

Introduction

The Challenge of Fractions

The losses that occur because of the gaps in conceptual understanding about fractions, ratios, and related topics are incalculable. The consequences of doing, rather than understanding, directly or indirectly affect a person's attitudes toward mathematics, enjoyment and motivation in learning, course selection in mathematics and science, achievement, career flexibility, and even the ability to fully appreciate some of the simplest phenomena in everyday life.

Susan J. Lamon (2012, p. xi)

The need for better teaching and learning of fractions is one of the few topics in the curriculum with which mathematics educators at every grade level would agree. At conference after conference, teachers bemoan students' resistance to fractions, the trouble they have making sense of them, and their ineptitude at solving problems involving fractions. I offer three fundamental reasons we must take a closer look at how we teach fractions in the United States:

1. *Fractions play a key role in students' feelings about mathematics.*

For many students, fractions present a first mathematical

stumbling block. Students begin disliking mathematics when they must surrender their sense making and yield to sense-less memorization.

2. *Fractions are fundamental to school math and daily life.* Although fractions underpin many complex mathematical topics, including ratios, rates, percents, proportions, proportionality, linearity, and slope, their importance is not limited to mathematical study. As the quote on the previous page indicates, fluency with fractions is also required for many activities of daily life: following recipes, calculating discounts, comparing rates, converting measuring units, reading maps, investing money, and more.
3. *Fractions are foundational to success in algebra.* In its final report, *Foundations for Success*, the National Mathematics Advisory Panel (2008) concluded that (1) algebra is the gateway to success in high school and college, and (2) the main reason for U.S. students' failure in algebra is their poor proficiency with fractions. The worthy goal of "algebra for all" is not possible unless "fractions for all" is a reality. And in our present educational system, a solid grounding in algebra is foundational to a STEM career.

The selection of topics in this book, though by no means exhaustive, was made on the basis of research studies that address the teaching and learning of fractions, evidence from teacher practice, and my own work over the past 25 years with teachers, students, and parents (in both the United States and abroad) from which I have preserved recordings, questions, answers, insights, and samples of student work. It is my hope that readers will find that this book enhances their knowledge of fractions, deepens their appreciation of the complexity involved in teaching them, and perhaps even challenges some long-held beliefs.

Appreciate the Fraction Challenge

No area of elementary mathematics is as mathematically rich, cognitively complicated, and difficult to teach as fractions, ratios, and proportionality.

John P. Smith III (2002, p. 3)

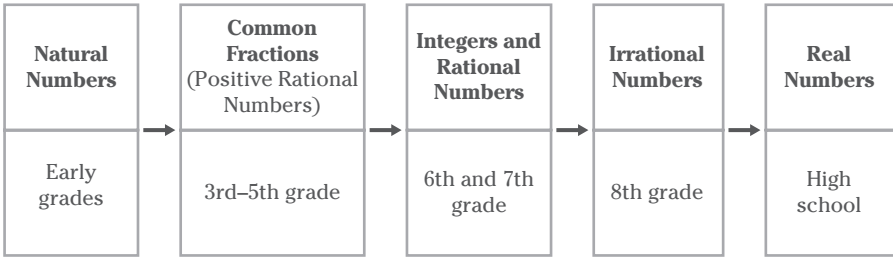
In this section, I introduce the principal reasons fractions are so difficult for students. In Chapters 1 through 7, we'll look at ways to help students move past these difficulties, using strategies and problems that foster understanding of underlying concepts.

From Natural Numbers to Real Numbers

In order to tackle students' greatest challenges with fractions and to feel confident in trying new pedagogical moves, it is important for professionals who work with teachers or students in any grade to know how the number system builds from natural numbers to real numbers (Figure 0.1).

Natural numbers. During early childhood and up to about age 8, children engage with counting numbers, or *natural numbers*—also called *positive whole numbers*. Natural numbers are denoted in mathematics by the symbol \mathbb{N} . In set notation, we write $\mathbb{N} = \{1, 2, 3, 4, \dots\}$. In the United

FIGURE 0.1
**Grade-by-Grade Progression from
 Natural to Real Numbers**



States, 0 is not considered a natural number, but the exclusion of 0 from the set of natural numbers is not universal.

Integers. The next important set of numbers is generated by appending to \mathbb{N} zero and all the “opposites” or “negatives” (*additive inverses*) of the natural numbers. These numbers are called *integers*, denoted by the symbol \mathbb{Z} , and are typically introduced to students in the middle grades. In set notation, we write $\mathbb{Z} = \{ \dots -4, -3, -2, -1, 0, 1, 2, 3, 4 \dots \}$. Since the natural numbers are a subset of the integers, *every natural number is an integer*.

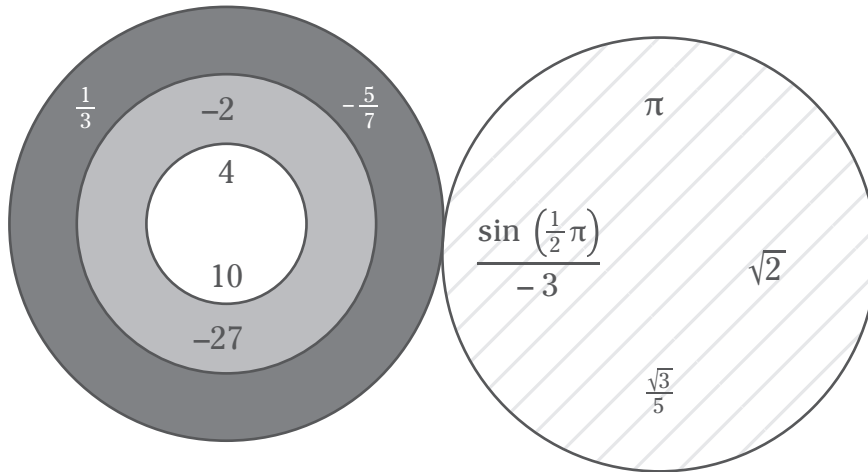
Rational numbers. The common fractions introduced in the upper elementary school grades, such as $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{2}{3}$, are a subset of the *rational numbers*. Rational numbers, denoted by the symbol \mathbb{Q} , are numbers that can be expressed in the form $\frac{a}{b}$, where a and b are integers, provided $b \neq 0$. In set notation, we write $\mathbb{Q} = \left\{ \frac{a}{b}, \text{ where } a \text{ and } b \text{ are members of } \mathbb{Z}, \text{ but } b \neq 0 \right\}$. Every rational number has an equivalent decimal form—for instance, $\frac{1}{2} = 0.5$. Thinking of the symbol $\frac{a}{b}$ as a *quotient* of two integers helps students remember the symbol \mathbb{Q} .

Notice that any integer can be written in the form $\frac{a}{b}$ in many ways. For example, -5 can be written as $\frac{-5}{1}$ or $-\frac{10}{2}$, and $+7$ can be written as $\frac{+7}{1}$ or $\frac{21}{3}$. Therefore, *every integer is a rational number*.

Irrational numbers. In 7th or 8th grade, students learn about a whole new set of numbers, such as π or $\sqrt{-2}$, which cannot be written as quotients of two integers. These are known as *irrational numbers*, because they didn’t make sense to the ancient Greeks who discovered them. Irrational numbers do not have a universally accepted symbol, although I is often used. Unlike the two preceding relationships, the rational numbers are not a subset of the irrational numbers; rather, the two sets are mutually exclusive.

Real numbers. Rational and irrational numbers together form the set of *real numbers*, denoted by the letter \mathbb{R} . By high school, the universe of numbers within which students operate has grown to include all real numbers as shown in Figure 0.2 on the next page.

FIGURE 0.2

The Real Number System

Note: The real number system contains \mathbb{N} , the natural numbers symbolized by the white set; \mathbb{Z} , the integers symbolized by the light gray set; \mathbb{Q} , the rational numbers symbolized by the dark gray set; and \mathbb{I} , the irrational numbers (whose symbol is not universal), symbolized by the hatched set.

At this point, you may be wondering, “Aren’t rational numbers a *middle school mathematics topic*?” Yes, but not exclusively. The higher expectations of the K–12 Common Core State Standards for Mathematics (CCSSM) require a more profound exposure to rational numbers before the middle grades. In fact, the CCSSM formally introduce fractions in 3rd grade, building on students’ prior informal experiences, such as cutting apples into equal halves or sharing a chocolate bar fairly among four people. A recent National Council of Teachers of Mathematics (NCTM) publication for teachers explicitly states, “Rational numbers compose a major area of school mathematics that is crucial for students to learn but challenging for teachers to teach. Students in grades 3–5 need to understand rational numbers well if they are to succeed in these grades and in their subsequent mathematics experiences” (Barnett-Clarke, Fisher, Marks, & Ross, 2010, p. 1).

A Word on the Word *Fraction*

Unlike the term *rational number*, *fraction* does not have a universal mathematical definition. In elementary school mathematics, fractions refer to positive rational numbers, such as $\frac{2}{3}$ or $\frac{9}{7}$. They are grouped into *proper* and *improper* fractions, depending on whether they are less than or greater than 1. By middle school, though, symbolic expressions such as $\frac{\sqrt{5}}{2}$ and $\frac{\pi}{4}$ are also called fractions, as they represent quotients of two quantities—albeit non-integer quantities—written in fraction form. If a fraction is defined as a symbolic expression of the form $\frac{N}{D}$, where numerator N and denominator D can be any non-zero quantity, then *any rational number can be written as a fraction*, but *not every fraction is a rational number* (Figure 0.3).

FIGURE 0.3
Fractions That Are and Are Not Rational Numbers

| Rational Numbers | | | | Not Rational Numbers | | | |
|------------------|---------------|-----------------|----------------|----------------------|----------------------|---------------------------------|--------------------------|
| $\frac{7}{13}$ | $\frac{3}{2}$ | $\frac{-5}{11}$ | $-\frac{4}{1}$ | $\frac{\pi}{2}$ | $\frac{\sqrt{2}}{3}$ | $\frac{\sin(\frac{\pi}{6})}{4}$ | $\frac{-5}{\sqrt[3]{7}}$ |

That said, for the rest of this book, whose main focus is mathematics as taught in grades 2–6, a fraction will designate a *non-negative rational number*, meaning one that is positive or zero. And, since most elementary school teachers use the term *whole numbers* instead of *natural numbers* with their students, I will use *whole number* and *natural number* interchangeably. In the United States, whole numbers usually include zero.

Two other forms: decimals and percents. Every rational number can be expressed in three forms: as a fraction, as a decimal, and as a percent. Although elementary school students begin to explore decimals, and middle school students are introduced to percents, this book focuses

primarily on the conceptual development of fractions, only a little on decimals, and not at all on percents. This is partly due to limited space but also because, as Collins and Dacey (2010) note, “One of the greatest errors fifth- and sixth-grade teachers typically make is introducing conversion of fractions to decimals *before* students have developed mastery with fractions on a conceptual level” (p. 16).

Cognitive Shifts to Consider

Multiplicative thinking, a new big idea students encounter in Grade 3, is the foundation for an entire network of interconnected concepts including multiplication, division, fractions, ratios, rational numbers, proportional relationships, and linear functions—all of which are central to algebra.

Monica Neagoy (2014, p. 7)

The advent of fractions engenders important shifts in students’ ways of thinking about these new numbers that we must be aware of and sensitive to in our pedagogy. I would like to address two major ones: the shift from additive to multiplicative thinking, and the shift from whole numbers to rational numbers.

From additive to multiplicative thinking. From their earliest notions of numbers through roughly age 8, children live in an additive world. They experience situations that involve adding to (joining together or composing) and subtracting from (taking apart or decomposing; taking away or removing). Counting itself is an additive process; we add 1 to each number to obtain the next number. Even when solving comparison problems—*How much more? How much less? How much longer? How much shorter? How much older? How much younger?*—students think additively. The very way that questions are formulated leads students down the

additive path. For instance, “How much taller is Dante than Ashley?” invites a student to reason, “How many inches [or other units] must I add to Ashley’s height to get Dante’s height?” or “How many inches do I get if I subtract Ashley’s height from Dante’s?” Either way, this is additive thinking.

Consider another comparison problem: Plant *A* went from a height of 2 feet to a height of 8 feet in one year. Plant *B* went from 6 feet to 12 feet in the same year. If you ask your students which plant grew more, they will probably answer that both grew the same amount—namely, 6 feet. But that’s looking at the problem additively and considering the *absolute* growth: $8 - 2 = 6$ and $12 - 6 = 6$.

Let’s now examine the *relative* growth and ask, “How many times its original height is each plant at the end of the year?” Plant *A* quadrupled in height, whereas Plant *B* only doubled in height. Therefore, Plant *A* grew more, *relative to its original height*. An equivalent but more sophisticated way of saying this is “Plant *A* grew three times its original height, whereas Plant *B* grew once its original height.”

Most real-world numbers aren’t always so nice and neat, with whole-number multiples. If, say, Plant *A* grew from 2 to 3 feet, and Plant *B* grew from 6 to 8 feet, then we would say that Plant *A* grew $\frac{1}{2}$ of its original height, whereas Plant *B* only grew $\frac{1}{3}$ of its original height. Such reasoning exemplifies *multiplicative thinking* and necessarily involves rational numbers.

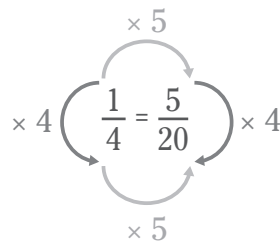
Consider a final example. If you ask a rising 6th grader to compare $\frac{13}{15}$ and $\frac{14}{16}$, chances are that the student will say they are equal, because in both cases the numerator and denominator differ by 2. The student’s explanation might be, “I add the same number, 2, to the top number to get the bottom number, so they’re the same.” This is a testimony to ingrained additive thinking. Despite learning the equivalent fraction algorithm, most

students leave elementary school unaware of the double multiplicative nature of equivalent fractions. Why? Because we don't take the time to unpack it for them and then revisit it in multiple contexts!

Figure 0.4 illustrates both the *between* (or *across*) *ratio* of 1 to 5 and the *within* (or *downward*) *ratio* of 1 to 4 in two equivalent fractions. If this multiplicative nature of fractions were cultivated during the last three years of elementary school, then students wouldn't think of comparing $\frac{13}{15}$ and $\frac{14}{16}$ additively. Multiplicative thinking underpins fractions, which in turn underpin the mathematics of ratios, rates, percents, proportions, linearity, and rational functions.

FIGURE 0.4

The Double Multiplicative Nature of Fraction or Ratio Equivalence



From whole numbers to rational numbers. The shift from whole numbers to rational numbers follows the shift from work with *discrete quantities* to work with *continuous quantities*. Simplistically put, discrete quantities are things we can *count*, such as blocks, cell phones, or people; continuous quantities are things we can *measure*, such as length, area, or time.

The action of measuring is a multiplicative process par excellence, but it almost never results in an exact whole number of units. Students begin to work with measurement in 3rd grade; hence, the need for an

understanding of rational numbers. But in this shift toward rational numbers, students unfortunately continue to apply their familiar whole-number thinking. Their reasoning often goes like this:

- $\frac{1}{4}$ is greater than $\frac{1}{3}$, because 4 is greater than 3.
- 0.157 is greater than 0.63, because 157 is greater than 63.
- $\frac{2}{3} + \frac{1}{2} = \frac{3}{5}$, because $2 + 1 = 3$ and $3 + 2 = 5$.
- $4 + 0.3 = 7$ or 0.7, because $4 + 3 = 7$ and $0.4 + 0.3 = 0.7$.
- $\frac{2}{3} \times 9$ can't be 6 because "multiplication makes numbers bigger."
- $4 \div \frac{1}{2}$ can't be 8 because "division makes numbers smaller."

The chapters that follow address these and other misconceptions, each of which is subtle and deserves attention. The key is to emphasize for students the deeper aspects of whole-number reasoning that remain unchanged (such as the meanings of operations) while simultaneously pointing out new mathematical ways of thinking ushered in by rational numbers—such as the multiplicative comparison of two quantities.

The Rush to Algorithms

Set building number sense for fractions among elementary school-aged students as the goal as opposed to building procedural skill with adding, subtracting, and multiplying fractions.

Kathleen Cramer and Stephanie Whitney (2010, p. 21)

The teaching and learning of fractions is notoriously associated with memorizing computational algorithms or procedures. A case in point is the ubiquitous rhyme "Ours is not to reason why, just invert and multiply," which gives the algorithm for dividing fractions. It is this lack of sense making, so pervasive in traditional mathematics instruction, that leads to frustration, surrender, and even failure.

For decades—if not centuries—“knowing” fractions has been synonymous with knowing how to perform fraction operations, and “knowing” ratios and proportions has meant knowing how to solve proportional equations, such as $\frac{3}{7} = \frac{10}{x}$. Traditional instruction of rational numbers has commonly been rule based. Consider these six rules:

- To add or subtract fractions, first find common denominators and then add or subtract numerators accordingly.
- To multiply fractions, just multiply across—both numerators and denominators.
- To divide two fractions, invert the second fraction and then multiply the fractions.
- To multiply or divide a decimal by a multiple of 10, move the decimal point to the right (for multiplication) or left (for division) as many digits as there are zeroes in the multiple of 10.
- You *cannot* have zero as the denominator.
- To solve a proportion (e.g., $\frac{3}{7} = \frac{10}{x}$), cross-multiply and then divide by the coefficient of x .

Human brains enjoy reasoning logically, finding meaning, discovering patterns, and making connections. Deprived of these actions, the naturally curious mind instead surrenders to passivity and accepts math as merely a set of meaningless numerical procedures (which—in the world of algebra—become a set of meaningless symbolic procedures). If I asked you “Why?” after each of the six rules, would you be able to explain? Could you illustrate each rule for your students with a real-world context?

Don’t feel bad if you cannot. Traditionally, teachers were not expected to unpack the whys—our focus was more on the how-tos. But in our technological world, with machines computing faster and more accurately than humans can and with the CCSSM setting higher expectations, the bar

has been raised for teachers and students alike. We are now expected to reason, understand, evaluate, explain, justify, and prove—in short, we’re expected to use higher-order thinking skills!

What Can You Expect from This Book?

The following chapters address seven big ideas in the teaching and learning of fractions. Each chapter begins with a vignette from a real classroom. The chapters then explore students’ most common misconceptions regarding the topic in question, based on my research and practice, followed by a thorough unpacking of productive mathematical thinking. Each chapter ends with seven challenging multistep and thought-provoking problems for teachers to explore with their students.

Featured in every chapter are four additional resources:



Bridges to Algebra establish an important connection between the fraction concept at hand and an algebraic notion to be encountered in later years.



Mathematical Practices illustrate one of the eight Common Core State Standards for Mathematical Practice using an aspect of fraction instruction.



Teaching Tips highlight a practice you are encouraged to adopt that could enhance your fraction teaching.



Fraction Apps relate to the chapter topics and are offered free of charge at www.apps4math.com. The apps are designed to enhance teaching and facilitate learning while making both more enjoyable!

Chapter 1 discusses the dense web of meanings that surround the concept of fractions. We begin with students’ own mental images and ideas about fractions and show how we can help them construct new knowledge

by building on their existing informal knowledge of fractions through careful observation and focused conversations.

Chapter 2 describes multiple uses of visual and tactile models in fraction instruction. We address questions such as the following: *Which models are effective? Are all models equivalent? What limitations might they present? Should we use one or many models?*

Chapter 3 examines common student misconceptions related to the concept of the *whole* or *unit* and consider the importance of the unit in developing foundational fraction concepts.

Chapter 4 takes on one of our greatest pedagogical challenges in teaching fractions: helping students both see with their eyes and understand with their minds that equivalence between two fractions can be maintained, despite numeric or symbolic transformation. It shows how we can build on students' intuitive methods and understandings of equivalence to address their principal stumbling blocks.

Chapter 5 focuses on building number sense. In order to compare two quantities or numbers judiciously, we must first have good number sense—and an important part of number sense is recognizing the relative magnitude of numbers. When it comes to fractions, however, this ability is direly lacking; many students don't or can't compare or order fractions without resorting to a memorized algorithm.

Chapter 6 explores how students' computation patterns can emerge organically in well-designed tasks with a teacher's guidance. A premature focus on the memorization of algorithms reinforces the erroneous but prevalent belief that mathematics is more about memorizing procedures than about reasoning about powerful ideas. We emphasize the importance of giving students time to develop good fraction and operation sense before rushing to teach computation algorithms.

Chapter 7 unpacks on the whole number–decimal connection, the fraction-decimal connection, common student misconceptions, and recommendations for overcoming them.

The concluding chapter summarizes seven habits of mind that foster good fraction sense, recalls the dangers of the long-standing rule-based approach to teaching fractions, and looks ahead to ratios and proportions.

It is my sincere hope that *Unpacking Fractions* will inspire educators to help students shift from fear to enjoyment and from meaningless memorization to deep understanding. It seems reasonable that the shift from failure with fractions to success with fractions will follow naturally.