

# Mathematical Reasoning and Sense Making<sup>1</sup>

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Reasoning and sense making are the foundation of mathematical competence and proficiency, and their absence leads to failure and disengagement in mathematics instruction. Thus, developing students' capabilities with reasoning and sense making should be the primary goal of mathematics instruction. In order to achieve this goal, all mathematics classes should provide ongoing opportunities for students to implement these processes.

What are mathematical reasoning and sense making? *Reasoning* is the process of manipulating and analyzing objects, representations, diagrams, symbols, or statements to draw conclusions based on evidence or assumptions. *Sense making* is the process of understanding ideas and concepts in order to correctly identify, describe, explain, and apply them. Genuine sense making makes mathematical ideas “feel” clear, logical, valid, or obvious. The moment of sense making is often signaled by exclamations such as “Aha!” “I get it!” or “Oh, I see!”

## Why Focus on Reasoning and Sense Making?

Reasoning and sense making are critical in mathematics learning because students who genuinely make sense of mathematical ideas can apply them in problem solving and unfamiliar situations and can use them as a foundation for future learning. Even with mathematical skills, “[i]n order to learn skills so that they are remembered, can be applied when they are needed, and can be adjusted to solve new problems, they must be learned with understanding [i.e., they must make sense]” (Hiebert et al. 1997, p. 6).

Sense making is also important because it is an intellectually satisfying experience, and not making sense is frustrating (Hiebert et al. 1997). Students who achieve genuine understanding and sense making of mathematics are likely to stay engaged in learning it. Students who fail to understand and make sense of mathematical ideas and instead resort to rote learning will eventually experience continued failure and withdraw from mathematics learning.

## Understanding How Students Think

An abundance of research describing how students learn mathematics indicates that effective mathematics instruction is based on the following three principles (Battista 2001; Bransford, Brown, and Cocking 1999; De Corte, Greer, and Verschaffel 1996; Greeno, Collins, and Resnick 1996; Hiebert and Carpenter 1992; Lester 1994; NRC 1989; Prawat 1999; Romberg 1992; Schoenfeld 1994; Steffe and Kieren 1994):

1. To genuinely understand mathematical ideas, students must construct these ideas for themselves as they intentionally try to make sense of situations; their success in constructing the meaning of new mathematical ideas is determined by their preexisting knowledge and types of reasoning and by their commitment to making personal sense of those ideas.
2. To be effective, mathematics teaching must carefully guide and support students as they attempt to construct personally meaningful mathematical ideas in the context of problem solving, inquiry, and student discussion of multiple problem-solving strategies. This sense-making and discussion approach to teaching can increase equitable student access to powerful mathematical ideas, as long as it regularly uses embedded formative assessment to determine the amount of guidance each student needs. (Some students construct ideas quite well with little guidance other than well-chosen sequences of problems; other students need more direct guidance, sometimes in the form of explicit description.)
3. To effectively guide and support students in constructing the meaning of mathematical ideas, instruction must be derived from research-based descriptions of how students develop reasoning about particular mathematical topics (such as those given in research-based learning progressions).

Consistent with this view on learning and teaching, professional recommendations and research suggest that mathematics teachers should possess extensive research-based knowledge of students' mathematical thinking (An, Kulm, and Wu 2004; Carpenter and Fennema 1991; Clarke and Clarke 2004; Fennema and Franke 1992; Saxe et al. 2001; Schifter 1998; Tirosh 2000). Teachers should "be aware of learners' prior knowledge about particular topics and how that knowledge is organized and structured" (Borko and Putnam 1995, p. 42). And because numerous researchers have found that students' development of understanding of particular mathematical ideas can be characterized in terms of developmental sequences or *learning progressions* (e.g., Battista and Clements 1996; Battista et al. 1998; Cobb and Wheatley 1988; Steffe 1992; van Hiele 1986),

teachers must understand these learning progressions. They must understand “the general stages that students pass through in acquiring the concepts and procedures in the domain, the processes that are used to solve different problems at each stage, and the nature of the knowledge that underlies these processes” (Carpenter and Fennema 1991, p. 11). Research clearly shows that teacher use of such knowledge improves students’ learning (Fennema and Franke 1992; Fennema et al. 1996). “There is a good deal of evidence that learning is enhanced when teachers pay attention to the knowledge and beliefs that learners bring to a learning task, use this knowledge as a starting point for new instruction, and monitor students’ changing conceptions as instruction proceeds” (Bransford et al. 1999, p. 11).

Beyond understanding the development of students’ mathematical reasoning, it is important to recognize that to be truly successful in learning mathematics, students must stay engaged in making personal sense of mathematical ideas. To stay engaged in mathematical sense making, students must be successful in solving *challenging but doable* problems. Such problems strike a delicate balance between involving students in the hard work of careful mathematical reasoning and having students succeed in problem solving, sense making, and learning. Keeping students successfully engaged in mathematical sense making requires us to understand each student’s mathematical thinking well enough to continuously engage him or her in *successful* mathematical sense making. Furthermore, to pursue mathematical sense making during instruction, students must *believe*—based on past their experiences—that they are capable of making sense of mathematics. They must also believe that they are supposed to make sense of all the mathematical ideas discussed in their mathematics classes.

Finally, as part of the focus on reasoning and sense making in mathematics learning, students must adopt an inquiry disposition. Indeed, students learn more effectively when they adopt an active, questioning, inquiring frame of mind; such an inquiry disposition seems to be a natural characteristic of the mind’s overall sense-making function (Ellis 1995; Feldman and Kalmar 1996).

## Reaching All Students

The principled, research-based instruction described above not only helps all students maximize their learning but also benefits struggling students (Villasenor and Kepner 1993). In fact, this type of teaching supports all three tiers of Response to Intervention (RTI) instruction. For Tier 1, high-quality classroom instruction for all students, research-based instructional materials include extensive descriptions of the development of students’ learning of particular mathematical topics. Research shows that teachers who understand such information about student learning teach in ways that produce greater student achievement. For Tier 2, research-based instruction enables teachers to better understand and monitor each student’s mathematics learning through

observation, embedded assessment, questioning, informal assessment during small-group work, and formative assessment. They can then choose instructional activities that meet their students' learning needs: whole-class tasks that benefit students at all levels or different tasks for small groups of students at the same level. For Tier 3, research-based assessments and learning progressions support student-specific instruction for struggling students so that they receive the long-term individualized instruction sequences they need.

Because extensive formative assessment is embedded in this type of teaching, support for its effectiveness also comes from research on the use of formative assessment, which indicates that formative assessment helps all students—and perhaps particularly struggling students—to produce significant learning gains, often reducing the learning gap between struggling students and their peers.

## What Does Sense Making During Learning and Teaching Look Like?

The following two examples are illustrations of the development of students' reasoning and sense making about particular mathematical ideas. We examine obstacles to sense making, variations in student sense making, and how teaching can support sense making at various levels of sophistication.

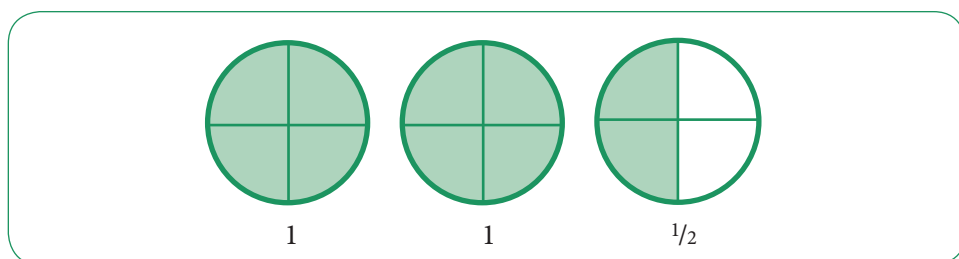
### Making Sense of Division of Fractions

To illustrate the nature of mathematical sense making, reasoning, and understanding, consider two different ways that students might reason about and make sense of the problem “What is  $2\frac{1}{2}$  divided by  $\frac{1}{4}$ ?” (Battista 1999). Many students solve this problem using the “invert and multiply” procedure they memorize and almost never understand:

$$2\frac{1}{2} \div \frac{1}{4} = \frac{5}{2} \times \frac{4}{1} = \frac{20}{2} = 10$$

They do not make conceptual sense of this procedure, and the only way they can justify it is by saying something like “That’s the way my teacher taught me.”

In contrast, students who have made sense of and understand division of fractions do not need a symbolic procedure to compute an answer to this problem. They can think about the symbolic problem physically as one that requires finding the number of pieces of size one-fourth that fit in a quantity of size two and one-half (see fig. 1.1). They reason that, since there are 4 fourths in each 1, and 2 fourths in  $\frac{1}{2}$ , there are 10 fourths in  $2\frac{1}{2}$ .



**Fig. 1.1.1. Finding the number of fourths in  $2\frac{1}{2}$**

Furthermore, having this mental-model-based intuitive understanding of division of fractions can help students start to make personal sense of the symbolic algorithm. In the problem  $2\frac{1}{2} \div \frac{1}{4}$  why do we change division by  $\frac{1}{4}$  to multiplication by 4? Because there are 4 fourths in each whole, to determine how many fourths are in the dividend  $2\frac{1}{2}$ , we must multiply the number of wholes in the dividend (including fractional parts) by 4.

$$2\frac{1}{2} \div \frac{1}{4} = 2\frac{1}{2} \times 4 = 8 + 2 = 10$$

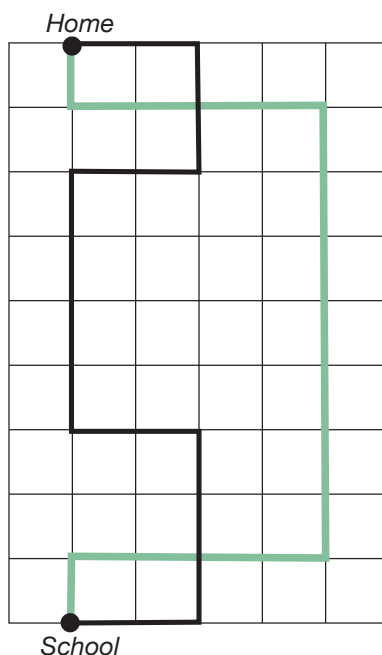
As another example, what is 10 divided by  $\frac{1}{4}$ ? Because there are 4 fourths in each 1, and there are 10 ones in 10, there are 10 times 4 fourths in 10. So the answer is found by multiplying the dividend 10 by 4; that is,  $10 \div \frac{1}{4} = 10 \times 4 = 40$ . To have students continue this reasoning, we can ask them to describe how to find the quotients for problems like  $12 \div \frac{1}{5}$  and  $8\frac{1}{2} \div \frac{1}{2}$  and to describe in words why their solution procedures works.

## Students' Reasoning and Sense Making About the Concept of Length

To further illustrate the ideas previously described, we examine students' sense making and reasoning about the concept of length. We look at the different ways that students make sense of and reason about this topic and how instruction can encourage and support students' increasingly more sophisticated reasoning about it. There are three key steps to helping students make sense of a formal mathematical idea. First, determine empirically how they currently are making sense of the idea. Second, hypothesize how their understanding of the idea might progress. Third, choose problems and representations that can potentially help them progress to more sophisticated ways of reasoning.

The Home to School problem (fig. 1.2) provides an excellent assessment of how well young students understand the concept of length. We first examine how several students made sense of this problem; then we examine the kinds of instruction each student needs.

*Which sidewalk from home to school is longer, the black one, the green one, or are they the same (Battista 2012b)?*

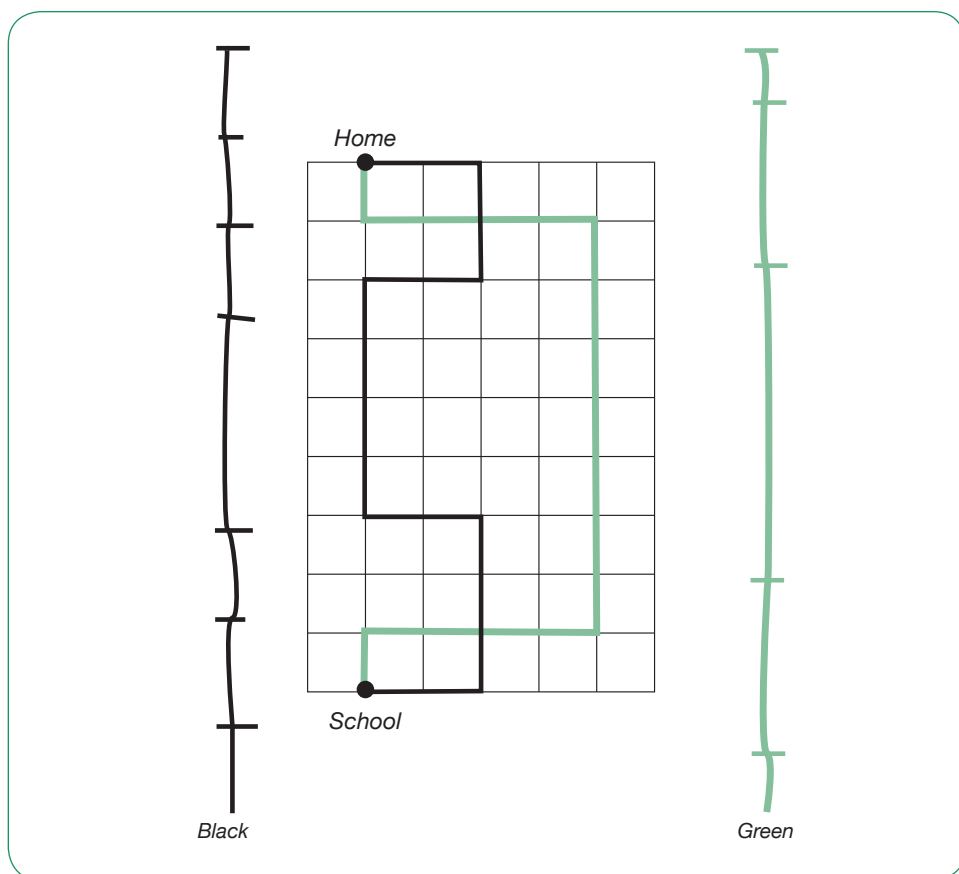


**Fig. 1.2 Diagram of the two sidewalks from home to school**

### *Investigating Students' Reasoning and Sense Making*

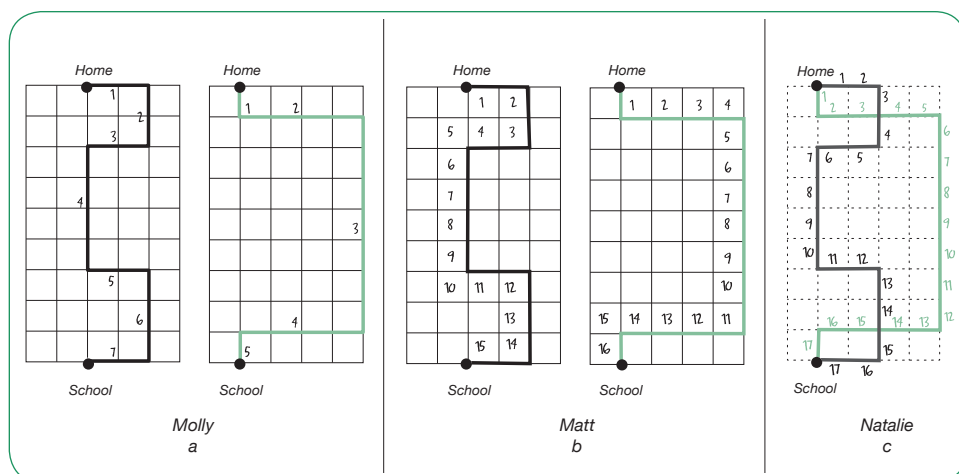
Deanna says that the black sidewalk is longer because it is more “curvy.” She made personal sense of the problem by relating it to her experience of walking paths, which tend to take longer when they have many turns (often students confound the length of a path with the time it takes to walk the path).

David uses the spread between his thumb and finger to draw a straight version of the green sidewalk, one straight component at a time (drawing on the right in fig. 1.3). He uses this same procedure to construct a straight version of the black sidewalk (drawing on the left in fig. 1.3). He then compares his drawings and says that the black sidewalk is longer. This student made personal sense of the problem by straightening the paths and comparing them directly, side by side. Note that this reasoning suggests the beginnings of a valid understanding of the concept of length, and if it were done precisely (say piece by piece on a large grid), it would be mathematically correct.



**Fig. 1.3. Straightening the sidewalks**

The next three students make personal sense of the problem by reasoning that they should count *something*, a strategy they have often seen used in their classrooms (see fig. 1.4). However, Molly and Matt do not yet understand exactly what to count. Molly counts whole straight sections of the sidewalks and concludes that the black sidewalk is longer. Matt has observed people counting squares along paths on similar tasks, but he does not recognize how counting squares must be done in a way that corresponds to counting unit-lengths; he concludes that the green path is longer. Only Natalie correctly counts 17 unit-lengths along each sidewalk to conclude that the two paths have the same length.



**Fig. 1.4 Three students' solutions to the Home to School problem**

### Research Note

Difficulties in reasoning about this problem are widespread among elementary students. In individual interviews with students in grades 1–5, Battista (2012) found that there were more than twice as many students who used non-measurement strategies as those who used measurement strategies even when measurement strategies are most appropriate. Non-measurement strategies do not use numbers (like Deanna and David); measurement strategies use numbers (like Molly, Matt, and Natalie). Furthermore, no first- or second-grade students and only 6 percent of third graders, 12 percent of fourth graders, and 21 percent of fifth graders used correct measurement reasoning on this task (like Natalie). Even some adults have difficulty with this problem.

### Instruction Focused on Individual Student Needs

Choosing instruction to help students like Deanna, David, Molly, and Matt make progress in understanding length measurement requires us to understand the thinking each student can build on (see Battista 2012 for a detailed description of instructional activities). For instance, we can help Deanna by having her straighten paths and compare them directly.<sup>2</sup> We can help David by giving him precut sticks or rods that match the straight segment lengths on the two paths, then having him make straight paths from each set of sticks to see that the paths are the same length when carefully straightened.

We can help Molly and Matt by having them physically place unit-length rods along each path, counting the unit-lengths then using the set of unit-lengths from each path to make straight versions of the paths. Because there are 17 unit-lengths in each path, when we straighten the paths, they have exactly the same length. Furthermore, most young children do not *logically* understand why



counting the number of unit-lengths in the two paths tells us which is longer when straightened. They come to this conclusion *empirically* by repeatedly observing that counting predicts which path will be longer when they physically straighten the paths. It is also useful to have students compare unit-length and square iterations along grid paths to see that these iterations generally produce different counts (unless the paths are straight). Some students make sense of this discrepancy by saying that the plastic squares have to be held “sideways”—perpendicular to the student page—to give a correct count.

The key here is to help students build on what they know to make sense of increasingly more sophisticated reasoning about length. If our instruction is too abstract for students’ current levels of understanding, they will not be able to make the appropriate jump in personal sense making.

### *Integrating Conceptual and Procedural Knowledge in Reasoning About Length*

Examining the conceptual and procedural knowledge needed to solve the Home to School problem sheds additional light on student reasoning and sense making. Both types of knowledge are critical for mathematical proficiency (Kilpatrick, Swafford, and Findell 2011).

A *concept* is the meaning a person gives to things and actions. Concepts are the building blocks of reasoning: we reason by manipulating, reflecting on, and interrelating concepts that we have made sense of. *Conceptual knowledge* enables one to identify examples of and define concepts, to see relationships among concepts, and to use concepts in reasoning. For instance, we might *conceptualize* length as the linear extent of an object when it is straightened or as the distance you travel as you move along a path.

*Procedural knowledge* enables us to perform and use mathematical *procedures*, which are repeatable sequences of actions on objects, diagrams, or mathematical symbols. Procedural knowledge includes more than computational skill. For instance, to solve many length problems, students use the procedure of iterating and counting unit-lengths.

In the Home to School problem, Molly and Matt did not properly connect conceptual and procedural knowledge in their reasoning. These two students have not yet developed a conceptual understanding of length measurement. Molly does not understand that the iterated units must be the same length. Matt does not understand that the iterated units must be length units, not squares. Or he thinks that iterating squares is the same as iterating unit-lengths “because they are the same size.” Even more confusing for students is the fact that squares can be used to properly iterate unit-lengths for straight paths and for non-straight paths *if done very carefully*. Indeed, understanding the difference between iterating squares as squares and using squares to iterate unit-lengths is extremely difficult for many students. If Matt thinks that counting squares

iterates unit-lengths, he may not have sufficient conceptual understanding of unit-length iteration to guide the correct use of squares to iterate unit-lengths; that is, his iteration of squares will be correct only if the whole sidewalk path is covered by a sequence of unit-lengths, with no gaps or overlaps in the sequence.

### Summary

This discussion of the Home to School problem illustrates that students often make sense of the same formal mathematical idea in different ways during the process of learning and that if instruction is to promote and support reasoning and sense making, it must be chosen to help each student build on his or her current mathematical ideas.

### Observing How Instruction Can Help Students Develop the Concept of Unit-Length Iteration

We now examine how teaching can help students develop a concept of unit-length iteration that is abstract and powerful enough to deal correctly with the Home to School problem. First we look at instruction that preceded the presentation of the Home to School problem. Then we examine how the students made sense of the problem in light of the concepts about unit-length iteration they had already developed.

Six students are working together with their teacher. The students are working on a sequence of problems of increasing difficulty, starting with the following problem.

**Unit-Length Problem.** *Which wire is longer, or are they the same length? Suppose I pull the wires so they are straight. Which would be longer? Can you check your answers with inch-rods? Can counting anything help you solve this problem?*

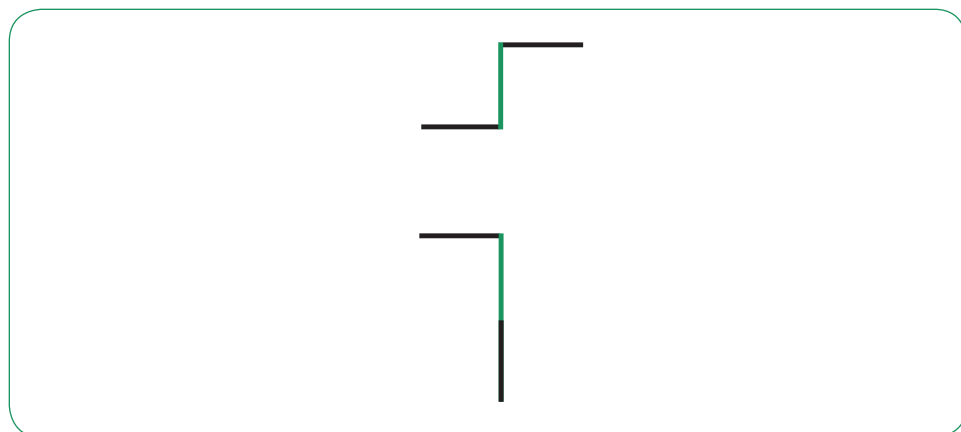


Fig. 1.5. Which wire is longer, or are they the same?

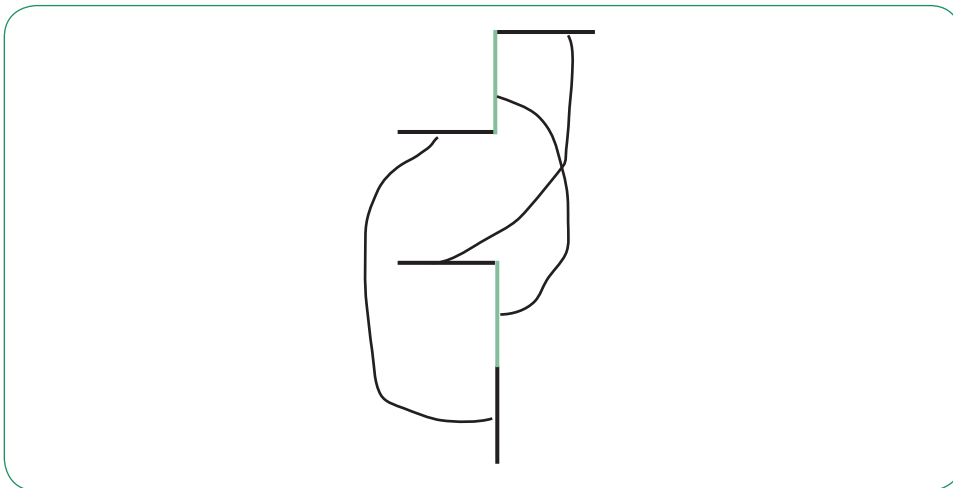
The students made sense of and reasoned about this problem in a variety of ways.

Michael: I think they might be the same length because it's curved.

Kerri: *[Uses the eraser end of her pencil to move upward on the vertical segment of the bottom wire]* 1, 2, . . . 6, *[pointing at the horizontal segment of the bottom wire]* 7. *[Uses the eraser to count the 3 unit segments on the top wire]*. 1, 2, 3; 3 plus a little more. The bottom wire is longer.

Zack: Two on this *[puts pencil horizontal then slightly vertical on the bottom wire]*. And 3 on this *[puts the pencil horizontal, vertical, then horizontal on the top wire]*. I think the top one is longer. *[Note that this is the same type of straight-section reasoning used by Molly.]*

Brandi: Wait! I would say they're the same. Like this would be *[points at the bottom black vertical segment of the bottom wire]* like this *[points at the bottom black horizontal segment of the top wire (see fig. 1.6)]*. And this line *[points at the green vertical segment of the bottom wire]* would fit here *[points on the green vertical segment of the top wire]* and this line *[points at the horizontal black segment of the bottom wire]* would fit here *[points at the top black horizontal segment of the top wire]*.



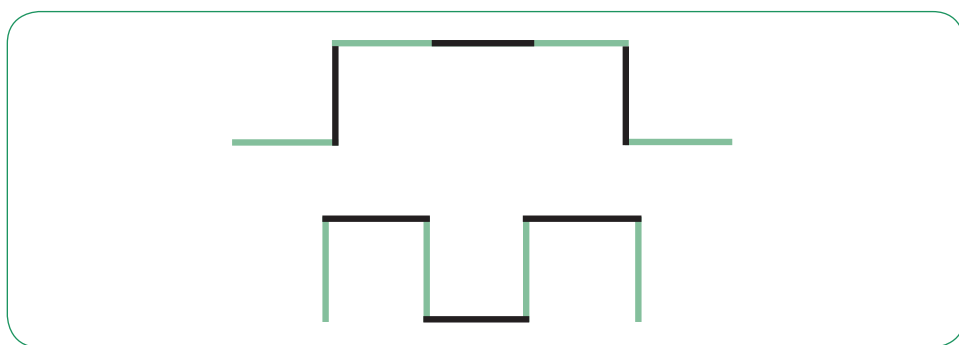
**Fig. 1.6. Brandi's comparison of the unit-lengths**

Gerald: I thought this one was longer *[top wire]* because I measured by using my fingers *[shows a finger spread]*.

- Kerri: The line might be an inch [*pointing at the top segment of the bottom wire*]. Then I would know that it would be 3 inches on there [*goes over the bottom wire*] and maybe 3 inches on there [*goes quickly over the top wire*]. So the same.
- Teacher: Do you want to check with the inch-rods [*straws cut into 1-inch pieces*]? Each one of these is 1 inch long.
- Kerri: 1, 2, 3 [*moves an inch-rod along the bottom wire*]. 1, 2, 3 [*moves an inch-rod along the top wire*]. They're the same length.
- Teacher: Without using those inch-rods, is there anything that you could use to solve this problem by counting?
- Brandi: 1, 2, 3 [*pointing to segments on the bottom wire*]. 1, 2, and 3 [*pointing to segments on the top wire*].

There is a wide variety of sophistication in students' reasoning about this problem, from Michael's non-measurement reasoning, to Kerri and Zack's incorrect counting, to Brandi's non-measurement but correct one-to-one correspondence, to Kerri's correct counting. To encourage and support the students in moving toward more sophisticated reasoning, the teacher not only provides students an opportunity to check their answers but also has them reflect on how they could have solved this problem by counting unit-lengths. Note, however, that some students' sense making would have benefitted from putting 3 unit-lengths on two separate wires, counting the unit-lengths, and then actually straightening each wire. Also note how Brandi seemed to have made sense of Kerri's counting procedure and incorporated it into her own reasoning.

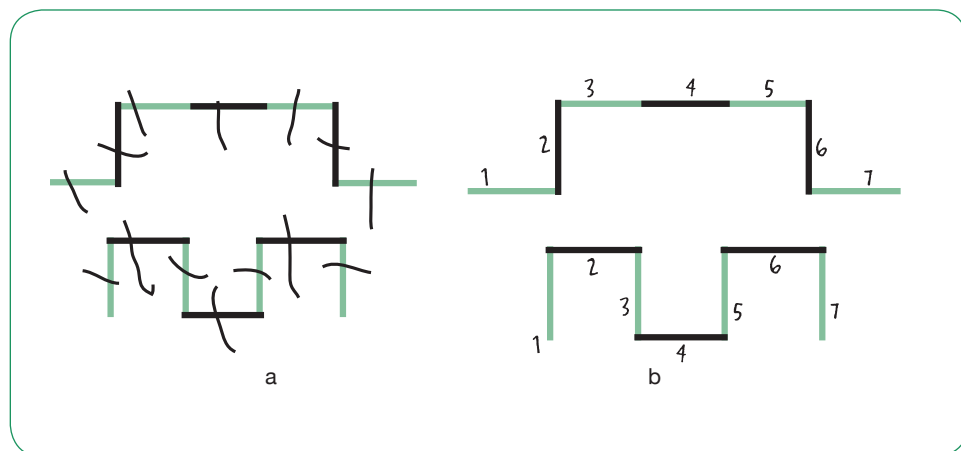
After several other problems, students are given the problem represented in figure 1.7.



**Fig. 1.7**

- Zack: 1, 2, 3, 4, 5, 6, 7 [*points at segments on the top wire*]. 1, 2, 3, 4, 5, 6, 7 [*points at segments on the bottom wire*]. Both are the same.
- Kerri: [*Counts 7 unit-segments on each wire*] Yep.

Serena makes a slash on each segment on both wires as she counts (fig. 1.8a), while Michael writes numbers on the segments for both wires as he counts (fig. 1.8b).



**Fig. 1.8. Serena's work (a) and Michael's work (b)**

Teacher: And do you want to check them with the inch-rods?

Kerri: Yeah.

Several students: No.

By the end of this session, the students were routinely counting unit-lengths to compare the lengths of the wires in problems like those previously shown.

Two weeks later the teacher returned to the topic of length and had her students work on the Home to School problem (fig. 1.2), each with his or her own two activity sheets, one path per sheet. Although the students routinely used unit-length (inch) counting on the previous set of problems, in the new context of the Home to School problem, students abandoned this strategy. Because the sidewalk paths are drawn on square inch-grids, squares become visually salient for the students. Their concept of unit-length iteration was not abstract and general enough to apply in this new situation. The dialogue below illustrates how students' sense making in this new situation evolved.

Teacher: When we're trying to figure out the lengths of the sidewalks, what should we count?

Gwen: I think we should count the squares because they're like an inch.

Kerri: The squares are as long as the segments. [*Points at a square along a sidewalk, then at its side*] So they're the same length, which means that if you chose either one of them it wouldn't be wrong, because they're the same length. [*Pause*] Well, you won't for sure come up with the same answer. Cause there's more squares than

segments. Oh wait! Then you could just like count the squares that are nearest [*pointing at the sidewalk*].

Gwen: And you wouldn't count ones near the corner because they're not near a segment; it's just a corner touching the line.

Kerri: You would like want to count all the ones that have a segment on them. [*Points at a segment on a sidewalk path*]

Teacher: [*Deciding that the students should all be looking at the same thing, shows Serena's sheet for the black path*] Now this part of the sidewalk you're saying is 4 blocks [*points to the squares numbered 1–4 in figure 1.9*], right?

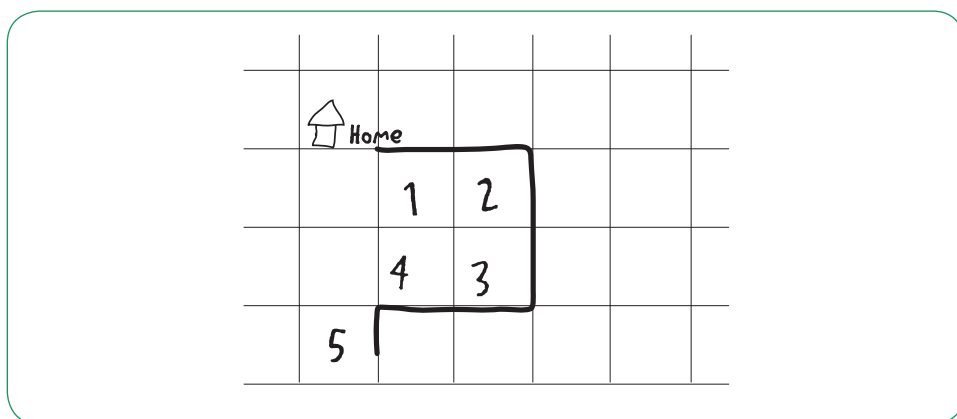


Fig. 1.9. Serena's initial count is the same as Gwen's.

Teacher: What would happen if we were measuring and we used our 1-inch straws?

Gwen: That's a problem. You can't count the squares because like they would be sharing one; this and this [*pointing at the second and third unit-segments, starting at Home*] each need a square, and we only counted one [*the 2*]. So you would need to count the sides. So we would count 1, 2, 3, 4, 5, 6, 7 around here [*correctly counting unit-lengths on the section of the sidewalk shown in figure 1.9*].

Teacher: So how long do you think the whole black sidewalk is?

Gwen: [*Correctly pointing to and counting unit-lengths along the black sidewalk*] 1, 2, . . . , 16, 17.

Teacher: 17 what?

Gwen: Sides, same as our inch straws.

Note the different ways that the students made sense of this problem during the discussion. At first, Kerri thought that using squares in the grid would work because “the squares are as long as the segments.” Kerri revised her reasoning and then claimed that they should count only the squares that have a side on the sidewalk path. When Serena and Gwen seemed to do what Kerri suggested (fig. 1.9), they came up with an incorrect count. So the teacher asked a question that she thought might encourage students to revise their reasoning: “What would happen if we were measuring and we used our 1-inch straws?” After Gwen realized the difficulty with counting squares, she made sense of the correct method for iterating unit-lengths along the sidewalk paths.

This episode is an excellent example of the SMP Reason Abstractly and Quantitatively: the students had to decontextualize the unit-length counting strategy they used in the rod problems and re-contextualize it (transfer it) to apply it in the more difficult and complex context of the Home to School problem. This re-contextualizing did not occur automatically; it required reasoning and sense making above and beyond the reasoning they had applied in previous problems. In fact, it was reasoning about this new problem that led students to construct a more powerful concept of unit-length iteration that applied in more complex situations. Most often, students’ initial reasoning is context dependent; it is only by giving students a variety of contexts that students decontextualize and abstract this context-dependent reasoning so that it becomes generally applicable.

## Standards for Mathematical Practice and Process Standards in Sense-Making Episode

To relate our discussion of students’ reasoning and sense making about the concept of length to the CCSSM Standards for Mathematical Practice (SMP) and NCTM’s Process Standards (PS), we explicitly examine how the previous episodes on length are related to these practices and processes.

### Standards for Mathematical Practice

Students clearly tried to make sense of the problems, but not only did the sense making differ from student to student it also evolved over instructional time. Students made sense of the concept of length by straightening paths, by matching equal sublengths, and by using increasingly more sophisticated counting (SMP 1a, b, g). They translated between different representations—numerical counting and spatial unit-iteration—both concretely and pictorially (SMP 1f). They made ever-increasing sense of counted quantities (SMP 2a). They constructed and evaluated arguments, and gave explanations and justifications for their work (SMP 3a, d, f). They applied the mathematics of counting and the concept

of length to a real-world situation depicted in the Home to School problem (SMP 4a). They identified important quantities and made sense of the numerical results as their notions of what must be counted evolved (SMP 4c, d). They used appropriate tools of inch-rods (SMP 5). They attended to precision as they moved away from eraser estimations and became more precise about how counting could be used in a way relevant to the problem, in essence defining in action a definition for the appropriate unit to enumerate (SMP 6b). They communicated precisely (SMP 6a). They saw structure when they were able to use one-to-one matching of unit segments in two wires (SMP 7); that is, they saw that both wires were made from the same linear components and thus had the same length. Finally, they continually evaluated their methods (SMP 8d).

## Process Standards

Clearly the students built new mathematical knowledge through problem solving by implementing, discussing, and evaluating solution strategies (PS 1a, b). They reasoned and justified, developed mathematics arguments, and communicated and evaluated their thinking and strategies (PS 2a, c; PS 3a, b). They connected counting and spatial iteration and applied mathematics (PS 4a, b). They represented spatial unit-length iteration with counting (PS 5a, b). As shown below, some students even progressed from counting to reasoning with addition and fractions (PS 4a, PS 5b).

Kerri: I separated this one in half [*draws a vertical segment separating the bottom wire into two parts (fig. 1.10)*], and I knew 4 plus 4 was 8. And the top wire is 4 plus 4 [*circling the right and left sides of the top wire*] plus 1 in the middle [*pointing*]. The top is longer.

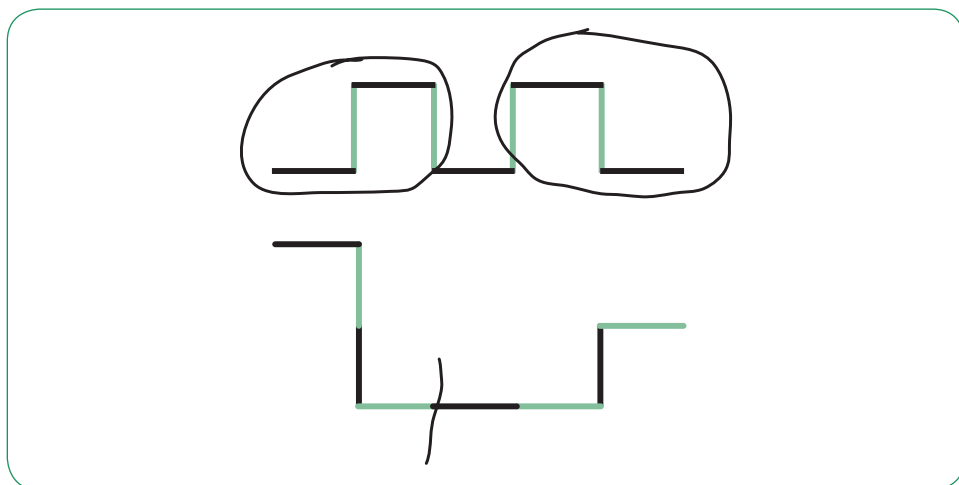


Fig. 1.10. Kerri's work



# Deepening the Reasoning

The reasoning students used as described above can be extended in a number of ways. One way is to consider missing-length perimeter problems. Start with problems that help students learn to use one property of rectangles (opposite sides are equal) to find perimeter.

**Perimeter Problem.** Find the perimeter of the rectangle in figure 1.11a. (The perimeter of a shape is the distance traveled as you trace around it.) Check your answer by drawing the shape on square grid paper (see fig. 1.11b for how a student might draw the situation).

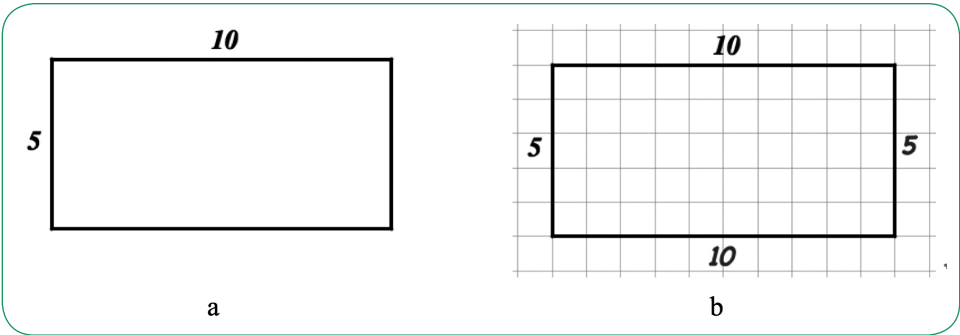


Fig. 1.11. Diagrams for the Perimeter problem

As students do several problems of this type, have them look for patterns in the measurements. They should discover that the opposite sides of rectangles are equal. Do enough problems so that students can correctly predict the perimeters before checking with grid paper (but let them check with the grids as long as they need them). After students have some proficiency with the rectangle problems, have them investigate more difficult problems such as the following.

**Comparing Perimeters.** Which shape has the greater perimeter, or do they have the same perimeter? (The perimeter of a shape is the distance traveled as you trace around it.) Finding rectangles in the shapes can help you.

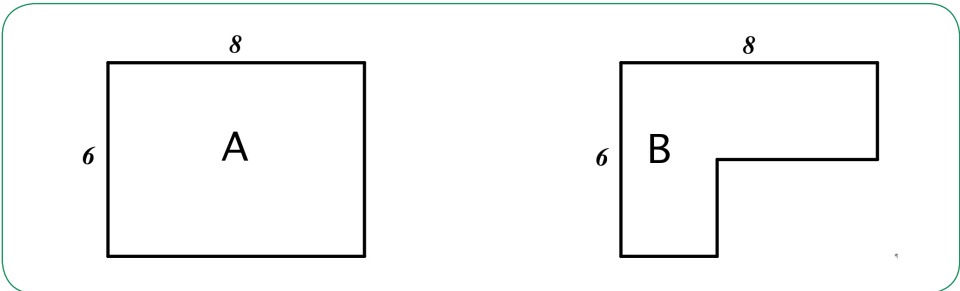


Fig. 1.12. Diagrams for Problem 3

Emily: They're the same perimeter. Because if you move this side down [motioning along arrow a as shown in figure 1.13], you get this segment [drawing the horizontal dotted segment]. And if you move this side over [motioning along arrow b], you get this segment [drawing the vertical dotted segment]. So the sides in B make the sides in A. They're equal.

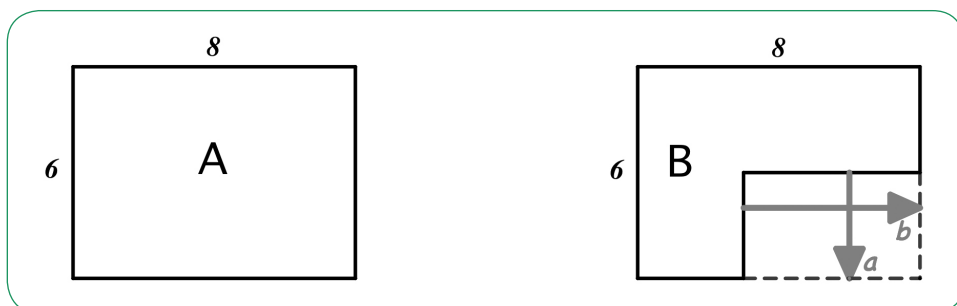


Fig. 1.13. A solution for Problem 3

Teacher: How can we check our answer?

Jordan: Maybe draw the shapes on graph paper.

Teacher: Okay, everybody do that.

Teacher: What did you find? Can somebody show us on the document projector? Jordan.

Jordan: Okay. I counted units on each side and got that A is 28 and B is 28. I also sort of checked what Emily said. See this 3 and 3 [numbers for unknown vertical segments on the right in B in figure 1.14] equals the 6 over here [right vertical segment marked with 6 in A]. And this 3 and 5 [numbers for bottom horizontal unknown segments in B] make 8 like the 8 down here [bottom 8 in A]. So Emily was right. I just needed to see the numbers to get it.

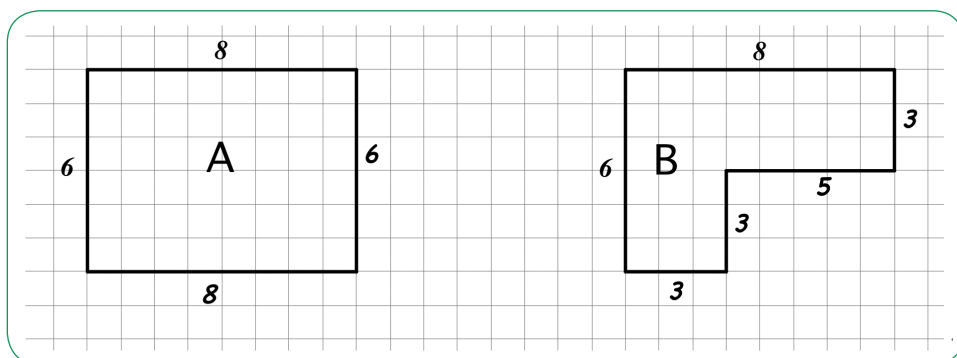


Fig. 1.14. Checking the solution on graph paper

Although some students can make sense of Emily's logical argument, many other students, like Jordan, need to see numbers to make sense of the argument. Showing both the empirical (Jordan) and logical (Emily) arguments on numerous problems will help students like Jordan start making logical arguments like Emily.

Some students will need to build shapes with interconnecting rods to help them make sense of the reasoning that if one shape can be made from the other shape by rearranging its sides, the two shapes have equal perimeters. For example, in the top portion of figure 1.15, students can use the rods to show that shape B can be made from shape A by moving rod  $p$  to the left and rod  $q$  up. For students who need more support in this sense making, we can even take the shapes apart and rearrange them in straight lines to show that they have the same length. Also, some students might reason more quantitatively that shapes A and B have the same length because they both can be made from 2 congruent long rods, 4 congruent medium rods, and 2 congruent small rods. Decomposing shapes into parts and rearranging the parts is a powerful form of geometric reasoning that is useful not only in length comparisons but also in area and volume comparisons.

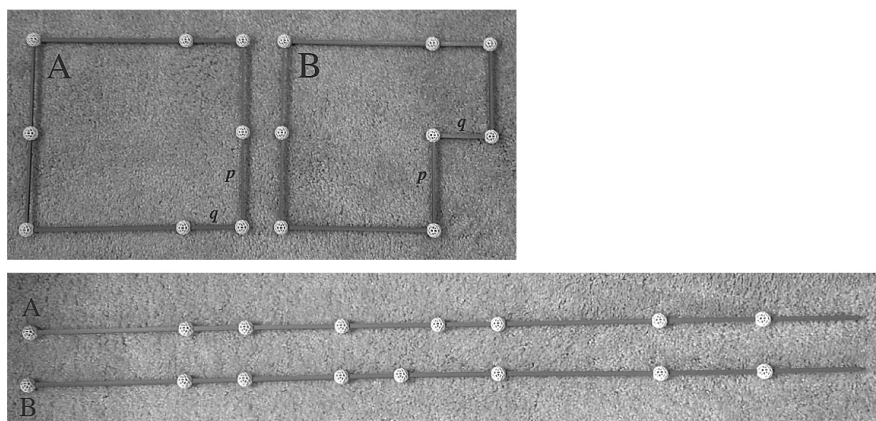


Fig. 1.15. Decomposing the shapes

## Concluding Remarks on Mathematical Reasoning and Sense Making

To use mathematics to make sense of the world, students must first make sense of mathematics. To make sense of mathematics, students must transition from intuitive, informal reasoning based on their interactions with the world to precise reasoning based on formal mathematical concepts, procedures, and symbols. The key to helping students make this transition is providing appropriate

instructional tasks that target precisely those concepts and ways of reasoning that students are currently ready acquire. And the key to providing this support is an understanding of research-based descriptions of the development of students' increasingly more sophisticated conceptualizations and reasoning about particular mathematical concepts. Understanding students' mathematical thinking is critical for selecting and creating instructional tasks, asking appropriate questions of students, guiding classroom discussions, adapting instruction to students' needs, understanding students' reasoning, assessing students' learning progress, and diagnosing and remediating students' learning difficulties. The chapters in this book use research on student learning to help teachers monitor, understand, and guide the development of students' reasoning and sense making about core ideas in elementary school mathematics.

## Notes

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2. For instruction on the Home to School problem, draw the paths on square-inch grid paper, and have individual inch-rods available. Also useful are sets of inch-rods strung on flexible wires so that students can make the problem paths and straighten them.

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