shared language for discussing instruction with their colleagues.

Consider, for example, the case of Theresa Bradford (a pseudonym), a teacher with whom we have worked for several years. Theresa routinely selected high-level tasks that afforded her students opportunities to explore mathematical ideas and concepts in meaningful ways. One such task was the "tape-roll toss." In this problem, students were asked to design a game for a fund-raising carnival at their school and were given some initial directions. A player would toss rolls of masking tape onto a game board. If the tape roll landed completely in a shape without touching any lines, the player would win a T-shirt. If the tape roll touched any line on the game board, he or she would lose. With the cost of the game at three tosses for \$1, and the cost of T-shirt prizes at \$4 each, students were left to decide how many shapes should be on the game board and what size they should be so that the fund-raiser would make a profit.

Theresa supplied a variety of materials—grid paper of various sizes, yardsticks, rulers, tape rolls, markers, scissors—for constructing the game boards, and her students worked for the entire class period, designing and testing them. Although Theresa had planned this lesson to be an exercise in mathematical exploration, one that would stretch students' thinking and allow them to come up with several possible solutions and corresponding justifications, the actual implementation was disappointing. Students appeared overwhelmed by the number of choices that needed to be made and the need to impose structure on the task. After the first twenty minutes, Theresa ended up guiding students in the design of their game boards. She found herself asking questions and then shortly thereafter answering them for the students. Not surprisingly, the game boards ended up looking more alike than different!

Several months after using tape-roll toss, Theresa attended a conference at which the Mathematics Tasks Framework was presented. When the speaker began to explain that tasks are not always implemented as intended, Theresa immediately turned to a researcher-colleague sitting behind her and eagerly announced, "That's what happened to tape-roll toss!" On further reflection and discussion, Theresa realized that the students' lack of prior experience with open-ended tasks made them uncomfortable when they were presented with a task that they did not immediately know how to solve. Their inclination—fortified by years of school experience—was to wait until someone, usually the teacher, *showed* them how to do it. Theresa was drawn unwittingly into this scenario because she was most comfortable with it. Was she not supposed to be the "sage on the stage"—the one with all the answers?

Before her acquaintance with the Mathematics Task Framework, Theresa had the general feeling that the activity could have been better, but she was not able to pinpoint the source of the difficulty. The framework gave her a language for describing events that had occurred in her classroom and for understanding why things may not have worked out as she had envisioned that they would.

Using the Framework for Reflection

The framework proved to be a powerful tool for Theresa and her colleagues at Ridgeway Middle School as they tried to introduce more cognitively complex and meaningful tasks to their students. To share ideas and give one another moral support, during the 1994–1995 school year they decided to meet once each month. During these meetings, most of the teachers simply described the lessons with which they wanted help; a few, however, had begun to share videotapes of their teaching.

The case of Ron Castleman: Part 1

At a meeting in early spring, Ron Castleman (a pseudonym), a seventh-grade teacher at Ridgeway, decided to share a videotape of a lesson in which he had set up the "doing mathematics" task shown in **figure 1**. Although students successfully solved the problem, he was left with the feeling that it all happened too quickly. On the basis of his conversations with Theresa, he had the sense that the Mathematics Task Framework might be a useful way to think about the lesson. He asked his colleagues for help in applying the framework to his lesson.

The tape began with Ron's setting up the task for his students. He carefully explained that he wanted them to shade six squares of the 10×4 rectangular grid and then to figure the percent, decimal, and fraction, in that order, of the rectangle that was shaded. As the task-implementation phase began, Ron reminded his students that they would need to explain whatever answer they came up with. Students became restless after only a short while of attempting to figure what percent of 40 the six shaded squares represented. Hands shot into the air as students began to realize that the algorithms that they had learned were useless. As Ron traveled from desk to desk, he was confronted with the same refrain, "How do you do this?"

For a short time, Ron turned the question back to the students, telling them that it was their job to figure it out. As the students became increasingly anxious about their lack of progress, however, Ron began to tell them that they should try starting with the fraction first. Most students had no difficulty figuring that six shaded squares would be 6/40. Then they found the decimal by dividing 6 by 40 to get 0.15 and then turned to the "tried and true" method of moving the decimal point two places to the right to convert from 0.15 to 15 percent. What had started as a completely intractable problem was solved in a matter of minutes!

When Ron asked for feedback on the lesson, one of his colleagues noted that by moving through the problem in this way, the students had completely divorced their thinking from the diagram and consequently from the meanings of decimal, percent, and fraction. Another teacher found it curious that the students showed no inclination to even check the plausibility of the answers that they came up with against the diagram. After more discussion, the teachers agreed that by succumbing to the students' requests of "how to do it," Ron had reduced or eliminated the challenging, sense-making aspects of the task, thereby robbing students of the opportunity to develop thinking and reasoning skills and meaningful mathematical understandings. Using the Mathematics Task Framework, the teachers decided that the task had been set up at a high level but had been implemented at a much lower level; in the end, students were left with a task that required only that they apply a procedure without making any connection to the underlying meaning.

The case of Ron Castleman: Part 2

Ron appreciated his colleagues' comments. Although he might initially have wanted to hear, "It was a great lesson," he realized that such feedback really would not have been very helpful. Before the teacher meeting, he had not thought about what impact his actions were having on students' learning. When he reviewed the student work later, he realized that he saw no evidence of students' actually having paid attention to the diagram. After reflecting on the lesson with his colleagues, he then realized that he had contributed to their departure from the diagram by stepping in and suggesting that they start with the fraction.

Later that same week, Ron set up the same "doing mathematics" task in a different class. This time he was clearer in his own mind about the kind of student thinking he wanted to encourage during the task-implementation phase. To keep the task at a high level, he wanted to help his students come up with their own ways of solving the task by using the diagram as opposed to relying on learned procedures. If his students proposed and tested strategies based on the diagram, he reasoned, meaningful engagement with the concepts of percent, decimals, and fractions would come naturally.

This time, instead of giving in to students' pleas for simplification, Ron suggested that they look carefully at the rectangle, noticing both the total number of squares and the ways in which the squares were organized into columns and rows. As he walked around the room, he noticed that those students who were making the most progress had observed that each column represented one-tenth of the rectangle and had shaded in six squares, almost as if they were "filling up" one and one-half columns. If a column was one-tenth, or 10 percent, then a "column and a half," they reasoned, would be 15 percent. The students who were having the most difficulty were working with rectangles in which the shaded squares were not in columns but rather in some other configuration. He helped these students find other ways to figure the percent by asking questions that would allow them to build on the particular configuration that they had shaded. Several examples of students' strategies and Ron's questions appear in **figure 3**.

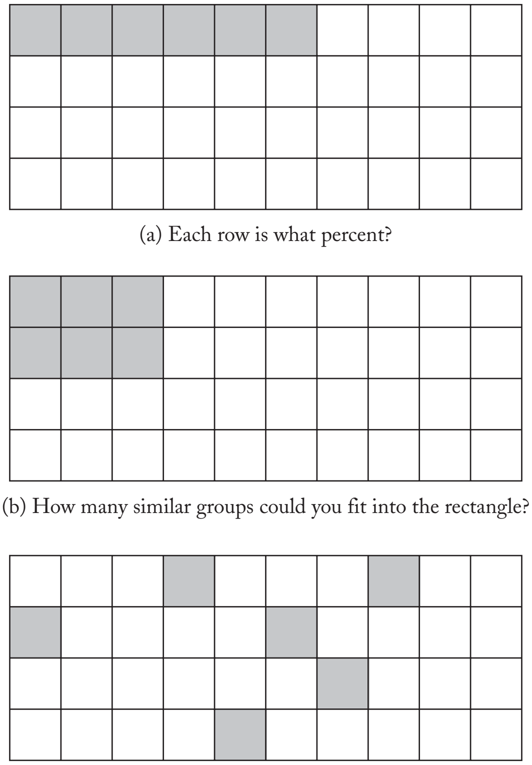
Ron's assistance encouraged the students to persist with figuring percent and, more important, made the students think about what percent meant in relation to this particular diagram. Although it took nearly the entire class period to get through this one problem, Ron found spending the time to be worthwhile. By the end of the lesson, several students had presented alternative strategies at the overhead projector in the front of the classroom. Even Ron was surprised at the many different ways in which the students solved the problem!

At the close of the lesson, Ron was tired, but satisfied. He had never listened so hard to students—and then tried to assist them on the basis of their pre-existing knowledge. He was pleased with what his students were able to do, especially with how they were able to use their understanding of percent to solve the task.

The Ridgeway teachers discuss why

Shortly thereafter, we made arrangements to meet with Ron, Theresa, and their colleagues at Ridgeway to discuss the ways in which the Mathematics Task Framework had been helpful to them. Ron was eager to share his experiences as previously described. He stressed how important it was to be able to focus his attention on some aspect of his teaching. By looking at the tasks he used and how he and the students went about working on the tasks, he felt that he had been able to focus more squarely on what students were learning. He commented that it was easy to get so tied up in what *you did* that you lost sight of what students were learning from the experience.

During the course of the conversation, several other teachers described episodes from their own classrooms, both of tasks that were implemented in ways that supported high-level thinking and of ones that were not. We then asked teachers why they thought that tasks did or did not play out as intended. Could they identify factors that were associated with the maintenance or decline of a task? Ron began by indicat-



(a) Each row is what percent?

(b) How many similar groups could you fit into the rectangle?

(c) Each block is what percent?

Fig. 3

ing that in the class in which he had first used the percent-decimal-fraction task, the major factor associated with task decline was that he had told students to start with the fraction. Once they did that, he explained, they could rely totally on previously learned procedures. Theresa commented that what she had done with the tape-roll toss was very similar to what Ron described. Although students could not use a simple procedure for solving the task, she explained, she did basically give them a step-by-step description of what needed to be done. The teachers went on to suggest other factors, such as classroom management, too little or too much time, and not holding students accountable, as being associated with task decline.

The teachers then began to speculate about the factors that would support maintenance of a task at a high level. They began by saying that some of the factors would be the "opposite" of the first list—not proceduralizing a task, allowing sufficient time, and holding students accountable for high-level thinking. In addition, they added that the most important thing was finding a way to help students make progress without giving away the solution or solution path. Ron explained how hard this approach was to maintain, but that in the end he realized how much more students learned from working through a problem rather than being handed a procedure to follow.

At this point in the discussion, we indicated that in our research we had identified factors associated with the maintenance and decline of high-level demands that included all the factors that they had identified, plus a few more. Our list (shown in **fig. 4**), we explained, was derived from a study of nearly 150 tasks that had been used over a three-year period at four different schools. Teachers nodded while reviewing the list, signaling their agreement with the factors we had identified. One teacher commented that she agreed with all the factors we had identified and could think of situations in which each had contributed to the success or failure of a particular lesson. She went on to say, however, that she would not have been able to articulate each factor. She explained, for example, that she often gave students a rubric that specified the criteria that would be used to evaluate a particular problem, hence "providing students with a means of monitoring their own progress," but that she had never thought about this rubric as a factor that contributed to maintaining high-level cognitive engagement. In retrospect, she admitted that "things went better when students had a rubric."

Factors Associated with the Maintenance of High-Level Cognitive Demands

1. Scaffolding of student thinking and reasoning is provided.
2. Students are given the means to monitor their own progress.
3. Teacher or capable students model high-level performance.
4. Teacher presses for justifications, explanations, and meaning through questioning, comments, and feedback.
5. Tasks build on students' prior knowledge.
6. Teacher draws frequent conceptual connections.
7. Sufficient time is allowed for exploration—not too little, not too much.

Factors Associated with the Decline of High-Level Cognitive Demands

1. Problematic aspects of the task become routinized (e.g., students press the teacher to reduce the complexity of the task by specifying explicit procedures or steps to perform; the teacher "takes over" the thinking and reasoning and tells students how to do the problem).
2. The teacher shifts the emphasis from meaning, concepts, or understanding to the correctness or completeness of the answer.
3. Not enough time is provided to wrestle with the demanding aspects of the task, or too much time is allowed and students drift into off-task behavior.
4. Classroom-management problems prevent sustained engagement in high-level cognitive activities.
5. Task is inappropriate for a given group of students (e.g., students do not engage in high-level cognitive activities because of lack of interest, motivation, or prior knowledge needed to perform; task expectations are not clear enough to put students in the right cognitive space).
6. Students are not held accountable for high-level products or processes (e.g., although asked to explain their thinking, unclear or incorrect student explanations are accepted; students are given the impression that their work will not "count" toward a grade).

Fig. 4 Factors associated with maintenance and decline of high-level cognitive demands

The list seemed to put into words a set of classroom factors and conditions with which teachers immediately identified. Although many of the factors reflected common practices among the teachers, such as drawing frequent conceptual connections and building on students' prior knowledge, they had not previously connected these actions and decisions with the successful implementation of a task.

Using the Framework in the Classroom

The framework is not meant to be a rigid prescription; rather, it is a tool for reflection. When used well, it should draw attention to what students are actually doing and thinking about during mathematics lessons. This focus on student thinking, in turn, helps the teacher adjust instruction to be more responsive to, and supportive of, students' attempts to reason and make sense of mathematics.

Ron Castleman found the framework helpful in his efforts to support students' engagement with high-level tasks. With the help of his colleagues, Ron came to understand how his actions in the classroom were influencing students' learning. Having supportive colleagues who can serve as sounding boards and provide nonjudgmental feedback is invaluable. However, the framework can be used in various settings. In the following sections, we make two suggestions about how you can begin to use the framework as a tool for reflection on your practice.

Teachers observing teachers

Work with a colleague to set up a regular schedule for observing and being observed. Meet afterward to discuss the lesson and make suggestions for improvement. The framework can be used to guide the kinds of things you look for and what you talk about afterward.

When observing, think hard about what messages are being conveyed to the students about what they are expected to do, how they are to do it, and what resources they are to use. You might want to try quickly working the task yourself to make sure that you understand what is entailed in solving it. As students actually go about working on the task, roam around the room, going from desk to desk or from group to group, listening and watching closely to discern how deeply students are grappling with significant mathematical ideas. Are students dealing with mathematical meaning as they work? Is their talk grounded in mathematical reasoning and evidence? Or are they staying at the level of memorized procedures and symbols that are disconnected from underlying ideas?

Afterward, before the end of the day if possible, meet to discuss the observation. Begin by agreeing on the segment of instructional time that constitutes "the task" and on what will be considered as the setup and implementation phases. Then discuss the cognitive demands during each phase. This part of the conversation works best when the observer gives his or her judgments regarding the cognitive demands of the task *first*; the teacher then comments on those judgments, noting whether she or he agrees or disagrees and why. In this way, the observer will be forced to offer critical feedback and be less tempted to gloss over differences of opinion—differences that are important for growth.

If the two of you agree that one or more tasks were set up at a high level of cognitive demand, go on to discuss whether those demands were maintained at a high level during the implementation phase or declined into less challenging work. In either situation, an essential piece of this part of the conversation is identifying the classroom factors (see **fig. 4**) that influenced the maintenance or decline of the cognitive level of the task. Most teachers find this part of the framework the most fascinating, probably because it reflects most directly on things that they are doing well or that they can improve. You should also spend time discussing tasks that are identified to be at a low level at the setup phase, focusing on how the task might be altered to become more challenging.

Teachers observing themselves

If you do not have a colleague with whom you would feel comfortable observing and being observed, try videotaping your own teaching. Then you can reflect on your own instruction at a time that is convenient, unhurried, and private. Using videotape to reflect may, in fact, offer advantages that reflections based on memory or notes do not offer. For example, memories of classroom events are not as objective as what is recorded on videotape. Also, a videotape allows you to watch and rewatch a segment, trying to discern what exactly was going on in students' minds as they worked on a particular task.

So What Is the Payoff?

Evidence gathered across scores of middle-school classrooms in QUASAR middle schools has shown that students who performed the best on project-based measures of reasoning and problem solving were in classrooms in which tasks were more likely to be set up and *implemented* at high levels of cognitive demand (Stein and Lane 1996). For these students, having the opportunity to work on challenging tasks in a supportive classroom environment translated into substantial learning gains on an instrument specially designed to measure exactly the kind of student learning outcomes advocated by NCTM's professional teaching standards.

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