NAVIGATIONS SERIES

GRADES 9–12

NAVIGATING *through* GEOMETRY

Chapter 1

Transforming Our World

Geometry has played a vital role in mathematics for centuries. It formed the basis for much of the study of mathematics in the past and continues to allow students to explore topics that extend to other branches of mathematics and other disciplines. To explore the visual appeal of geometry and to think about the usefulness of the subject to engineers, scientists, artists, mathematicians, and others, we set the stage in this chapter for a systematic approach to geometry using transformations. *Principles and Standards for School Mathematics* (NCTM 2000) states that students in grades 9–12 should be able to "apply transformations and use symmetry to analyze mathematical situations" (p. 308). The initial, distance-preserving transformations (isometries) used by high school students to examine geometric ideas are translations, reflections, rotations, and glide reflections, as shown in table 1.1.

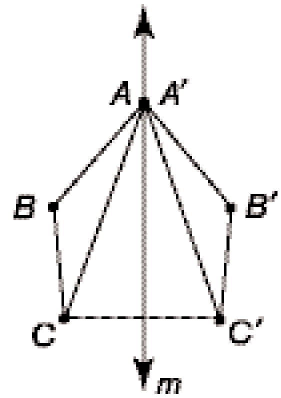
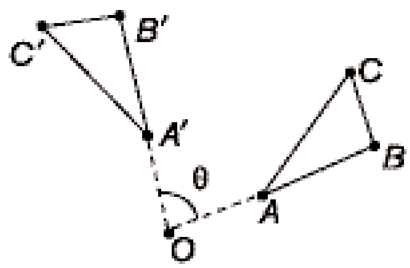
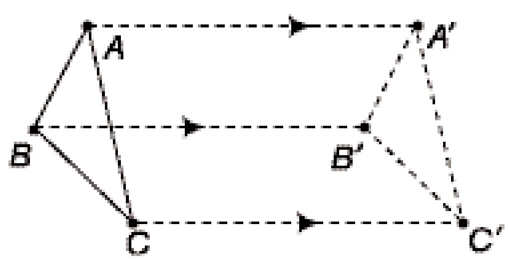
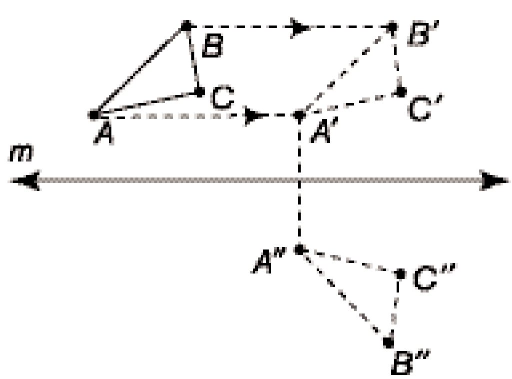
As teachers acquaint their students with the information in the table, they may wish to ask students why it is customary for studies of the properties of transformations to consider what happens to just three noncollinear points of the plane (or the triangle determined by those three points). To help students respond, teachers might ask them how a plane is determined. If they know what will happen to three noncollinear points, can they determine what will happen to other points of the plane?

Two of the expectations for students in grades 9–12 expressed in *Principles and Standards* are that they be able to—

- understand and represent translations, reflections, rotations, and dilations [discussed in chapter 3 of the current book] of objects in the plane by using sketches, coordinates, vectors, function notation, and matrices;

Table 1.1

Representations of reflection, translation, rotation, and glide reflection

Type of Transformation	Representation	Properties
Reflection in line m		Any point A on m is its own image. For any point C not on m, m is the perpendicular bisector of $\overline{CC'}$ where C' is the image of C. We write $R_m(\triangle ABC) = \triangle A'B'C'$.
Rotation with center O and with measure θ		Each point A is rotated $\theta°$ on a circle whose center is O and whose radius is OA. (If θ is positive, the rotation is counter-clockwise; if θ is negative, then the rotation is clockwise.) We write $R_{O,\theta}(\triangle ABC) = \triangle A'B'C'$.
Translation by vector $\overrightarrow{AA'}$		Each point A is moved along a vector $\overrightarrow{AA'}$ so that $AA' = BB' = CC'$ and $\overleftrightarrow{AA'} \parallel \overleftrightarrow{BB'} \parallel \overleftrightarrow{CC'}$. We write $T_{A \to A'}(\triangle ABC) = \triangle A'B'C'$.
Glide reflection with translation by vector $\overrightarrow{AA'}$ and reflection in line m		A point A is translated along a vector $\overrightarrow{AA'}$ and then reflected in line m, where $\overleftrightarrow{AA'} \parallel m$. We write the glide reflection as a composition of functions as $R_m(T_{A \to A'}\triangle ABC) = R_m(\triangle A'B'C')$ $= \triangle A''B''C''$.

- use various representations to help understand the effects of simple transformations and their compositions. (NCTM 2000, p. 308)

To understand and represent transformations mathematically, students need to see and use the necessary mathematical objects. This requirement does not preclude students from seeing the geometry in a real-world setting but demands that they work with different representations of the objects in order to understand what they are seeing from a mathematical point of view. To use transformations to analyze a mathematical situation, consider the wallpaper sample in figure 1.1.

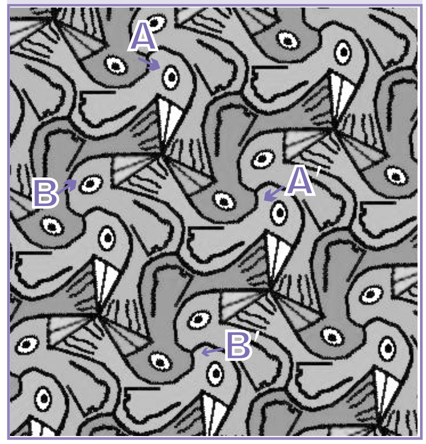
If the shading in the figure is ignored, transformations can be used to create any bird from any other bird. For example, the bird marked A can be translated to obtain the bird marked A', and the bird marked B can be rotated to obtain the one marked B'.

A transformation approach to geometry provides a formal approach to Euclid's technique of moving one figure on top of another to determine congruence. The mathematical study of transformations was organized by Felix Klein (1849–1925) in the *Erlanger Programm*, in which he described geometry as the study of properties that remain unchanged (invariant) under the group of transformations of a plane (Eves 1969, pp. 187–88). Though transformations were systematized around the beginning of the twentieth century, they were not widely included in secondary school curricula for many years, until their appearance in *Modern Mathematics*, Volumes 1–2 (Papy 1968); *Motion Geometry*, Books 1–4, from the University of Illinois Committee on School Mathematics (Phillips and Zwoyer 1969); *Geometry: A Transformational Approach* (Coxford and Usiskin 1971); and *Geometry: Constructions and Transformations* (Dayoub and Lott 1977). Since the publication of these books, a transformation approach has appeared in many texts, such as those by Hoffer (1979) and Serra (1989).

Although reflections, translations, and rotations are the primary transformations considered in these texts and in *Principles and Standards*, examination of an additional function, the glide reflection, helps to complete the picture. To consider various representations of transformations in this chapter, we present activities that range from paper folding to matrix operations to geometric constructions using technology.

Fig. **1.1.**

Wallpaper created by Tessellation Exploration.

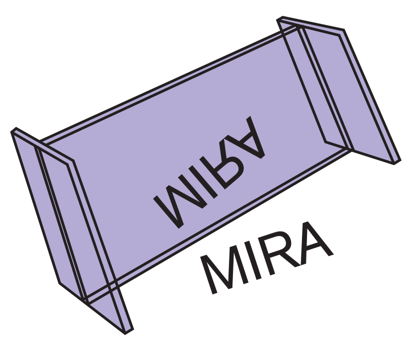


Tessellation Exploration is available in a demonstration version on the CD-ROM, courtesy of Tom Snyder Productions.

What Are the Transformations?

Reflections

A reflection is a primary building block for all isometries, and a variety of concrete methods can be used to study it. In middle school, a student may have constructed a reflection by folding paper or using a reflective device such as a Mira (a small plastic instrument that students can use to create a reflection of a figure and draw its image by looking through the plastic). This concrete approach is reviewed in the following activity, Fold Me! Flip Me!



Fold Me! Flip Me!

Goals

- Draw and construct two-dimensional geometric objects using a variety of tools
- Understand and represent reflections
- Establish conjectures and proof by means of constructions

Materials and Equipment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- A copy of the activity pages for each student
- Paper for paper folding
- A Mira or an equivalent reflective device for each student

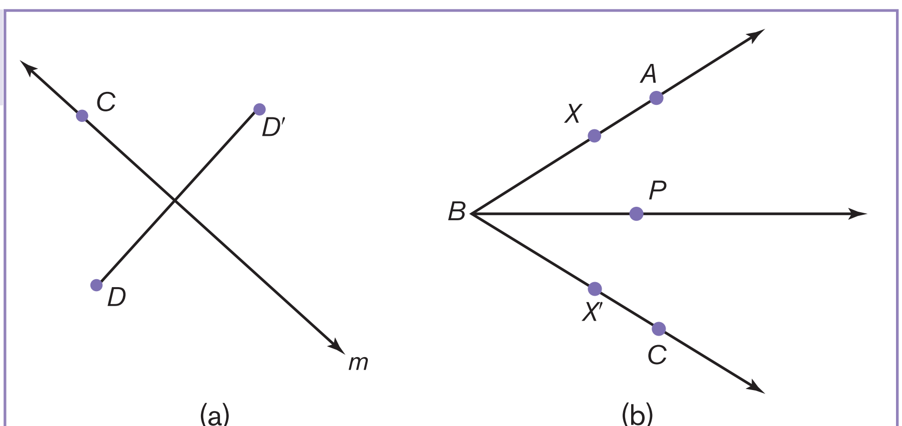
Discussion of the Activity

The activity shows that properties of a reflection, including the fact that the reflecting line is the perpendicular bisector of the segment connecting a point and its image, aid in the study of diverse topics in mathematics—such as line symmetry in a figure and the notion of an even function in algebra—as well as in archaeology. The activity directs students to use both paper folding and a reflective device to construct reflection images. Students should consider that, as shown in figure 1.2a, the reflecting line m is the perpendicular bisector of every segment connecting a point and its image under the reflection. Either folding paper or using a Mira (or similar device) will convince a student intuitively that this is true.

Other observations in the early stages of this activity are that (1) if any part of a figure is drawn on the same side of line m as D, then the image of that figure reflected in line m must be on the same side of m as D'; and (2) the *orientation* of a figure changes when it is reflected. That is, if the labels on a figure appear in the order A, B, C when they are read clockwise, then the labels on the image of that figure under a reflection will appear in the order A', C', B' when they are read clockwise as well.

An application of the fact that the reflecting line is the perpendicular bisector of a segment connecting a point and its image can be used to prove the following result: *Every point on an angle bisector is equidistant*

Fig. **1.2.**
Reflection in line m

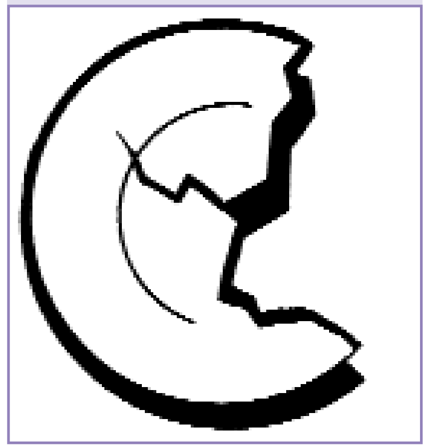


from the sides of the angle that it bisects. To see that this is true, consider figure 1.2b. Let $\overline{PX}$ be perpendicular to $\overrightarrow{BA}$, and let $\overline{PX'}$ be perpendicular to $\overrightarrow{BC}$. As a result, $\triangle X'BP$, the image of $\triangle XBP$ in the line containing $\overrightarrow{BP}$, is congruent to $\triangle XBP$. We know that this is so because the reflection preserves both angle size and distance. Therefore, the image of $\overrightarrow{BA}$ must be $\overrightarrow{BC}$; the image of $\angle ABP$ must be $\angle CBP$, and $\overline{BP}$ is its own image. Thus, the triangles are congruent by Side-Angle-Side (SAS), and $\overline{PX}$ and $\overline{PX'}$ must be the same length. Hence, any point P on the angle bisector must be equidistant from the sides of the bisected angle. Students should be able to argue in a similar manner that *every point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment.*

Students use the last result when they try to locate the center of an ancient plate from the remaining pottery shard pictured in figure 1.3. This problem presents them with an application of reflection (or line) symmetry. Because a circle has infinitely many lines of symmetry and each line of symmetry can be considered a reflecting line that maps the circle onto itself, the lines of symmetry are the perpendicular bisectors of the chords of the circle, and all the lines contain the center of the circle. Thus, by finding the intersection of any two lines of symmetry of the circle that contains the shard, students can find the center of the circle and determine its radius.

Fig. **1.3.**

Pottery shard from a plate

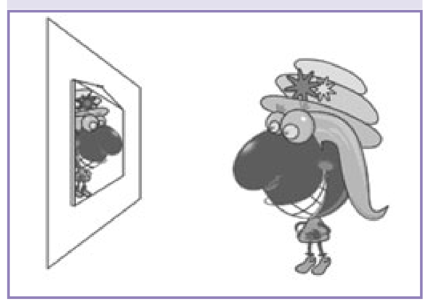


Exploring Mirror Images

Determining the size of an image in a mirror is a different application of a reflection, this time from the realm of science. Figure 1.4 shows the character named Polygon from the Figure This! Web site (www.figurethis.org). Polygon is looking at her reflection in a mirror hung flat on a wall. If the mirror image depicts Polygon from head to foot, how much of herself can she see as she walks toward the mirror or away from it? To answer this question, we use a property of a reflection in a mirror: *The angle of incidence is equal to the angle of reflection.* This property, depicted in figure 1.5, is explored in the activity Mirror, Mirror, on the Wall.

Fig. **1.4.**

Figure This! character Polygon reflected in a mirror



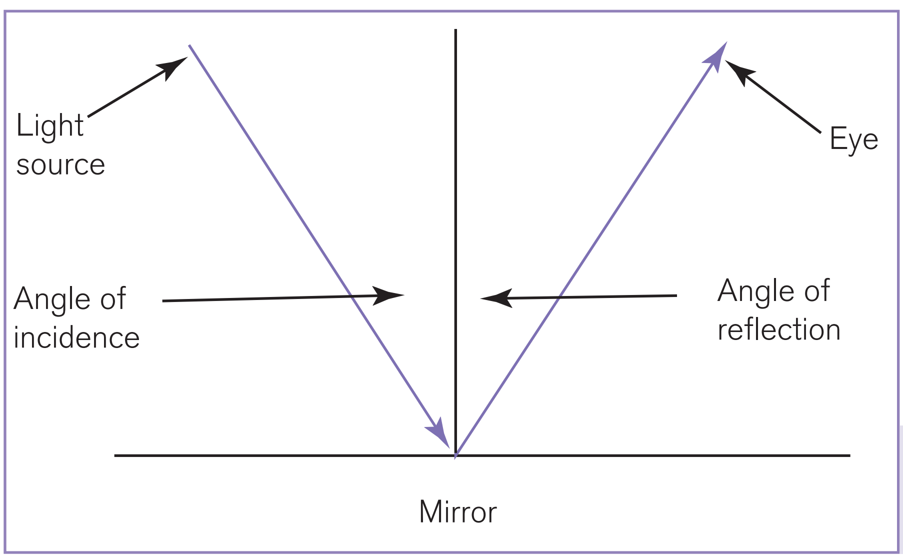


Fig. **1.5.**

Angle of incidence is congruent to the angle of reflection