CHAPTER 3

Mathematical Habits of Mind

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THE FIRST two chapters of this volume present a view of doing and learning mathematics as actively making sense of problematic situations. We extend that view by making the point that mathematical thinking involves more than just understanding important mathematical ideas and learning to apply useful methods and procedures. Modes of thought, or habits of mind, exist that transcend content knowledge and are taken for granted by mathematicians as essential facets of their work. These habits of mind are useful for reasoning about the world from a quantitative or spatial perspective and for reasoning about the mathematical content itself, both within and across mathematical fields.

Our teaching experience over the years has convinced us that these habits of mind should not, and probably cannot, be taught by studying the "top-ten habits of mind." Rather, the requirement is an integrated program of immersion in developing concepts and skills with accompanying reflection, solving problems of all types, making and using abstractions, and building and applying mathematical theories. In short, students develop these habits of mind as a by-product of learning mathematics through problem solving using approaches described in the other chapters of this volume. We begin by describing some of the most important mathematical habits of mind using sample problems to illustrate the meaning of each habit. We then discuss how students acquire these habits of mind and offer some suggestions for teachers who want to help students develop these habits.

Habits of Mind

In this section, we identify some of the most significant and useful habits of mind. Space limitations preclude any attempt on our part to present a comprehensive classification of habits of mind. Some habits on this list will overlap to some extent, and surely some habits exist that will be missing. For more exhaustive lists, see Cuoco and others (1996) and Goldenberg (1996). We present each habit with a brief description and an example of its use in a problem situation.

Guessing Is Not Necessarily Bad—We All Do It!

When students are engaged in mathematical problem-solving activities that are challenging to them, they should be encouraged to develop the habit of trying reasoned guessing to help them better understand the problem and move toward a solution. For example, consider the triangle-dissection problem:

> What triangles can be divided into two isosceles triangles?

Please do not read any further if you have not tried the problem!

With enough experience doing geometry problems, students learn that problems that involve triangles often have a solution that is special—maybe an isosceles or right triangle. So guessing that the solution to the triangle-dissection problem is special in some way is reasonable. You as the teacher should encourage this kind of guessing. Isosceles triangles would be a reasonable guess, but they lead to a dead end. They cannot generally be "cloned." In contrast, right triangles do lead to a solution. If we brainstorm what we know about right triangles, the image in figure 3.1 might come to mind. Adding a radial line to the right angle in figure 3.1 makes it clear how right triangles satisfy the conditions of the problem.

The answer to the triangle-dissection problem is all right triangles. Right? We return to this question in our discussion of the next habit of mind.

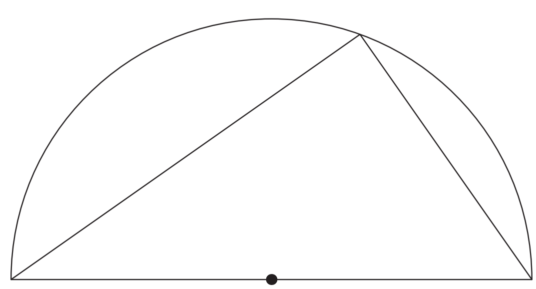

Fig. 3.1. Right triangle inscribed in a semicircle

Challenge Solutions, Even Correct Ones

Once students have found a potential solution to a problem, they need to develop the habit of challenging the solution by looking back over the problem. For example, “all right triangles” seems like a nice solution to the triangle-dissection problem. The solver is tempted to “check off” this problem and move on to the next, but another family of solutions to this problem remains to be found. Reflecting on the way right triangles solve the problem is probably the most natural way to discover the second family. With right triangles, the dissection line is a leg of both isosceles triangles. If we try to fit two isosceles triangles together so that the dissection line is a base of one triangle and a leg of the other triangle, we are forced to draw a picture something like figure 3.2. A new family of solutions, not totally disjoint from the first, emerges. Is that all there is? We leave this question to the reader and go on to the next habit of mind.

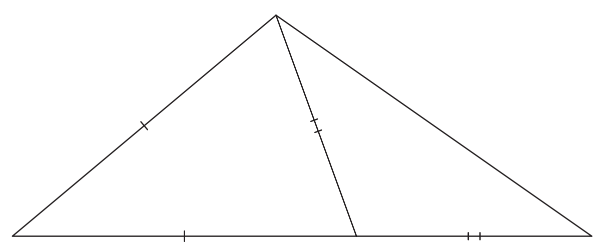

Fig. 3.2. Isosceles triangle dissected into two isosceles triangles

Look for Patterns

Students should be in the habit of looking for patterns. Consider the two-pan-balance problem with powers of 3:

> Given weights of 1, 3, 9, 27, 81, 243, and 729 grams, what weights can you measure? If you want to upgrade your set of weights, what is the next weight you might buy?

You would expect your students to try a few examples, but taking the step to create an organized table of examples is far more useful (see table 3.1). Once the results have been tabulated, students are then in a better position to recognize patterns. What patterns do they see here? Where are the natural dividing lines in the cases, and where will the next one occur?

Table 3.1. Small weight measurements using powers of 3

Weight	Left Pan	Right Pan	Weight	Left Pan	Right Pan
1		1	9		9
2	1	3	10		9, 1
3		3	11	1	9, 3
4		3, 1	12		9, 3
5	3, 1	9	13		9, 3, 1
6	3	9	14	9, 3, 1	27
7	3	9, 1	15	9, 3	27
8	1	9			

Work still remains to be done to describe how the patterns in table 3.1 help answer the question, but the patterns can be a key, and they are unlikely to be obvious as the solver starts working on the problem.

Conserve Memory

One habit of many mathematicians is that they try to memorize as little as possible. Students should be encouraged to develop the same habit. For example, most mathematicians use the identity for the cosine of the sum of angles infrequently. Instead of memorizing the identity, they might just remember the general pattern for the sin (or cosine) of a sum (or difference), which is

$$(\mathrm{SC})(\alpha \pm \beta) = (\mathrm{SC})(\alpha)(\mathrm{SC})(\beta) \pm (\mathrm{SC})(\alpha)(\mathrm{SC})(\beta),$$

where each (SC) is either sin or cos. They should not have difficulty remembering that exactly two of the SCs on the right are cosine and two are sine. Knowing a few of the more basic facts about the trigonometric functions lets us find the right combination if we really need it. For example, if we are looking for $\cos(\alpha + \beta)$ and we consider the specific case where $\alpha = \beta = 0$, we can rule out the case where each term on the right has sine as a factor.

Does this process seem like more work than just memorizing? Before deciding, two considerations should be kept in mind. First, this derivation is done on a personal level—it is internalized and does not have to play out completely in many cases. Second, the correct identity,

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$

is now in the short-term memory. It will be memorized for a while, and this temporary retention is sufficient because people tend to use identities and other facts in clusters.

A good understanding of underlying mathematical ideas is required to conserve memory, but the habitual practice of conserving memory in this way helps develop mathematical understanding. In fact, we believe that this sort of mutually supportive relationship with mathematical understanding holds true for each of the habits of mind.

Specialize—Sometimes Everything Is Special

Sometimes insight can be gained about a given problem by analyzing a special case. A classic example of turning a problem into a special case is completing the square. Instead of just learning the mechanics of completing the square, your students should be aware of its significance. In fact every quadratic equation can be reduced to the special case of solving $x^2 = c$.

Special cases also arise when mathematicians begin the solution of a problem with "Without loss of generality…." This statement is a kind of code for saying that we are going to make an assumption that does not restrict the problem in any way. Consider the commuter problem that follows:

> If I drive to work at 60 miles per hour and come back home along the same route at 50 miles per hour, what is my average speed for the two-way commute?

We are not told the distance between home and work; so we reasonably start by making an assumption about the distance: "Without loss of generality, we assume that the commute to work is [fill in almost anything] miles." With that assumption, the solution emerges, and upon reflection we realize that the commuting distance has no bearing on the average speed. In this example, the special case not only adds new insight but actually leads to a solution of the original problem.

Use Alternative Representations

Another mathematical habit of mind that we want to encourage students to develop is to represent a problem in various ways. For example, consider the postage-stamp-arithmetic problem that follows:

> Given an unlimited supply of 7-cent and 10-cent stamps, what postage amounts can you make? Which ones can you not make?

The postage-stamp-arithmetic problem can be approached using only arithmetic, but new insights can be gained by introducing coordinate geometry to the analysis. See figure 3.3. The problem essentially asks what positive numbers can be expressed in the form $7a + 10b$, where a and b are both nonnegative integers. These nonnegative combinations of 7 and 10 can be represented as lattice points on the first quadrant, including the axes.

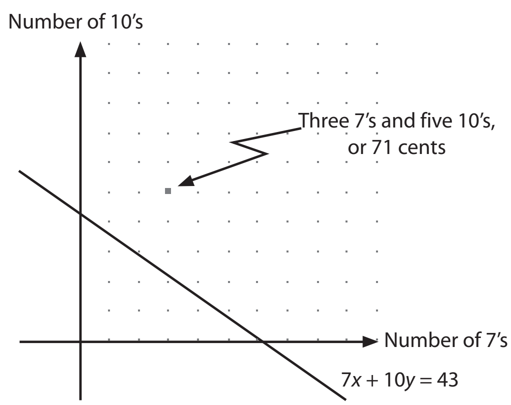

Fig. 3.3. Graph of solution set of the postage-stamp-arithmetic problem

Now the question reduces to asking, Which lines of the form $7a + 10b = k$ pass through these points? This inquiry is not a solution, but it is a different, sometimes useful way to look at the problem. Encourage your students to look for different representations of the same problem. Many more examples of this type appear in a recent yearbook of the National Council of Teachers of Mathematics (2001).

Carefully Classify

Another good mathematical habit is to carefully classify the outcomes in a problem, especially in one that involves complex counting. For example, the dial of a lock pictured in figure 3.4 was made by the Simplex Company. Simplex advertised "thousands of combinations." A combination is an ordered sequence of zero to five pushes. In each push, the user presses from one to five buttons. Once a button is pressed, it stays in, so it cannot be reused in a subsequent step. Does the Simplex Company's advertisement tell the truth?

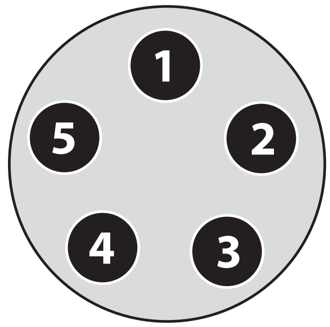

Fig. 3.4. Dial of a lock made by the Simplex Company

One way to solve the combination-lock problem is to classify the possible button sequences according to their "shapes." By shape, we mean the number of buttons that are pressed, such as 3-1-1 or 1-2-1. Counting the number of shapes is a manageable problem, and the number of button sequences with each shape is relatively easy to compute. For example the number of button sequences with the shape 1-2-1 is

$$5 \times C(4,2) \times 2 = 5 \times 6 \times 2 = 60,$$

where $C(4,2)$ is the binomial coefficient "4 choose 2." Counting all 1082 of the button sequences is mostly a matter of organizing this process. For more details, see Velleman and Call (1995).

Think Algebraically

Using the logic of algebra and perhaps, but not necessarily, algebraic symbol manipulation is also a good habit for students to develop. Consider the "arithmagon" problem from Mason (1985).

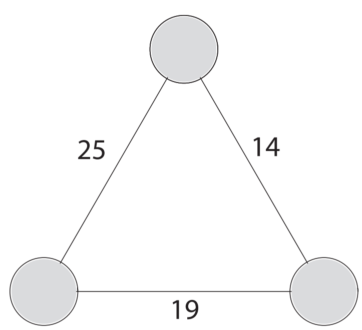

Fig. 3.5. Arithmagon problem with sums of secret numbers on the edges

> Each of the vertices of the triangle in figure 3.5 has been assigned a secret number. Each edge is labeled with the sum of the secret numbers of vertices it connects. What are the secret numbers?

The “arithmagon” problem is not really challenging for those of us who teach algebra. Starting from the top and going clockwise, if we let the unknown numbers at the vertices be x, y, and z, then the problem reduces to solving the linear system

$$x + y = 14,$$

$$y + z = 19,$$

$$x + z = 25.$$

However, thinking algebraically does not necessarily mean knowing how to create or solve this system. In fact, the observation that if we add the three numbers 14, 19, and 25, we will get two times the sum of the secret numbers leads to what some people might consider a more elegant solution. To make this observation, we must be able to think abstractly about the numbers, and the logic is definitely algebraic. It is also a line of reasoning that has been observed from, and understood by, “prealgebra” students. The point is that a teacher does not need to start with formal algebra to expose students to the logic that is used in the subject.

How Can Students Acquire Useful Habits of Mind?

The habits described here should not ordinarily be the explicit objects of our teaching; rather, each student should internalize them as they do mathematics. Part of the way this outcome can be achieved is by teachers’ modeling the very habits we want students to develop. However, the crucial element is that students be given the opportunity to develop mathematical understanding through problem solving. The problems should not all be difficult, but they should challenge students to think about, and make sense of, the problems and the mathematics that underlies them. Reflection on solution methods is also crucial. In short, classrooms in which mathematics is taught through problem solving as described in this volume are excellent settings for these habits of mind to develop.

Teachers can facilitate students’ development of these habits by making the habits explicit and by encouraging their students to reflect on them when opportunities arise. We have found that student journals are a good tool to foster explication of, and reflection on, habits of mind. We suggest that students record the highlights of their solution methods as they attempt a problem. They should not record everything that pops into their heads, but rather, they should aim for entries that show insights into their thinking as described in the foregoing habits of mind. Here are some items that you might want to suggest for your students' journals. Their connections with the various habits should be clear.

- A rewording of the problem, including an identification of variables, assumptions, and what is to be solved for or proved.
- A question of clarification about the problem. Even if as the teacher you are going to answer the question immediately, encourage students to record the question first.
- A strategy that might help solve the problem. For example, tabulating some data might lead to a solution to some problems. Encourage students to record what they are going to try before actually launching the attempt.
- Conjectures. Some conjectures are based on "gut feelings" that are difficult to explain, but you should encourage students to record them. Conjectures that are based on a plausible reason are even better. For example, suppose that you ask for the dimensions of a rectangle with given perimeter having maximal area. A guess that the solution is a square might be purely a guess, but it also might be based on an understanding that at the extreme (a very long skinny rectangle) the area is nearly zero.
- Guesses. Many nonroutine problems are "`inverses" of routine ones. For example, many students can easily solve this problem:

  > Mary drives from Boston to Washington, D.C., a trip of 500 miles. If she travels at an average of 60 mph on the way to Washington and 50 mph on the way back, how many hours does her trip take?

  But they have a difficult time with this one:

  > Mary drives from Boston to Washington, D.C., and she travels at an average of 60 mph on the way to

Washington and 50 mph on the way back. If the total trip takes 18.5 hours, how far is Boston from Washington?

Both problems involve the same functional relationship, but the second one requires that the relationship be inverted. Repeatedly guessing at an answer to the second problem and then checking each guess allow students to formalize the underlying relationship so that it can be inverted (see Cuoco [1993]).

- A drawing, symbolic notation, or alternative representation of anything that relates to the question. Suppose that a student is working on a problem that involves a line of people who are facing left or right. A representation using left and right arrows reveals something about the student's facility with abstraction.
- Solutions to special cases of the problem. For example, in the triangle-dissection problem, a description of all isosceles triangles that can be divided into two isosceles triangles is, in a sense, an extremely small step. Yet in many problems, such small steps can be scaffolded to a complete solution.
- The statement of another related problem, with or without a solution. What triangles can be divided into two similar triangles or two right triangles? Such a problem may prove to be either challenging or trivial, but in either situation, thinking of related alternative problems should be encouraged.

As students compose, enter, and reexamine journal entries about many different problems, they are likely to witness the emergence of many of the described habits of mind as patterns in their thinking. Their explicit recognition of the habits in their journals has the effect of further strengthening the habits for students.

Conclusion

Mathematical habits of mind develop as a by-product of teaching mathematics through problem solving. As you select problem sequences for teaching any particular content area, you can anticipate that certain habits will be strengthened. One aspect that makes the process fascinating is that students often come up with a surprising, unanticipated approach that fosters a different habit. Periodically, you might want to review a list of habits that you collect from this chapter and other references. If you note that your class has not encountered a particular habit as frequently as you would like, you might try to lead students in the direction of that habit through problem selection or by your own modeling of the habit. You might use the habit of mind as you introduce a concept or as you reflect on a problem that the class has done.

Finally, we want to emphasize a facet of the process that we did not have the space to address in this chapter, namely, the design and selection of orchestrated problem sets. Most of the problems we have used in this chapter can be extended to form such a problem set. For example, an introduction to geometric proof could be built around the triangle-dissection problem. For more details on selecting and creating problem tasks and task sequences, see chapter 5, by Marcus and Fey, in this volume.

Devising and perpetuating a problem-solving environment that fosters good mathematical habits of mind in students is a habit that all mathematics teachers should acquire.