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Editorial

Introducing the *JRME* Editor-Designate

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Once again, the wheels of change are turning at *JRME*. This issue marks the beginning of Volume 47, which will be the last volume under my editorship. To facilitate the transition to the new editor, there is a 1-year period during which the work of the journal is shared by the editor and the editor-designate. I am pleased to announce that Jinfa Cai, Professor of Mathematical Sciences at the University of Delaware, is the *JRME* editor-designate. His term as *JRME* editor will span Volumes 48–51, from 2017–2020.

Although volumes of the journal correspond to calendar years, the tasks of the editor follow a different timeline, which is governed by production schedules. Thus, the work of the editor-designate begins long before his or her name appears as editor on the journal cover. For example, Jinfa and the University of Delaware editorial team began handling all new manuscript submissions on September 1, 2015. The Illinois State team continued to process manuscripts that were already in the system and conducted reviews for resubmissions of manuscripts that had previously received *revise and resubmit* decisions. As of January 1, Jinfa and his team will have assumed responsibility for all aspects of the submission and review process. During this final phase of the transition, my role, with the assistance of Amanda Fain, will be to work with authors to complete the final editing of articles that will appear in Volume 47.

Another aspect of the transition involves the special section editors, who are appointed by the *JRME* editor. As you may have noticed from previous issues, some new appointments have already occurred: Steve Williams is now serving as the Research Commentary editor, and Sarah Lubienski has taken on the role of Book Review editor. Randy Groth will continue as the Monograph editor.

The gradual shifting of responsibilities between the *JRME* editor and the editor-designate in this transition period allows for a seamless change in the stewardship of the journal. While the work at one editorial office is winding down, the work at the other is ramping up. I would like to thank the NCTM Board of Directors for supporting both editorial offices during this period of transition.

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Research Commentary

Educational Technology: An Equity Challenge to the Common Core

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The implementation of the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) has the potential to move forward key features of standards-based reforms in mathematics that have been promoted in the United States for more than 2 decades (e.g., National Council of Teachers of Mathematics, 1989, 2000; National Science Foundation, 1996). We believe that this is an especially opportune time to purposely focus on improving the mathematics education of students who have historically been denied access to a high-quality and rigorous mathematics education in the United States, specifically low-income students and students of color (e.g., Kitchen, DePree, Celedón-Pattichis, & Brinkerhoff, 2007; Leonard & Martin, 2013). We discuss a challenge to realizing standards-based reforms in mathematics in the United States: computer-based interventions in mathematics classrooms.

Key words: Computer-assisted instruction; Diversity and equity; Standards-based reforms

This is an important time in the history of mathematics education in the United States. As of this writing, 43 of 50 states, the District of Columbia, four U.S. territories, and the Department of Defense Education Activity have adopted the Common Core State Standards (Common Core State Standards Initiative, 2015). We believe that the implementation of the Common Core State Standards for Mathematics (CCSSM; National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010) has the potential to move forward key features of standards-based reforms in mathematics[1] that have been promoted in the United States for more than 2 decades (e.g., National Council of Teachers of Mathematics [NCTM], 1989, 2000; National Science Foundation [NSF], 1996). In particular, we consider this to be an opportune time to purposefully focus on improving the mathematics education of low-income students and culturally or linguistically diverse students[2] who have historically been denied access to a high-quality and rigorous mathematics education in the United States (e.g., Jacobsen, Mistele, & Sriraman, 2013; Kitchen,

We would like to acknowledge the helpful suggestions that Dr. Nicole Joseph, assistant professor at the University of Denver, made as a reviewer of a previous draft of this Research Commentary. We would also like to thank the anonymous reviewers of this commentary, who provided valuable suggestions as well.

DePree, Celedón-Pattichis, & Brinkerhoff, 2007; Leonard & Martin, 2013; Téllez, Moschkovich, & Civil, 2011). In this Research Commentary, we argue that computer-assisted instruction (CAI) presents a challenge to realizing standards-based mathematics reforms for underserved students in the United States.[3]

During the 2009–2010 school year, more than 21 million students in the United States attended schools that received supplemental federal funding (i.e., Title I funding) to improve the academic achievement of children from low-income families (U.S. Department of Education, 2014). This was approximately 44% of all students in Kindergarten to Grade 12 (Hussar & Bailey, 2013). Although these are the most recent data available, the current figures are likely to be similar. In recent years, there has been an influx of federal dollars for educational interventions for Title I schools as part of the No Child Left Behind (NCLB) Act of 2001.[4] The purpose of NCLB was "to ensure that all children have a fair, equal, and significant opportunity to obtain a high-quality education and reach, at a minimum, proficiency on challenging State academic achievement standards and state academic assessments" (U.S. Department of Education, 2004b, "Sec. 1001. Statement of Purpose," para. 1). Technology should be employed by schools and school districts to ensure that students achieve academic proficiency (U.S. Department of Education, 2004a). From 2007 through 2014, an average of $14.1 billion of NCLB funding was devoted to Title I grants to low-performing school districts throughout the United States (New America, 2015).

Given the influx of federal dollars into Title I schools and the growing educational technology industry—investments in educational technology companies nationwide have tripled in the last decade, from $146 million to $429 million in 2011 (DeSantis, 2012)—we believe it is important to understand the impact that technology is having on mathematics instruction in Title I schools. A concern for us is that Title I schools disproportionately use educational technology such as CAI to "learn or practice basic skills" (Gray, Thomas, & Lewis, 2010, p. 3); 83% of students attending a Title I school experience technology primarily for skill

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1 Standards-based reforms in mathematics refer to mathematics curriculum and instruction that promote the development of student reasoning through problem solving and discourse (e.g., National Council of Teachers of Mathematics, 1989, 2000; National Science Foundation, 1996). Although differences exist, for our purposes, we use "standards-based reforms in mathematics" in this commentary as being essentially synonymous with reforms promoted in the CCSSM.

2 We use "low-income students" to mean students who are classified as living in poverty in the United States (DeNavas-Walt & Proctor, 2014).

3 Throughout, we use "underserved students" to mean low-income students or culturally and linguistically diverse students. We use "computer-assisted instruction" throughout this commentary to mean computer-based interventions and computer-based training.

4 The No Child Left Behind Act (NCLB) was signed into law by President George W. Bush in January 2002. The NCLB law significantly increased the federal government's role in holding schools responsible for the academic progress of their students. The law placed a special focus on ensuring that schools improve the performance of certain groups of students, such as poor and minority students. States did not have to comply with the new law, but if they did not, they risked losing federal Title I funds (Klein, 2015).

development, compared to 61% of their counterparts at non-Title I schools (Gray et al., 2010). Because standards-based mathematics instruction may not be a priority at schools attended mainly by underserved students (Kitchen, 2003; Martin, 2013), we wonder what role CAI is playing with regard to the low-level, skills-based mathematics instruction that has been pervasive in these schools (Davis & Martin, 2008; Secada, 1995). More specifically, we pose the question: *How may CAI support or hinder standards-based education reform in mathematics (e.g., development of students' reasoning through problem solving and discourse), particularly in schools largely attended by underserved students?*

For the purposes of this commentary, we adopt the commonly held perspective that CAI is an instructional approach in which a computer, rather than an instructor, provides self-paced instruction, tests, and learning feedback (Seo & Bryant, 2009).[5] To be clear, it is not our intent to characterize CAI programs as uniform because large differences exist. For instance, programs vary in terms of interactivity, use of graphics, and versatility (Barrow, Markman, & Rouse, 2009). Some are software programs, whereas others are web-based. In addition, we recognize that there are at least three different applications of CAI in the classroom: supplemental, core, and computer-managed learning systems (Slavin, Lake, & Groff, 2009). The concerns we express here are intended to apply generally to any CAI intervention program designed for use in mathematics classrooms in U.S. schools, and some of these concerns may apply for some CAI programs and not for others. Our goal is to identify and discuss our apprehensions with regard to CAI in general and to attempt to explain why we believe our worries are particularly pertinent for underserved students.

To establish context, we give an overview of mathematical reasoning and discourse that are prominent in the CCSSM. We proceed to explore research about how underserved students are being denied access to a rigorous standards-based education in mathematics, provide some background on educational technology, and offer a brief review of the research on educational technology interventions that rely on CAI. We conclude with remarks about the need for more research to understand the influence of CAI on mathematics instruction at schools that serve students who have historically been marginalized by the U.S. educational system.

Mathematical Reasoning and Discourse

To implement standards-based reforms in mathematics, teachers need to possess a solid understanding of mathematics and have the pedagogical skills needed to support their students to learn mathematics with understanding (Franke & Kazemi, 2001; Hill, Rowan, & Ball, 2005). Reasoning is central to standards-based reforms in mathematics, specified among the five strands of mathematical proficiency presented in *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001).

[5] "Dynamic geometry software" (e.g., Jones, 2000) and video game interventions (e.g., Barab, Gresalfi, & Ingram-Goble, 2010) do not meet this definition of CAI.

For us, mathematical reasoning is synonymous with Blanton and Kaput's (2005) definition of algebraic reasoning as "a process in which students generalize mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways" (p. 413).

A focal point of the CCSSM is the Standards for Mathematical Practice (NGA & CCSSO, 2010), which advocate for developing students' abilities to reason mathematically across the K–12 curriculum. Among the eight practices, mathematical reasoning is prominent in three of them: Mathematical Practice 2, "Reason abstractly and quantitatively," Mathematical Practice 3, "Construct viable arguments and critique the reasoning of others" (NGA & CCSSO, 2010, p. 6), and Mathematical Practice 8, "Look for and express regularity in repeated reasoning" (p. 8). Reasoning and proof were also significant process standards in mathematics education reform policy documents that foreshadowed the Standards for Mathematical Practice recommended in the CCSSM (e.g., NCTM, 1989, 2000; NSF, 1996).

Another prominent feature of the CCSSM is the value placed upon mathematical conversations or discourse. During mathematical discourse, the teacher seeks to foster and continually engage in dialogue with her students (Cazden, 2001; Herbel-Eisenmann & Cirillo, 2009). Research has demonstrated that when students have opportunities to engage in mathematical discourse to explain their ideas to peers and to listen to and make sense of the ideas of others, their learning is enhanced (e.g., Herbel-Eisenmann & Cirillo, 2009; Webb, 1991). As students engage in mathematical discourse through participation in learning communities, they build on their prior experiences and knowledge to achieve more advanced understanding of challenging mathematical concepts (Franke & Kazemi, 2001; Lave & Wenger, 1991; Stein, Silver, & Smith, 1998).

Mathematics Education for Low-Income Students and Students of Color

In 2013, the poverty rate was 14.5%, with 45.3 million people living in poverty in the United States (DeNavas-Walt & Proctor, 2014). Moreover, the poverty rate for ethnic and racial minorities in the United States in 2013 was much higher than the national poverty rate of 14.5% (DeNavas-Walt & Proctor, 2014). In 2013, 27.2% of Blacks and 23.5% of Hispanics were poor, compared to 9.6% of non-Hispanic Whites and 10.5% of Asians (DeNavas-Walt & Proctor, 2014). Also in 2013, the poverty rate for children under the age of 18 was 19.9% (DeNavas-Walt & Proctor, 2014), but it was 39% for Black children and 32% for Hispanic children (National Center for Education Statistics, 2015). Massey (2009) contended that advantages and disadvantages procured from an individual's socioeconomic status (SES) are both reinforced and compounded by geographic concentration; Tate (2008) referred to this as the "geography of opportunity" (p. 397). Students from low-income communities attend schools in which pupil expenditures compare unfavorably to pupil expenditures in schools located in wealthy communities and achieve at lower levels than their wealthy counterparts (Payne & Biddle, 1999). For

example, Hogrebe and Tate (2012) found that the SES of local communities is significantly related to students' performance in algebra, with students from low-income communities achieving at lower levels than students from affluent communities. Brynes and Miller (2007) argued that SES not only has direct effects on mathematics achievement but also has indirect effects on both the opportunities students have to enroll in advanced mathematics classes in high school and on their propensity to take advantage of learning opportunities in mathematics.

In addition to poverty and SES, student access to a challenging standards-based mathematics education is influenced by race, ethnicity, and English-language proficiency (Diversity in Mathematics Education Center for Learning and Teaching [DiME], 2007; Gutiérrez, 2008; Martin, 2013). For instance, schools that enroll large numbers of African American students often have disproportionately high numbers of remedial mathematics classes in which instruction is focused on rote learning and strategies that are intended to help students be successful on standardized tests (Davis & Martin, 2008; Lattimore, 2005). In response to the NCLB Act of 2001 and the demands to increase test scores, Davis and Martin (2008) argued that the preponderance of skills-based instruction negatively "shape[s] the lives of poor African American students in more significant ways than middle-class or affluent students" (p. 18). In schools that serve large numbers of immigrant Latino or Latina students who speak with an accent, use English words incorrectly, or speak in Spanish as a means to express themselves, educators, peers, and community members may assume that they lack the capacity to perform well in mathematics (Gutiérrez, 2008; Moll & Ruiz, 2002; Moschkovich, 2007). Ability grouping or tracking is another widespread practice in the United States that has disproportionately hurt underserved students (Oakes, 2005; Secada, 1992). Tracking continues to "divide students by perceptions of 'ability' and communicate to students the idea that only some people—particularly White, middle class people—can be good at mathematics" (Boaler, 2011, p. 7).

Given the high percentage of students of color living in poverty in the United States and the extensive research base that demonstrates that low academic expectations and lower pupil expenditures have historically been the norm at schools attended by underserved students (e.g., DiME, 2007; Ferguson, 1998; Flores, 2008; Knapp & Woolverton, 1995; Payne & Biddle, 1999), it is not difficult to surmise that millions of students are being denied access to instruction in which mathematical reasoning and discourse are used to solve complex tasks (Davis & Martin, 2008; Kitchen, Burr, & Castellón, 2010; Téllez et al., 2011). Because standards-based mathematics instruction may not be a priority at schools attended primarily by underserved students (Kitchen, 2003; Martin, 2013) and because technology plays such a large role in student skill development in these schools (Gray et al., 2010), we worry about the role that CAI may play in exacerbating historical injustices in mathematics education for underserved students.

Some Background on Educational Technology and Research on CAI

The educational technology market is a big business. In a 2011 survey, the overall market value for prekindergarten to Grade 12 nonhardware educational

technology was $7.5 billion (Software & Information Industry Association, 2011). Since 2011, federal funding for educational technology in K–12 schools has been integrated into other funding streams in order to make technology expenditures more efficient for schools (Pascopella, 2012). This makes it difficult to track how much of the 2013 Education Department budget of $69.8 billion (Office of Management and Budget, 2012) is actually spent on educational technology (Pascopella, 2012). As the market share of educational technology companies grows (Software & Information Industry Association, 2011) and billions of dollars go to Title I schools (New America, 2015), there is a need for research that explores the consequences of CAI implementation (Säljö, 2010), specifically in Title I schools.

Research has demonstrated that CAI has both strengths and weaknesses, but overall its effect on student achievement is inconclusive. Among the strengths, students are provided with immediate feedback on their performance, instruction is individualized, and the program maintains evaluative information concerning students' progress (Kulik & Kulik, 1991; Lockard, Abrams, & Many, 1997). Hu et al. (2012) found that students who participated in an afterschool program in which they received tutoring via a CAI program performed significantly better on a standardized test than nonparticipating peer students and that these students' mean scores were equivalent to or higher than (although the difference was not statistically significant) scores of students receiving afterschool tutoring from teachers. Additionally, Slavin and Lake (2008) found in their review of CAI programs designed for use with elementary school students that "CAI effects in math, although modest in median effect size, are important in light of the fact that in most studies CAI was used for only about 30 minutes three times a week or less" (p. 481). Summarizing their findings, Slavin and Lake (2008) wrote, "A number of studies showed substantial positive effects of using CAI strategies, especially for computation, across many types of programs" (p. 481).

In terms of weaknesses, studies have also shown that use of CAI does not affect student achievement in mathematics. For example, the What Works Clearinghouse reported that a popular CAI program had "'no discernible effect' on student achievement" (as cited in Cavanagh, 2008, p. 4). As part of the NCLB Act of 2001, Congress requested a $15 million study by the U.S. Department of Education to examine the effectiveness of 10 different mathematics and reading educational software technology products (Gabriel & Richtel, 2011). The report was released in 2007 and was compiled from data collected during the 2004–2005 and 2005–2006 school years. The findings indicated that "after one school year, differences in student test scores were not statistically significant between classrooms that were randomly assigned to use products and those that were randomly assigned not to use products" (Campuzano, Dynarski, Agondini, & Rall, 2009, p. xv). This report focused on technology products in mathematics and reading, and none of the mathematics technology effects were statistically significant. Moreover, Slavin, Lake, and Groff (2009) determined in their extensive review of CAI programs designed for use with middle school and high school students that the "effect sizes were very small" (p. 839).

Given the inconclusive and, at times, contradictory research concerning the effects of CAI on mathematics learning and achievement, we wonder why schools are investing significant financial resources and valuable classroom time in these educational technology products. A 2010 survey (U.S. Government Accountability Office, 2010) asked school leaders and district officials how they chose curricula, and 58% responded that they had never heard of or consulted the What Works Clearinghouse, an initiative of the U.S. Department of Education that reviews education research and publishes findings relevant to school leaders. Many experts believe that decisions to purchase are based on politics, personal preferences, and marketing; it is more a matter of slick public relations pitches rather than the effectiveness of the actual products that influence sales (Gabriel & Richtel, 2011).

Addressing Our Question and Considering Our Concern

We now return to the question: How may computer-based interventions support or hinder standards-based education reform in mathematics in the United States, particularly in schools largely attended by underserved students? In order to address this question, we discuss four areas of concern that we have regarding the impact of CAI on mathematics education: (a) the individualized nature of CAI, (b) the tendency to utilize CAI for drill and skill development rather than to promote mathematics reasoning through discourse, (c) the lack of preparation and training teachers need to appropriately implement CAI and the consequences of CAI implementation on teachers' work, and (d) the potential to use CAI as a replacement for mathematics teachers.

Our first concern has to do with the individualized nature of CAI. Although a strength of computer-based training is that such interventions allow for instruction to be individualized (Kulik & Kulik, 1991; Lockard et al., 1997), this strength could also become a weakness if students consistently engage in mathematics alone. As a consequence of the individualized nature of CAI programs, students often work in isolation from peers while interventions are taking place. Teachers may have limited opportunities to understand how to use CAI programs (Snow, 2011) in ways that engage students in discourse with peers to collaboratively solve problems. If students have limited opportunities to engage in mathematical discourse with peers, they will also have limited opportunities to develop their mathematical reasoning and conceptual understanding (Franke & Kazemi, 2001; Stein et al., 1998). Interestingly, Slavin et al. (2009) concluded that CAI programs that support student interactions "have more promise" (p. 839) than those in which students interact with the technology alone. Given the potential challenges inherent in CAI to engage students in mathematical learning communities, research needs to be conducted to understand the type of assistance that teachers need in order to learn how to utilize CAI to support the development of their students' mathematical reasoning skills through discourse. This research should be undertaken in schools with large numbers of underserved students in which the tendency is for teachers to expect less of students and to focus more

on skills-based instruction (Davis & Martin, 2008; Flores, 2008; Kitchen et al., 2007).

A second concern is that CAI may be used more for drill and skill development than as a means to promote mathematical reasoning through discourse (Ganesh & Middleton, 2006; Snow, 2011). The research, though mixed and incomplete, suggests that CAI holds promise to support the development of students' calculations and skills in mathematics (Slavin & Lake, 2008). Desperate to improve test scores, school leaders may be willing to pay the high costs associated with purchasing, maintaining, and utilizing CAI with the hope that the program will, at a minimum, support the development of students' mathematical skills. Although this is understandable, we believe that it is also important to consider the potential ramifications of introducing CAI into a school. For instance, secondary teachers have the tendency to use computers primarily for mathematical drill and practice (e.g., Manoucherhri, 1999; Weiss, Banilower, McMahon, & Smith, 2001). This is particularly problematic in schools populated by underserved students, given the long history in the United States of focusing more on skills-based mathematics instruction in these schools (e.g., Kitchen et al., 2007; Lattimore, 2005).

Clearly, challenges exist regarding the use of technology and equity (Dunham & Hennessy, 2008), and we believe such challenges should be taken seriously by instructional leaders as they consider not only the benefits of introducing computer-based interventions in their schools but also the potential drawbacks. Specifically, we fear that the ongoing use of and potential overreliance on CAI, particularly at Title I schools, may privilege skills-based instructional formats in general over those that focus on developing students' mathematical reasoning through discourse to the continued detriment of underserved students.

Our third concern pertains to teacher training and how CAI may affect teachers and their work. While serving as vice president of the Software & Information Industry Association, Karen Billings described how schools often do not properly deploy the products or train teachers to use them (Gabriel & Richtel, 2011). Teachers are not being properly trained on the appropriate uses of CAI (Ganesh & Middleton, 2006; Snow, 2011), specifically on ways to support standards-based mathematics instruction. This is particularly a problem in diverse schools situated in low-income communities precisely because these schools have historically struggled to offer a challenging mathematics program (Ganesh & Middleton, 2006). Researchers (e.g., Heid, 1997) have argued that technology has the potential to act as a catalyst to promote reforms in mathematics education. As we have previously asserted, more research is needed to understand the supports teachers need in order to use CAI for the development of their students' mathematical reasoning skills through discourse. For instance, what types of coaching models (West & Cameron, 2013) may be most appropriate to support teachers in learning how to appropriately use technology to promote standards-based reforms in mathematics?

Finally, researchers have expressed apprehension that computers are increasingly being used as a replacement for teachers rather than in support of a rigorous

mathematics education program (e.g., Steele, Johnson Palensky, Lynch, Lacy, & Duffy, 2002). Certainly it may be the case that administrators turn to computers when they experience shortages of qualified mathematics teachers. Nevertheless, we worry that CAI programs may unwittingly contribute to the deskilling of teachers because such programs, particularly when used exclusively as a school's curriculum, may become the official mathematics curriculum (Apple, 2000). The deskilling of teachers combined with low student expectations contributes to and promulgates the historical trajectory of deprived mathematics pedagogy that has predominated in U.S. schools populated chiefly by underserved students (Flores, 2008; Kitchen et al., 2007; Tate, 1995). Frey, Faul, and Yankelov (2003) also found that although students value interventions such as CAI to help them learn course content, online and computer-based tools do not satisfy their desire to interact with each other and the instructor. We believe that it is especially important in schools largely attended by underserved students for students to have opportunities to collaborate with others, given the tendency in these schools to move toward controlled forms of instruction that limit students' opportunities to work collaboratively and learn from each other (Davis & Martin, 2008; Kitchen et al., 2010; Lattimore, 2005).

Final Remarks

Mathematics is best learned when students have opportunities to engage in discourse in which mathematical ideas are generated, shared, investigated, debated, and validated as a means to develop students' mathematical reasoning (Franke & Kazemi, 2001; Herbel-Eisenmann & Cirillo, 2009). The research literature has consistently documented that schools that predominantly serve low-income students and large populations of culturally or linguistically diverse students have taught mathematics in ways that do not align with rigorous standards-based mathematics instruction (e.g., Davis & Martin, 2008). A strength of CAI is that it supports skills-based instruction (Dunham & Hennessy, 2008; Gray et al., 2010; Manoucherhri, 1999), but this mode of instruction has failed students for years (Jacobsen et al., 2013; Leonard & Martin, 2013). Moreover, we question how well CAI interventions can promote standards-based instruction in mathematics for underserved students in which the development of students' mathematical reasoning is paramount. We also worry that schools serving large populations of underserved students that invest heavily in CAI may devote less human capital to ensuring that their students are also engaging in challenging problem-solving activities.

The companies that have developed computer-based interventions, like so many other consultants and for-profit companies, have clearly been a beneficiary of the NCLB legislation and the federal dollars that continue to be provided for educational interventions in Title I schools. However, it is not clear from research that students who regularly use these interventions have benefitted to the same extent (Campuzano et al., 2009; Slavin et al., 2009). CAI programs align best with skills-based interventions that have been tried for many decades in schools that

serve low-income, diverse communities in the United States, but these interventions have generally failed to produce positive results (Flores, 2008; Kitchen et al., 2007). Given that computer-based technologies and interventions that promise a positive impact on academic achievement are not supported by a solid research base (Säljö, 2010), more research needs to be done on how CAI best supports the learning of mathematics, particularly the rigorous mathematics promoted in the CCSSM and mathematical processes such as mathematical reasoning and discourse. We also need more research to understand how CAI programs are actually being used in diverse schools located in low-income communities as compared to schools in more affluent, White communities. Finally, we need to better understand how teachers should be trained to use CAI as a means to support standards-based reforms in mathematics.

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Brief Report

Teachers' Awareness of the Relationship Between Prior Knowledge and New Learning

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In this study, I examined the degree to which experienced teachers were aware of the relationship between prior knowledge and new learning. Interviews with teachers revealed that they were explicitly aware of when students made connections between prior knowledge and new learning, when they applied their prior knowledge to new contexts, and when they developed their prior knowledge as a result of applying that knowledge to new contexts. However, teachers were not explicitly aware of backward-transfer effects. Results from this study have implications for future research on backward transfer, as well as for teacher professional development.

Key words: Backward transfer; Prior knowledge; Teacher awareness

An assumption underlying much of mathematics education research is that "prior knowledge . . . serves a new foundational role in developing mastery" (Smith, diSessa, & Roschelle, 1993, p. 136). Interestingly, the influence that new learning in turn might have on prior knowledge has received less attention in the mathematics education research literature, especially in contexts in which the new learning is conceptually distinct from the prior knowledge. Researchers have conceptualized this type of effect as a form of transfer called *backward transfer* (e.g., Cook, 2003; Gentner, Loewenstein, & Thompson, 2005; Hohensee, 2014), which has been well established by language learning research (e.g., Cook, 2003). In multiple studies, researchers have found that learning a second language can influence bilingual students' production or comprehension of their native language.

As a new construct to mathematics education research, backward transfer offers a potential account for a growing body of findings pertaining to the influences of new learning on prior knowledge (e.g., Arzi, Ben-Zvi, & Ganiel, 1985; Hohensee, 2014; MacGregor & Stacey, 1997; Moore, 2012; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2004; Young, 2015). For example, the findings of a study by Van Dooren, De Bock, Hessels, Janssens, and Verschaffel (2004) indicated that new learning about nonproportional relationships unproductively influenced

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eighth graders' prior knowledge about proportional relationships. These researchers employed experimental lessons on nonproportionality with a treatment-group/control-group design. Pre- and post-tests revealed that the control group exhibited no change in reasoning but that the treatment group "suddenly started to apply non-proportional strategies to proportional problems" (Van Dooren et al., 2004, p. 485).

In another study, MacGregor and Stacey (1997) found that new learning about solving equations interfered with 11–15-year-olds' prior knowledge about simplifying algebraic expressions. They examined the interpretations of algebraic symbolism by a large sample of students (n = 1,473) from 22 schools in a 3-year longitudinal study and found several instances of "interference from new learning," such as "widespread misuse of exponential notation (e.g., x^3 instead of $3x$)," which "increased steadily over the four year levels, from 5% at Year 7 to 18% at Year 10" (MacGregor & Stacey, 1997, p. 11).

In a previous research report, I reported that new learning about quadratic functions either unproductively or productively influenced seventh graders' prior knowledge about linear functions (Hohensee, 2014). I used semistructured interviews to examine, at a fine-grained level, students' conceptual understanding of linear functions before and after instruction in quadratic functions. Evidence showed that a particular set of quadratic function lessons muddled several students' conceptual understanding of linear functions, but a revised set of the same lessons with a different group of students resulted in a productive effect on their conceptual understanding of linear functions.

The findings of these studies suggest that influences of new learning on prior knowledge may be an important consideration for the field of mathematics education research. Because backward transfer is underresearched in the context of mathematics education, research in this area could reveal new insights into learning and transfer. As a preliminary step to conducting such research, I sought to examine the degree to which experienced teachers were aware of backward transfer and to draw on teachers' experiences to generate a list of potential mathematics topics for which influences of new learning on prior knowledge are commonly realized. The study reported in this article addressed the following research questions:

1. Of what aspects of the relationship between students' prior knowledge and their new learning are teachers explicitly aware?
2. For which mathematics topic pairs are backward-transfer effects realized?

In this Brief Report, I present the findings of this study in order to increase awareness in the mathematics education research community of the construct of backward transfer and its role in the teaching and learning of mathematics.

Theoretical Foundation

The intersection of three important ideas creates the theoretical foundation for this study. First, new knowledge construction may result in additional changes or

effects throughout an individual's knowledge system. Second, influences of new learning on prior knowledge can be conceptualized as the transfer of learning. Third, teachers' awareness about particular instructional events can range from implicit to explicit.

Knowledge Construction and Changes Throughout the System

According to the constructivist perspective that guided this study, individuals construct knowledge by assimilating new experiences to existing schemes, making accommodations to schemes, or reflectively abstracting schemes to higher levels of abstraction (von Glasersfeld, 1995). Furthermore, knowledge construction "may entail changes of massive scope . . . creating very large ripple effects through the system" (Smith et al., 1993, p. 148). Influences of new learning on prior knowledge would be an example of a ripple effect.

Based on prior research, I assumed that influences on prior knowledge can produce effects that range from temporary to stable. In Hohensee (2014), I initially found that new learning about quadratic functions unproductively influenced or interfered with middle school students' prior understanding of linear functions. I interpreted this unproductive influence as a potentially temporary effect. However, when I revised the quadratics lessons to emphasize covariational reasoning and repeated the study with a new group of students, I found that after quadratics instruction, the students exhibited a deeper understanding of linear functions. I interpreted the deeper understanding as a potentially more stable effect.

In this article, when I distinguish between new learning and prior knowledge, I specifically mean contexts in which the new learning entails a topic that is *conceptually distinct* from the prior knowledge topic (as opposed to new learning that is a continuation of the prior knowledge topic). Barnett and Ceci (2002) referred to conceptually distinct topics as "dissimilar context[s]" (p. 615). Detterman (1993) called moving from one topic to a conceptually distinct topic taking "the learner out of the appropriate problem space" (p. 18).

Perspective on Transfer

The second foundational idea is that influences of new learning on prior knowledge can be conceptualized as the transfer of learning. According to the *actor-oriented transfer* (AOT) perspective, transfer is defined as "the influence of a learner's prior activities on her activity in novel situations (Lobato, 2008a)" (Lobato, 2012, p. 233). In other words, from an AOT perspective, the ripple effects described by Smith, diSessa, and Roschelle (1993) that result from assimilation, accommodation, or reflective abstraction would be conceptualized as instances of transfer.

The AOT perspective differs from the traditional perspective on transfer in that what counts as transfer is not predetermined. Instead, any influence, including those that lead to nonnormative outcomes, are counted as transfer. In other words, the AOT perspective takes the actor's, rather than the expert's, point of view. This perspective relates well to the study reported here because the influences that I was considering could have been either mathematically productive or unproductive.

Although I adopted the AOT perspective, my study examined influences of new learning on prior knowledge rather than on reasoning about novel tasks. I conceptualized these influences as backward transfer, which I defined as "*the influence that constructing and subsequently generalizing new knowledge has on one's ways of reasoning about related mathematical concepts that one has encountered previously*" (Hohensee, 2014, p. 136).

Explicit and Implicit Awareness

The third foundational idea is that *awareness* can be explicit or implicit. Mason (1998) described awareness as "comprising both conscious and unconscious powers and sensitivities which enable people to act freshly and creatively in the moment" (p. 243). The conscious power involves being "explicitly aware" (Mason, 1998, p. 254), whereas the unconscious power involves the "subconscious foci of our attention (whether directed inwardly or outwardly) . . . . largely implicit[ly]" (p. 256). Mason (2002) also defined implicit awareness as *ordinary-noticing*, which is "easily lost from accessible memory. It is only available through being re-minded (literally) by someone or something else" (p. 33). I hypothesized that teachers might be implicitly (rather than explicitly) aware of backward-transfer effects.

Methods

Eight teachers from the mid-Atlantic region were recruited to participate in one-on-one, semistructured interviews (Bernard, 1988). Teachers were recruited during professional development events and through university faculty recommendations. Both mathematics and science teachers were recruited because there are many mathematics concepts that overlap with science.[1]

Of the four mathematics teachers, three taught Pre-Algebra and Algebra 1 at the Grade 7–8 level, and one taught Integrated Mathematics 2 at the Grade 10 level. Of the four science teachers, two taught General Science at the Grade 6–7 and 7–8 levels, one taught Physics and Chemistry at the Grade 9–10 level, and one taught Chemistry at the Grade 11–12 level. The mean number of years of teaching experience was 12.35 years.

Data Collection

Data were collected from a three-part, video-recorded interview designed to stimulate teachers' recall of instances in which new learning influenced students' prior knowledge. The parts were progressively more direct in asking teachers to recall backward-transfer effects. For Part 1, teachers were asked the following:

> Imagine that you are teaching students a new math/science topic. Let's call that Topic B. After you finish teaching Topic B, you give students an activity that involves Topic A, which was a topic covered at an earlier time. Can you recall a time when students'

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[1] My prior backward transfer research looked at the learning of the mathematics of the science topics speed and acceleration (Hohensee, 2014).

> thinking about Topic A appeared to have been influenced or changed because they learned about Topic B?

For Part 2, teachers were presented with six Common Core State Standards for Mathematics topics—ratios and proportions, rational and irrational numbers, expressions and equations, functions, geometry, and statistics and probability (National Governors Association for Best Practices & Council of Chief State School Officers, 2010)—or six Next Generation Science Standards—structure of matter and chemical reactions, motion-force-energy, earth in the universe, heredity and evolution, waves, and ecosystems and human impact on earth (Next Generation Science Standards Lead States, 2013)—and asked to order the topics according to how they would typically be taught. Teachers were then asked, "Have you ever noticed that the way students reason about Topic 1 changed after they learned about Topics 2, 3, 4, 5 or 6?" Teachers were asked the same question about Topic 2 in relation to Topics 3, 4, 5, or 6 and so on.

For Part 3, all teachers were shown the same illustrative example of backward transfer. The example showed that before quadratics instruction, a student correctly interpreted a linear data set, but after instruction, he incorrectly interpreted the same linear data set. Teachers were asked if this example reminded them of a time when a new topic influenced students' prior knowledge about a different topic. This mathematics example was also given to the science teachers to help stimulate their recall of any mathematics examples of backward transfer in their science classes.

Data Analysis

Initially, I watched the recorded interviews and catalogued the topics that teachers reported as being involved in backward-transfer effects. Next, I transcribed and rewatched the interviews and wrote descriptive accounts (with minimal inference) about what transpired during each interview. Then, I used the descriptive accounts to develop a coding scheme that was grounded in the data (Strauss, 1987) and captured those aspects of the relationship between prior knowledge and new learning of which teachers were explicitly aware. Next, I coded each example that teachers provided about prior knowledge. Throughout the analysis, I used the constant comparison method to refine codes until they fit the data well (Strauss & Corbin, 1994). To assess reliability, I asked another researcher and a graduate student to use my refined codes to check a random sample of the data (i.e., they each coded 10 of the 51 teacher transcript passages that I had coded). Agreement between how the researcher and graduate student coded the data using my codes and how I coded the same data was 80% and 90%, respectively (i.e., 8/10 and 9/10 passages were coded the same as I had coded them).

Results

Teachers provided ample evidence that they were explicitly aware of when students did or did not (a) make connections between prior knowledge and new

mathematics topics, (b) apply prior knowledge in new mathematics contexts, and (c) develop prior knowledge through application in new contexts.[2] In contrast, teachers initially provided little evidence that they were explicitly aware of backward-transfer effects. However, the illustrative example in Part 3 of the interview helped teachers to recall backward-transfer effects. Note that the evidence presented next, including the evidence not directly about backward transfer described in the three points above, came from the teachers' responses to questions that were exclusively about backward-transfer effects. This highlights the challenges that teachers might face with orienting themselves to backward-transfer effects.

Explicit Awareness of Making Connections

Six of the eight teachers indicated that they were explicitly aware of when students did or did not make connections between prior knowledge and new learning. For example, Ms. Sutton[3] explained:

> I think because they don't connect that these two are connected and they're doing the same thing. But I think sometimes they don't realize that they are doing the same thing and that they're connected. They think this is shapes [referring to geometric similarity]. They think this is just numeric number sense and scaling [referring to ratios] but they wouldn't call it scaling here [referring to geometric similarity] even though it's still scaling.

In this example, Ms. Sutton recalled being explicitly aware of students not making connections between ratios (prior knowledge) and geometric similarity (new learning). Other teachers made similar statements about students making (or not making) connections.

Explicit Awareness of Using Prior Knowledge in New Contexts

Each of the eight teachers indicated that they were explicitly aware of when their students did or did not use prior knowledge in new contexts. For example, Ms. Keane explained:

> We teach them dimensional analysis. I go through this a couple of times in the beginning of the year, just straight up dimensional analysis . . . to go from unit to unit . . . it goes away for a good bit of the year and then it comes back. Only now it's an application. Instead of just converting within the metric system, now we're converting from one measuring system to another . . . Some of them get it, some of them don't . . . when they can't just move a decimal and they really have to use [dimensional analysis].

In this excerpt, Ms. Keane reported that although some students applied prior knowledge about dimensional analysis of the metric system to convert between measurement systems (new learning), others did not. The other teachers made similar statements about students' use of prior knowledge (or lack thereof) in new contexts.

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[2] Although findings specifically related to science were also identified, I primarily present mathematics findings in this article.

[3] All names are gender-preserving pseudonyms.

Explicit Awareness of Development of Prior Knowledge in New Contexts

Each of the eight teachers indicated that they were explicitly aware of when students did or did not develop their prior knowledge through applying that knowledge to new contexts. For example, Ms. Plant explained:

> I just find that when we start putting together equations, that they don't really understand what is *x*, what is *y*, they're just letters. And what is that number in front of *x*, what is that other number? They're just numbers too. And then, when we apply a word problem to it and they're given *x* to find *y*. That's when they see the meaning behind the symbols.

Here, Ms. Plant described a case in which students' prior knowledge of linear functions developed as a result of applying it to a new context (word problems). Teachers also recalled instances in which students did not further develop their prior knowledge through experiences in a new context.

Lack of Explicit Awareness of Backward-Transfer Effects

Most teachers did not appear to be explicitly aware of backward-transfer effects. Before I showed teachers the illustrative example of backward-transfer effects, I interpreted only two teacher observations as referring to backward transfer. In one observation, Mr. Henry described students who were asked to work on metric conversions (prior knowledge) after having learned about scientific notation (new learning) as "put[ting] all of their metric stuff into scientific notation when not asked for it." In the other observation, Ms. Sutton said:

> I do see a lot of lower students . . . learn a new concept and go back and apply wrong concepts to something they learned previously . . . We're talking about unit rates [prior knowledge]. So then we're going to go talk about complex fractions [new learning]. So when they go back [to unit rates] . . . they mix up the concepts.

The reasons that I interpreted these two observations as referring to backward transfer are because the teachers talked about *influences on* rather than *connections to* or *applications of* prior knowledge and because the new learning involved topics that were conceptually distinct from the prior knowledge.

Illustrative Example Orients Teachers to Backward Transfer

After seeing the illustrative example of backward transfer, all but one teacher reported observing backward-transfer effects or provided an example. For instance, Ms. Keane said, "So yeah . . . I do. I see it a lot." Similarly, Ms. Hayes noted, "Yeah I've seen kids do that . . . I have all kinds of different instances where kids come back and it's like, guys we've seen this before . . . and they can't answer the question," and Ms. Plant said, "Right, I know it's happened before . . . I know that's happened for us before."

Some teachers had difficulty recalling specific examples of backward-transfer effects. For example, Ms. Earl commented, "It's kind of hard to think of some specifics with that but yes, I've seen it." Two teachers also reported that attending to backward-transfer effects was not something they did. Ms. Plant stated, "I don't

think that's a method that we go back in and pull something back up after we've taught something new to see if it has been impacted or changed very often." Similarly, Ms. Kennedy said, "Have I really ever questioned a kid to ask them have they rethought that [prior knowledge], the answer would be I have not questioned them to rethink that."

Interestingly, Ms. Sutton did not acknowledge the possibility of backward transfer in the illustrative example. Instead, she had the following unique interpretation of the example:

> This would not be because I taught quadratics, this would be because of [the students] not counting by a constant rate . . . The concept that would be difficult for the student would be not counting by something consistent in the time, not learning the new topic.

According to Ms. Sutton's interpretation, the student's response could be attributed to a misconception about linear functions rather than to a backward-transfer effect from the quadratics instruction. Ms. Sutton's interpretation was focused on explaining what the student did not understand and not on the change in the student's reasoning. It is not surprising that experienced teachers, when first shown such changes in reasoning, might develop interpretations other than that new learning was an influence on prior knowledge. Based on this evidence, I hypothesize that had Ms. Sutton been explicitly aware of instances of backward-transfer effects in her classroom, she may have not interpreted them as such.

Mathematics Topic Pairs for Which Backward-Transfer Effects Were Observed

Although the teachers had discussed only two mathematics topic pairs associated with backward-transfer effects during Parts 1 and 2 of the interview, three additional pairs were identified in Part 3. After reading the illustrative example, the teachers were asked whether this reminded them of a similar situation that they had experienced with their students. Ms. Plant recalled that when students learn about linear functions before exponential functions, and then later try to work with linear functions again, "the concept of multiplying the growth factor or decay factor [from exponential functions] sometimes confuses them." Ms. Plant also stated that students learn the rules for adding and subtracting positive and negative numbers before learning about multiplying and dividing and that "when we go back to adding and subtracting, they'll get themselves confused with the rules of the two." Finally, Mr. Henry recalled that when his students learn about speed as the slope on a distance–time graph before learning about speed on a speed–time graph, they subsequently become confused about the slope of the distance–time graph, and he has to show them that the "slope of a speed versus time graph is not speed." All five mathematics topic pairs are presented in Table 1.

Discussion

Prior research on backward transfer has looked directly at students' thinking (e.g., Hohensee, 2014; Moore, 2012; Young, 2015), whereas the current study

drew on teachers' experiences to gain a fuller understanding about backward-transfer effects in classrooms. This study shows that, although teachers may be explicitly aware of multiple features of the relationship between prior knowledge and new learning, there may be an aspect of this relationship that teachers might not be explicitly aware of. These findings suggest that teachers' awareness of backward-transfer effects may be another aspect of teacher noticing (cf. Jacobs, Lamb, & Philipp, 2010).

Table 1
Mathematics Topic Pairs for Which Backward-Transfer Effects Were Observed

Prior knowledge	Description of influence of new learning
Distance–time graphs	Speed–time graphs confuse students about distance–time graphs
Linear functions	Exponential function growth factors confuse students about linear functions
Add/subtract integers	Multiplication and division of integers confuses students about addition and subtraction of integers
Metric conversions	Scientific notation confuses students about metric conversions
Unit rate	Complex fractions confuse students about unit rate

Of course, there might be alternate explanations for backward transfer. For example, one might consider that what I have referred to as backward transfer is instead evidence of prior knowledge being composed of *instrumental understanding* (i.e., knowing "rules without reasons," Skemp, 1978, p. 9), which can easily be misapplied or forgotten. However, for the studies cited earlier in which influences of new learning on prior knowledge were observed, the backward-transfer explanation seems more compelling. In several of the studies, the researchers went well beyond simply assessing rule following and instead gathered complex measures of learning (e.g., interviews that probed for conceptual understanding; Hohensee, 2014; MacGregor & Stacey, 1997; Moore, 2012; Young, 2015). By probing for conceptual understanding, these researchers were able to identify ways in which, at the conceptual level, prior knowledge took on characteristics of the new learning. Also, some of these studies included control groups (Van Dooren et al., 2004) and large sample sizes (Arzi et al., 1985, $n = 1{,}176$) and were thus able to establish causal links between new learning and changes in prior knowledge. Finally, some of the studies found productive effects on prior knowledge (Arzi et al., 1985; Hohensee, 2014; Moore, 2012; Young, 2015), which seem better explained by new learning than explained by time, forgetting, or some random effect. For these reasons, the influence of new learning on prior knowledge is a compelling explanation for the findings in those studies and suggests that similar

influences are likely to occur in other classrooms.

The findings generated by this study have relevance for future research. Recall that the second goal of this study was to generate a list of potential backward-transfer topics. My plan to build on these findings involves investigating the topics that were identified by the teachers in this study in both controlled and naturalistic instructional settings. I also plan to use what I learned about the contexts in which backward transfer occurs to help me capture examples of backward-transfer events on video. This will further the work on teachers' awareness and interpretation of backward-transfer effects.

The findings from this study also have relevance for mathematics teacher professional development. My plans to build on this study involve further examination of the use of illustrative examples during professional development as a way to orient teachers' awareness toward backward-transfer effects. Additionally, I hope to work with teachers to develop formative assessment strategies that monitor students' understanding of previously covered topics in order to help teachers determine if and when topics need revisiting.

In conclusion, this study incorporated teachers' perspectives into an examination of mathematics backward-transfer effects that occur during instruction. Results suggest that teachers are implicitly but not explicitly aware of backward-transfer effects. The findings point to new directions for research that could further the understanding of this effect in the context of mathematics learning. My hope is that by raising awareness of backward transfer among teachers and within the research community, effective new ways of improving mathematics learning are realized.

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Development of Probabilistic Understanding in Fourth Grade

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We analyzed the development of 4th-grade students' understanding of the transition from experimental relative frequencies of outcomes to theoretical probabilities with a focus on the foundational statistical concepts of variation and expectation. We report students' initial and changing expectations of the outcomes of tossing one and two coins, how they related the relative frequency from their physical and computer-simulated trials to the theoretical probability, and how they created and interpreted theoretical probability models. Findings include students' progression from an initial apparent equiprobability bias in predicting outcomes of tossing two coins through to representing the outcomes of increasing the number of trials. After observing the decreasing variation from the theoretical probability as the sample size increased, students developed a deeper understanding of the relationship between relative frequency of outcomes and theoretical probability as well as their respective associations with variation and expectation. Students' final models indicated increasing levels of probabilistic understanding.

Key words: Expectation; Probability models; Relative frequency; Statistical literacy; Theoretical probability; Variation

Data and chance permeate all aspects of our daily lives, including media advertisements, economic reports, and opinion polls. Dealing with forecasts of health, financial issues, or security risks necessitates interpreting, reacting to, and working with various levels of predictability and uncertainty (Gal, 2005). Increasingly, these chance situations demand that consumers be analytical, critical, and knowledgeable decision makers. Indeed, one cannot participate effectively in democratic discourse and public decision making without an appreciation of foundational statistical and probabilistic ideas.

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Young students are very much part of our data-driven and chance-laden society. They have early access to a vast array of computer technology, daily exposure to questionable advertising on mass media, and opportunities to participate in risk-taking ventures. It is thus imperative that their school mathematical experiences, beginning in the earliest school years, lay the foundations of the statistical and probability literacy needed for life beyond school (Gal, 2005; Greer & Mukhopadhyay, 2005).

Although numerous definitions of statistical literacy abound (e.g., Gal, 2002; Watson, 2006), "probability literacy" (Gal, 2005, p. 40) often does not feature prominently and tends to remain implicit in definitions of statistical literacy. We have argued in our research that statistical literacy encompasses statistics and probability, both of which must be targeted in developing young students' facility with chance situations (e.g., English & Watson, 2013). To this end, we view statistical literacy as the "meeting point" of statistics and probability with the everyday world (Watson, 2006, p. 11), in which encounters involve unrehearsed contexts, chance phenomena, and spontaneous decision making. The ability to apply statistical tools, probabilistic understanding, general contextual knowledge, and critical literacy skills is essential here. Similarly, Gal's (2005) notion of probability literacy incorporates the *knowledge elements* of big ideas, calculating probabilities, language, context, and critical questioning. These knowledge factors interact in complex ways with the *dispositional elements* including critical perspectives, beliefs and attitudes, and personal sentiments about chance situations.

Although we acknowledge the importance of all the knowledge components, we confine our discussion to those of variation and expectation as foundational for developing probability in the elementary school. The terms *variation* and *expectation* are used here in a wider sense of students' initial experiences with data (e.g., Watson, 2005; Watson, Callingham, & Kelly, 2007) rather than as shorthand for a calculated variance or expected value in theoretical probability. The creation and interpretation of probability models is an important feature of this development (e.g., Batanero, Henry, & Parzysz, 2005). We also consider Gal's (2005) dispositional elements pertaining to beliefs and sentiments but view these in terms of subjective beliefs regarding probability (Hawkins & Kapadia, 1984).

Within this perspective, we report on a study in which fourth-grade students experienced the core constructs of variation and expectation in their investigations of chance outcomes in tossing one, then two, coins multiple times both with and without computer simulation. Specifically, we focus on the students' initial expectations of the outcomes of tossing one and two coins, how their expectations changed with repeated numbers of trials, and how they related relative frequency based on large numbers of trials to the theoretical probability. We also examine students' creation and interpretations of probability models.

Variation and Expectation as Basic Concepts Underlying Probability

Variation and expectation, as concepts underlying probability and statistics, have had mixed recognition in curriculum documents that include the two topics.

Variation, as the fundamental phenomenon underlying all of statistics (e.g., Moore, 1990), and expectation, interpreted as prediction, are noted in the *Australian Curriculum: Mathematics* (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2015c). For example, in the description of the content for Foundation (pre-Grade 1) to Grade 2, it is stated, “Children have the opportunity to access mathematical ideas . . . by developing an awareness of the collection, presentation and variation of data and a capacity to make predictions about chance events” (ACARA, 2015b, “Foundation–Year 2,” para. 2). Across the grades, however, variation is only mentioned in Grade 3 and Grade 6 with respect to probability and in Grade 8 and Grade 10 Advanced with respect to statistics. In the *Common Core State Standards for Mathematics* (CCSSM; National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010) in the United States, probability and statistics are not included until Grade 6, in which statistical variability is the first topic. In Grade 7, random sampling to draw inferences on populations is introduced: Students “generate multiple samples (or simulated samples) of the same size to gauge the variation in estimates or predictions” (NGA & CCSSO, 2010, p. 50). Although not explicitly stated, these documents imply a recognition of the relationship of variation and expectation in the need to find (or predict) a summary message within the variation in a data set.

Konold and Pollatsek (2002) suggested the metaphor of signal in noise, identifying “noise” with variation and “signal” with the expectation or central tendency that can be extracted from the noise. In addition to Konold and Pollatsek’s seminal work exploring measurement data, repeated measures, and rates (as probabilities), other researchers have considered the juxtaposition in terms of variation and expectation. Watson and Kelly (2003, 2004, 2006) considered the dilemma of relating variation and expectation in predicting outcomes when tossing dice, when spinning a spinner, or when making decisions in a sampling environment. Further, the work of Reading and Shaughnessy (2004), which focused on variation in a chance setting drawing lollies from a bowl, considered expectation in terms of middles and proportions.

Young students’ exposure to variation in educational settings has been more limited, given that their experiences with probability tend to focus on chance events for well-defined sample spaces in which equally likely outcomes are assumed, such as those of a die, a spinner, or a coin (e.g., Jones, Langrall, & Mooney, 2007; Rubel, 2007). The variation in outcomes is often not emphasized in these learning experiences, in which expected outcomes are addressed before, or even to the neglect of, variation. Although there has been substantial research on the teaching of probability, the literature still appears lacking on young students’ understanding of the links between variation and expectation when learning about probability (Jones et al., 2007; Langrall & Mooney, 2005).

Perspectives on Probability

The tension between variation and expectation is mirrored in the ways that probability has been approached over time (e.g., Batanero et al., 2005; Chaput,

Girard, & Henry, 2011). The classical, Laplacian probability is derived from a sample space of equally likely outcomes by calculating the proportion of favorable outcomes compared to the total possible. This was the traditional approach to teaching probability as mathematics, usually labeled theoretical probability. In contrast, relative frequency estimates of probability are based on data collected through a random experiment or simulation of the behavior of the phenomenon in question. These estimates stabilize when the experiment or simulation is repeated a large number of times and the variation seen in the relative frequency estimates reduces. Hence, the theoretical probability is the expectation or expected value of the phenomenon. If there is no way of conceiving of equally likely outcomes for a theoretical probability, then the relative frequency approach may be the only way of estimating what Konold et al. (2011) called the "true" probability. The theoretical probability in other cases may or may not equate to the true probability, depending on how well the assumptions (e.g., equal likelihood of outcomes) fit the phenomenon. In this study, the idea of true probability, distinct from theoretical or estimated probability, was not introduced to the Grade 4 children because it was considered unnecessarily complex, given the general lack of previous exposure of the students to instruction related to probability.

Few studies have attempted to develop young students' understanding that relative frequency estimates approach theoretical probability as the number of trials is increased. Studies by Polaki (2002) and Pratt (2000) investigated this understanding, with Pratt's research yielding more effective outcomes largely because of students' control over the computer-based resources in generating their own data. Ireland and Watson's (2009) study with a Grade 6 class indicated similar benefits of student-controlled resources using simulations in TinkerPlots™ (Version 2.0; Konold & Miller, 2011) to model tossing coins with the class and later to model rolling a die with individual students. These activities were effective in developing an understanding of increasing sample size as linking relative frequency estimates of probability and theoretical probability.

The ability to appreciate the connections between experimental estimates of probabilities and theoretical probabilities can sometimes be hampered by students' intuitions about probability (Hawkins & Kapadia, 1984), yet this aspect appears not to have received the required attention in instructional efforts (Sharma, 2014). Intuitive ideas about probability may be viewed subjectively as "an expression [or expressions] of personal belief or perception" (Hawkins & Kapadia, 1984, p. 349), which in many cases are influenced by cultural factors (Amir & Williams, 1999; Greer, 2001; Sharma, 2014). Out-of-school biases toward certain outcomes can dominate students' beliefs about chance events: for example, getting a six on the throw of a die is perceived as being less likely than getting a one because the former outcome is more valued in many board games. Hence, from personal experience some numbers may be perceived as less likely to occur than other numbers. Intuitive beliefs about random devices can further influence students' responses. For example, beliefs that luck can contribute to the outcome of tossing a coin, that how one tosses it can have an impact, or that the preferred outcome

(e.g., obtaining a head) will be produced can all impinge on children's probability growth (Amir & Williams, 1999).

An objective example of intuitive probability that can detract from developing the experimental–theoretical links is the *equiprobability bias*, that is, the tendency to view any random trials of an experiment as sufficient indication of equally likely outcomes. The expectation that there is the same chance of obtaining a five and a six as two sixes when rolling two dice is one example of this bias, which appears to be common across age groups regardless of the context being considered or a person's background or gender (Amir & Williams, 1999; Khazanov, 2008; Lecoutre, 1992; Lecoutre, Durand, & Cordier, 1990).

Although the need to target students' intuitive beliefs about probability in instructional programs has long been noted, research addressing this issue with younger students remains scant. Past research has suggested exploring students' use of heuristics in dealing with probability situations. For example, the heuristic of representativeness is common in both students and adults when making predictions about random events such as viewing successive heads on a fair coin as more likely to be followed by a tail than by another head (e.g., Fischbein & Schnarch, 1997). Other research (Konold, Pollastek, Well, Lohmeier, & Lipson, 1993; Rubel, 2007), however, has indicated that such heuristics do not adequately account for the decisions that students make in determining probabilities.

One approach, which we have followed in this study, engages students in predicting the outcomes of trials, whether based on a previously "known" theoretical probability or not, and then confirming or disconfirming their predictions with experimentation. Such experiences enable students to appreciate the variation that occurs in random trials. We argue that the fundamental relationship between expectation and variation is critical to students' development of formal probability models. Our concern is that students' intuitive beliefs often lead to expectations that are confirmed by experimentation expressed in relative frequencies, which are then treated as "the answer" without an appreciation of the other relationships involved. The activities in this study were meant to reinforce appreciation of the core constructs and relationships among them, leading to a sophisticated model with general application for other situations (see Figure 1).

Probability Models

Instructional recommendations for advancing students' probability understanding also include emphasizing the generation and application of models (e.g., Batanero et al., 2005; Fischbein, 1975, 1987; Greer, 2001; Greer & Mukhopadhyay, 2005; Pfannkuch & Ziedins, 2014; Steinbring, 1991). Indeed, Greer and Mukhopadhyay maintained that "probability is all about modeling" (p. 315) and considered, for example, that assigning the probability of 1/2 to obtaining a head on tossing a coin is a process of construction and modeling.

Models are typically conveyed as systems of representation in which structuring and displaying data are fundamental; the structure is constructed, not inherent (Lehrer & Schauble, 2007). The importance of generating representations and

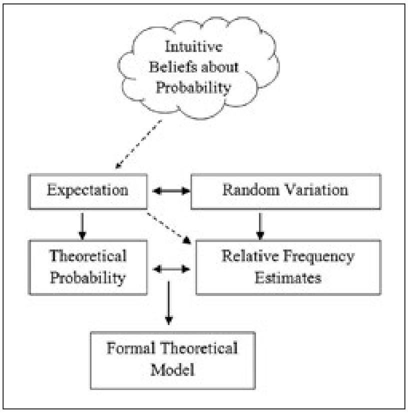

Figure 1. Pathway for developing probability understanding. Core constructs are presented in rectangles. The dashed-arrow path represents how students' intuitive beliefs lead to expectations that are confirmed by experimentation expressed in relative frequencies. Solid-arrow paths represent relationships between constructs.

models in developing students' probabilistic reasoning has been highlighted over the years (e.g., Batanero et al., 2005; Fischbein, 1975, 1987; Greer, 2001; Langrall & Mooney, 2005; Prodromou, 2014; Steinbring, 1991). Yet, limited attention has been devoted to young students' capabilities. The majority of the research on modeling has been concerned with the secondary and tertiary levels (e.g., Garfield, delMas, & Zieffler, 2012; Prodromou, 2012), which is not surprising given the traditional assumption that elementary school children are unable to develop their own models and sense-making systems for dealing with complex situations (Greer, Verschaffel, & Mukhopadhyay, 2007). Some research (e.g., English, 2012; Lehrer & Schauble, 2005) has revealed how very young students can engage in data modeling in which they investigate meaningful phenomena; identify the various attributes; and subsequently organize, structure, visualize, and represent the data collected.

Of relevance to the present study are the nature and the interpretation of the representations and models students produce in exploring probabilistic situations involving student-directed, hands-on, and simulated activities. In particular, the inclusion of computer-generated data in such experiences appears particularly effective in challenging students' intuitive notions in dealing with common chance events (Paparistodemou, Noss, & Pratt, 2002; Pratt, 2000). Batanero, Henry, and Parzysz (2005) discussed how simulation can serve as a *pseudoconcrete model* for various real-life situations, enabling opportunities for students to work without the formalism typically associated with the teaching of probability. In essence, the use of simulation "can act as an intermediary between reality and the mathematical model," facilitate students' learning of the different processes of modeling, and assist them in discriminating between the model and the reality it represents (Batanero et al., 2005, p. 31).

Modeling Processes

Drawing on the work of Henry (e.g., 2001), Batanero et al. (2005) identified three different phases in modeling probability, which we have adapted for this study in terms of developmental phases for working with young students. In our first phase, that of concrete experimentation, students observe the outcomes of a hands-on activity and describe their findings in their own words. Batanero et al. referred to this as the "pseudoconcrete model level" in which a "working hypothesis" is explored (p. 32). This first phase corresponds to the beginning processes of the data modeling approach in which students commence with an investigative question about a meaningful phenomenon, identify its attributes, decide what is relevant to the situation being investigated, and make predictions of possible outcomes. The gathering and initial recording of data (outcomes) through simple experimentation completes this first phase. At this point, students' intuitions about possible outcomes might begin to be challenged, such as assuming that a six is more difficult to obtain than a one on the roll of a die or that how one holds the die influences the outcome.

The second phase, which is less abstract than Batanero et al.'s (2005) "formalization of the model" (p. 33), involves organizing, structuring, visualizing, and representing the data produced. We view this as a transitional, predominantly representational, phase in which students' initial predictions or expectations are refuted, confirmed, or modified with the generation and representation of increasingly more data. Ways in which students record and display their data can include various forms such as lists, tally charts, tables, dot plots, bar and circle graphs, or pictorial representations.

Interpreting the messages conveyed by these representations is a key process in students' reflections on their initial prediction or working hypothesis. By generating more data via simulation, students are better able to observe the relationship between their expected outcomes and the emergence of, and changes to, the experimental estimates of the probability. The simulation is a core process in transitioning the students to an appreciation of how the experimental estimates are approaching the theoretical probability. By directly controlling and observing the progress of this alignment through simulation, students can better appreciate the role of variation in making decisions about chance events.

We view the third phase of our model development as the construction of a formal, mathematical (theoretical) model in which the findings from the previous phase are represented symbolically and diagrammatically. This construction phase is designed to link the initial concrete experimentation with the abstract concepts drawn from the findings. Importantly, students need to be able to interpret their formal model, explain what it is conveying, and relate it to their investigative question. Batanero et al. (2005) highlighted the importance of this model validation process in their third phase in which students can appreciate the relevance of their findings to the real-world situation. Similarly, Konold and Kazak (2008) stressed "model fit" in their study introducing the Sampler in TinkerPlots and focusing on data in chance and chance in data. This core process of translating between a model generated and the world it represents has long been emphasized

in the literature (e.g., Blum & Leiß, 2007; Blum & Niss, 1991). In the present study, the real-world situation was the tossing of coins, which has been a popular source of experiments for exploring students' understanding of probability (e.g., Konold et al., 1993; Moritz & Watson, 2000; Rubel, 2006, 2007).

Research on Coin Tossing

Although there have been some studies focusing totally or mainly on tossing various numbers of coins (e.g., Konold et al., 1993; Moritz & Watson, 2000; Rubel, 2006, 2007), of interest to the present study is research with a focus on tossing one and two coins. Prominent among these studies is that of Carpenter, Corbitt, Kepner, Lindquist, and Reys (1981) involving two coins. Reporting on student outcomes from the National Assessment of Educational Progress (NAEP) on predicting the probability of obtaining a head and a tail, Carpenter et al. found that 60% of 13-year-olds and 69% of 17-year-olds gave the correct answer of 1/2; however, when asked about the probability of obtaining two heads, 58% of 13-year-olds and 50% of 17-year-olds again gave the response of 1/2. Carpenter et al. attributed this inconsistency to "the students' reliance of 1/2 as being correct" (p. 344). It would seem that this observation was a precursor to Lecoutre's (1992) description of the equiprobability bias.

Further illustration of this inappropriate thinking is evident in Rubel's (2006, 2007) studies with 173 students in Grades 5, 7, 9, and 11 in which she adapted the question for predicting the probability of obtaining a head and a tail reported by Carpenter et al. (1981). Fifty-four percent of students gave the correct answer of 1/2, 23% stated 1/3, 13% said 1/4, and 11% gave another or no answer. Generally, the students answering 1/3 explained that there were three equally likely outcomes: 2 heads, 2 tails, and 1 of each. Of those who answered 1/4, some considered the order of the 2 coins and the compound event of one outcome followed by the other, hence misinterpreting the question. Of the 93 students who responded 1/2, 49% (mostly in Grades 9 or 11) explained their response based on some type of sample space argument. Thirty-two percent (mostly in Grades 5 and 7) justified their responses of 1/2 with a fifty-fifty type answer without addressing the sample space. Assigning a probability of 1/3 to tossing a head and a tail, assuming an equally likely sample space of three possibilities (2 heads, 2 tails, or a head and a tail), is historically a common response even among adults including mathematicians, as Hawkins and Kapadia (1984) noted.

Research Questions

In investigating young students' development in understanding the relationship of experimental estimates of probabilities based on relative frequencies and the theoretical probabilities of tossing one or two coins, we were interested in the following questions. Question 1 corresponds to Batanero et al.'s (2005) first phase of modeling probability, Questions 2 and 3 are related to the second phase, and Question 4 is associated with the third phase.

- *Research Question 1*: What are students' initial expectations, expressed as predictions, and experiences of the outcomes of tossing one and two coins?
- *Research Question 2*: How do students' expectations, expressed as predictions, emerge as they increase the sample size of their trials, experience reduced variation, and represent their findings?
- *Research Question 3*: How do students relate the observed relative frequencies to the theoretical probabilities?
- *Research Question 4*: How do students construct and interpret formal models of the theoretical probabilities for the coin outcomes?

Method

Participants

Fourth-grade students from a large, middle-socioeconomic, government school in Australia participated in the first year of a 3-year longitudinal study (2012–2014). Specifically, we report here on the responses of students from four Grade 4 classes as well as Grade 4 students from one Grade 4/5 class (*n* = 91; mean age = 9.75 years).[1] English was a second language for 43% of these students but most appeared of the same mathematical ability as the English-speaking students. Only students whose parents had given ethical approval for their participation are included in our results.

In terms of probability, the students had had previous experiences in relation to disjoint and independent events, as well as ordering everyday events. These experiences reflected the following descriptors for Grade 4 from the *Australian Curriculum: Mathematics*:

- Describe possible everyday events and order their chances of occurring
- Identify everyday events where one cannot happen if the other happens
- Identify events where the chance of one will not be affected by the occurrence of the other. (ACARA, 2015a, "Year 4 Content Descriptions: Statistics and Probability: Chance," para. 1)

Design

The 3-year longitudinal study adopted one form of design-based research (Cobb, Jackson, & Munoz, 2016) involving three phases conducted in each year of the study, with subsequent years informed by the outcomes of the previous years. The first phase focused on the development of the instructional materials with input from the teachers. The second phase took the form of a teaching experiment with the teachers taking primary responsibility for implementing the instructional program. The third phase was devoted to retrospective analyses of the data and the overall activity design and implementation.

[1] For some items analyzed, not all students responded; hence, the reported *n* may be less than 91.

Phase 1: Activity design. The activity reported here was the third of seven major activities in the project. Each activity was adapted according to the learning outcomes observed in the previous activity. Each activity was created in collaboration with the teachers and was part of their regular mathematics program in the area of statistics and probability.

Students had experienced two activities in the project prior to the one described here: (a) a benchmarking activity involving student-constructed surveys about their new playground (English & Watson, 2015b), which was designed to assist the researchers in subsequent planning, and (b) a measurement activity focusing on variation in students' arm span measurements (English & Watson, 2015a). The computer software TinkerPlots was introduced in the second of these prior activities for students to explore and compare variation in two data sets of arm span measurements using the Plot feature.

Probability activity. The third activity for the first year of the project, the focus of this article, addressed probability and introduced students to the Sampler in TinkerPlots to enable them to conduct large numbers of simulations of trials of tossing one coin and later two coins. The activity commenced with a review of everyday events that are *certain, uncertain*, or *impossible* and was followed by a brief discussion of the possible outcomes of tossing dice and coins and the certainty of each outcome. This included events that were disjoint or independent, but in most classes, this language was not introduced; rather, the language from the curriculum was used.

Prior to the tossing of one coin, students were to predict the outcomes. The trials were then to be carried out within the groups of students and recorded in their individual workbooks. After tossing the coin once, students were asked, "Does the type of coin (50¢, 20¢, 10¢, etc.) you toss influence the outcome?" and "Do you think the first person's toss influenced the second person's toss? Why or why not?" Students were then asked to predict the outcomes for 10 tosses. They were to record the outcomes of the 10 tosses, combine and plot outcomes for the entire class, and discuss how close the results came to their expectations expressed in their predictions. Students were to subsequently repeat their trials with the TinkerPlots Sampler, which would enable them to create a much larger number of trials and observe the variation in outcomes generated. This one-coin component of the task concluded with the students being introduced to a formal probability model, namely, a modified tree diagram comprising a drawing of a 10¢ coin with arrows and labels of 1/2 for each outcome. This model is shown in Figure 2.

Next, the students were to investigate the outcomes of tossing two coins, commencing by listing their predictions of all possible outcomes and the chances of each outcome and then recording a fraction for each outcome. This was followed by predicting the outcomes of tossing two coins 12 times and then trialing and recording the outcomes. The number of times the coins were tossed was increased from 10 to 12 to create the possibility of describing some outcomes more easily as the fractions 1/3 and 1/4. Students subsequently were to graph their results in

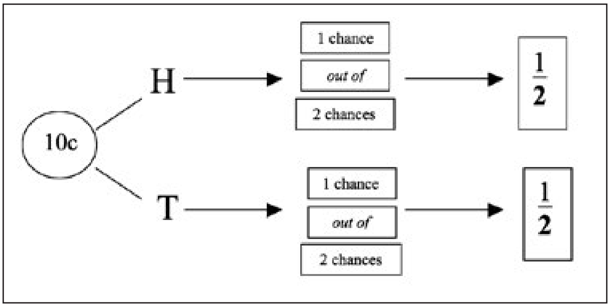

Figure 2. Model presented to students for the outcomes of tossing a single coin.

their workbooks and then create a whole-class plot that combined data from all the groups. The students were to compare their group results with that of the combined class and record their responses to the following prompts: “Explain why the center column (of the plot, displaying the outcomes of a head and a tail) is approximately twice the size of the HH or TT columns” and “Write a sentence describing the difference between the class outcomes and your own outcomes.” The students were next to conduct 500 tosses of two coins using the TinkerPlots Sampler and create a Plot in the software of their group outcomes. This Plot had four columns because the software reported separately results for HT and TH; hence, students were to answer the question “Is the plot different from the one we created for the class results? Why/why not?”

After dragging an icon from the HT column into the TH column, the students’ plot would then display three columns, the same format as the class display. They were asked, “Is this more like the graph created in class? Why?” and “What do you observe about the size of the combined HT/TH column compared to the HH and TT columns?” The students were then to run the Sampler again to generate another set of 500 tosses of the two coins, discussing any variation in their results with their group. The groups were to repeat the process with five sets of 500 tosses and record their outcomes (HH, one of each, TT) in a table. The activity concluded with the students constructing a model of their own to show the possibilities of the outcomes when two coins are tossed and then explaining how their model “works.”

Phase 2: Activity implementation. For the implementation of the activity, a detailed lesson outline was prepared for the teachers as well as a workbook for the students. The lesson plan included the relevant descriptors from the *Australian Curriculum: Mathematics* (ACARA, 2015c) for Grades 4 and 5 for probability and links to a state resource curriculum, which the teachers were being asked to implement that year. The student workbook consisted of instructions for the hands-on activities with coins together with questions about the outcomes requiring students’ written responses. The workbook also included instructions with screenshots for setting up the Sampler in TinkerPlots, which the students were expected to create for themselves.

The teachers implemented the activity across a whole school day. The authors were in attendance for the entire activity to observe the students' learning in each of the five classrooms and to offer guidance to the teachers when necessary. Given that the teachers' involvement in the study was vital, professional development sessions were conducted in preparation for the implementation of the activity. These sessions were followed by debriefing sessions during which the teachers reflected with us on the students' learning and their own professional growth.

The students worked in small groups, mostly pairs, using a laptop with TinkerPlots installed. Although the activity of coin tossing and carrying out simulations was conducted in a group situation, students were asked to write their own answers and explanations in their individual workbooks.

Data collection included the video- and audio-recording of all class discussions, which were subsequently transcribed for analysis. Other data for the activity included photographs, student workbook responses, and notes of researcher observations. When time permitted at the end of the activity, students made oral presentations to the researchers, providing further insight into their understanding. During the course of the study, 27 students gave oral presentations.

Phase 3: Retrospective analysis. The data analyzed in addressing the research questions were sourced from selected items in the students' workbooks together with selected extracts from transcribed class discussions. Because not all students answered all questions, the numbers of responses analyzed per workbook item varied, as indicated in the results section. The data were clustered according to similarities in the understanding displayed in the workbook responses. The coding of the responses generally reflected their correctness together with increasing levels of sophistication in understanding. Data were coded independently by the second author and an experienced researcher, and differences were resolved by negotiation.

For two of the items analyzed from the student workbooks, a hierarchical coding was developed. These items related to explaining the class results for tossing two coins and to producing a model for the probabilities associated with tossing them. This coding was informed by the Structure of the Observed Learning Outcome (SOLO) model of Biggs and Collis (1982, 1991). Of interest here is the concrete symbolic mode of functioning in the SOLO model, applicable in the middle years of schooling based on empirical elements and concrete materials. Within this mode of functioning, the Prestructural level does not employ any elements, and responses are likely to be idiosyncratic. At the Unistructural level, single elements may be employed but are unrelated and may contradict each other. At the Multistructural level, separate elements are employed in a sequence, whereas at the Relational level, all necessary elements are combined in an integrated fashion to produce a conclusion. Engagement and correctness as well as structure were considered in making decisions on levels of response. The hierarchical levels (0 to 4) explained in the Results section are related to the SOLO levels defined here, with Levels 0 and 1 (both Prestructural) distinguished to

indicated that although all Prestructural responses did not engage with the task set, some (Level 0) were idiosyncratic and did not even recognize that outcomes were involved.

Results

In this section, we report on the findings for each of the four research questions, drawing on the data sources indicated. In each case, we give consideration to students' responses from the one-coin investigation followed by those for the two-coin investigation.

Research Question 1: What Are Students' Initial Expectations, Expressed as Predictions, and Experiences of the Outcomes of Tossing One and Two Coins?

For this question, we first consider students' predictions based on one toss and then 10 tosses of the one coin followed by their predictions of the outcomes of tossing two coins just once.

One coin. After a discussion of how easy or difficult it is to predict the outcome when tossing a coin, the process of obtaining a fair toss by shaking the coin in a cup was described. Then students answered Q1 in their workbooks: "With your partner predict one toss and record your prediction in the table below. How certain are you that your prediction is correct? Tick a box □ Certain □ Partially certain □ Uncertain." Students' predictions of the outcomes, their certainty of their predictions, and how these compared with the actual outcomes indicated a basic understanding of chance events. Seventy-five percent of students' responses (n = 89) indicated that they were *partially certain*, and 24% indicated that they were *uncertain* in regard to their prediction. In explaining how their outcomes compared with their predictions, however, only 4% could offer a reason related to chance (e.g., "I predicted tails, and there was a 50 percent chance of getting it"). Most responses referred to the outcome only. In responding to the questions that addressed the influences of type of coin and previous toss, students appeared to display a greater appreciation of chance events, with 42% offering reasons pertaining to chance or to all coins having two sides (e.g., "No, because it doesn't matter what type of coin it is you can flip and it has a 50/50 chance"). Likewise, 88% of responses (n = 87) indicated that one person's toss would not influence another's, with 55% of students providing a reason related to chance or independence (e.g., "The first toss didn't influence the second toss because the events are independent and there is an equal chance").

In investigating students' initial predictions of the outcomes of tossing a coin 10 times, they were asked to predict the number of heads and tails and justify their response and then predict whether exactly the same number of outcomes would be obtained on yet another 10 tosses and why. Nearly half (45%) of the students' responses (n = 86) indicated that they predicted a definite equal chance of obtaining heads or tails, whereas 35% expressed uncertainty or indicated some

form of variation or randomness, such as "There[']s no predicting what you would get but it would probably be close to 5/5 because there is an equal chance on both sides." Very few responses ($n = 5$) indicated an equiprobability bias, as in, "Well, I could get any mixed number of all of them so it doesn't exa[c]tly matter."

In considering whether the same outcomes would occur on yet another 10 tosses of the one coin, the majority of students (77%, $n = 86$) did not believe that this would be the case. However, only 29% offered an explanation highlighting an awareness of randomness or independence, such as "because the activity is independent" or "I don't think that they will be the same because it is always random."

In predicting how many heads out of 10 they would actually toss with the one coin before undertaking 10 tosses, a surprising 42% ($n = 88$) gave no response or an idiosyncratic reason. This might have reflected their uncertainty in actually making such a prediction. Twenty-seven percent stated a definite five heads or equal chance, whereas the remaining 31% expressed uncertainty due to chance, with 20 students referring to randomness or some form of variation. Responses of the last type included "4/10, there is an equal chance so it would be around 5."

All but eight students ($n = 87$) offered an acceptable answer in describing how close their prediction was to their outcome, with students noting that these aligned or differed. Sample responses included "Very close there was only 1 off" and "My prediction was exactly right because I rolled 6 heads and 4 tails." Not many, however, went on to acknowledge the overall expectation and range of values. One student whose result was 6H4T said, "My prediction was right because I said it might be equal or it will be a close number."

Two coins. In their workbooks, students were asked to "List the **outcomes** that could occur" for "tossing two coins at the same time." Then they were asked "to create a **model** of all of the possibilities" including "fractions for each." Not surprisingly, for students with little experience in analyzing the possible outcomes from tossing two coins, the teacher's initial mention of three outcomes (2 heads, 2 tails, or 1 of each) prompted many students (74%; $n = 91$) to suggest three outcomes but with a mixture of associated probabilities. Although 23% of students could list four outcomes, only two students could correctly associate the probability of 1/4 with each. Table 1 details the subcategories of responses comprising either three or four outcomes, with 35% of students not assigning specific probabilities to their outcomes and the same percentage assigning inappropriate values, often not totaling 1.

Research Question 2: How Do Students' Expectations Emerge as They Increase Data Generation and Represent Their Findings?

One coin. The next part of the one-coin component involved collating the class outcomes of the 10 tosses, with students plotting the total number of heads from each group on a number line in their workbooks. The students were to describe the shape of the plot, whether this is what they would have expected and why or

why not, whether every group obtained five heads out of 10 and why not, and where the values in the plot were clustered and why. Examples of student plots for two different classes are shown in Figure 3.

Table 1
Preliminary Expectations for Outcomes of Tossing Two Coins

Category	Description	Subcategory	Description	Frequency
A	Four outcomes			21
		A1	Distinguishes 4 outcomes with probabilities	2
		A2	Distinguishes 4 outcomes, no probabilities	13
		A3	Distinguishes 4 outcomes, incorrect probabilities	6
B	Three outcomes			67
		B1	Distinguishes 3 outcomes, probabilities 1/3, 1/3, 1/3	22
		B2	Distinguishes 3 outcomes, no probabilities	19
		B3	Distinguishes 3 outcomes, inappropriate probabilities	26
NA	Idiosyncratic/ No response			3
Total				91

Note. This table is from "Confronting Expectation in Grade 4: Tossing Two Coins," by J. Watson and L. English, 2015, *Jornal Internacional de Estudos em Educação Matemática* [International Journal for Studies in Mathematics Education], 8(3), p. 11. Copyright 2015 by the *Jornal Internacional de Estudos em Educação Matemática*. Reprinted with permission.

Just over half of the students (53%, n = 88) gave an acceptable description of the class plot, with reference to a mountain shape, geometrical properties, and other features (e.g., "Triangle looks like there[']s big apartment and little houses but there[']s not many things in this town," and "2 small lines and then 1 big line

then it declines back to 1 agen [again]; it looks like a snoonamy [tsunami].” Fewer students (33%) mentioned a middle value or a clustering of values on the plot, with comments including “There is a big line of people in the middle but it dies out any direction you go” and “Big and tall in the middle and on the outside it gets really small.”

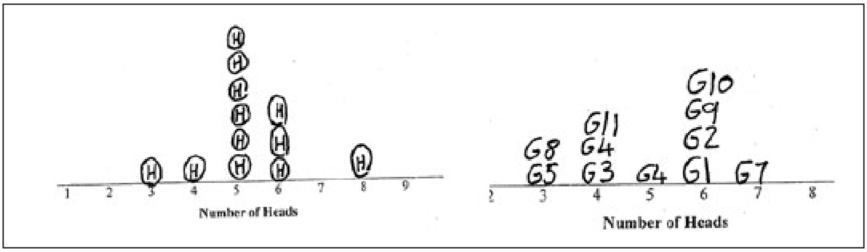

Figure 3. Number of heads from 10 tosses for groups in two different classes.

Student responses ($n = 88$) were mixed on whether the results shown in the plots were what the students expected, depending to some extent on the individual class’s data. Some students still expected values to be more widely spread than shown in the plots. For the class data represented on the left side of Figure 3, most students agreed that the results were as they expected and did not mention the relatively large number of times 5H occurred. For the class data represented in the right panel, only three students expressed explicit surprise that there was only one result of 5H. Overall, just under half (43%) provided an explanation that indicated recognition of a middle value or clustering. Their responses included “Yes, the majority of people had 5 heads and I expected that,” “Yes because it was likely there will be a lot nubmers [numbers] between 4 + 6 cause there [they’re] near five,” and “[Yes], Because it was the average number.”

When specifically asked where the values in the plot were clustered and why, just over half of the students (51%, $n = 88$) gave an explanation related to a clustering in the middle, such as “In the middle (5/10) because lot of people scored equally.” Only 30% of responses included a specific reference to chance, such as “They are clustered around 5 and 6 because those were the most likely and 1, 3, 7, and 8 were less likely.”

Two coins. For the two-coin component, students predicted the outcomes for 12 tosses of two coins, conducted the trials, recorded the outcomes in a table, and graphed the results in any way they preferred. No instructions were given for the representation. Several examples of the graphs produced are provided in Figure 4. Students’ main experience with graphing prior to this study was using bar graphs, although they had been introduced to stacked dot plots through their use of TinkerPlots earlier in the study. These were the two most frequently used types of

graphs for this activity, along with variations on tally tables and pictographs. When asked to explain any difference between their predictions and results, students generally focused on the detail rather than a deeper interpretation: "The difference is I peredicted [predicted] 4 for all of them and I got 2 HH's 2 TT's 8 TH's"; "The differance [difference] between the two is a lot more H/T happened than I expected."

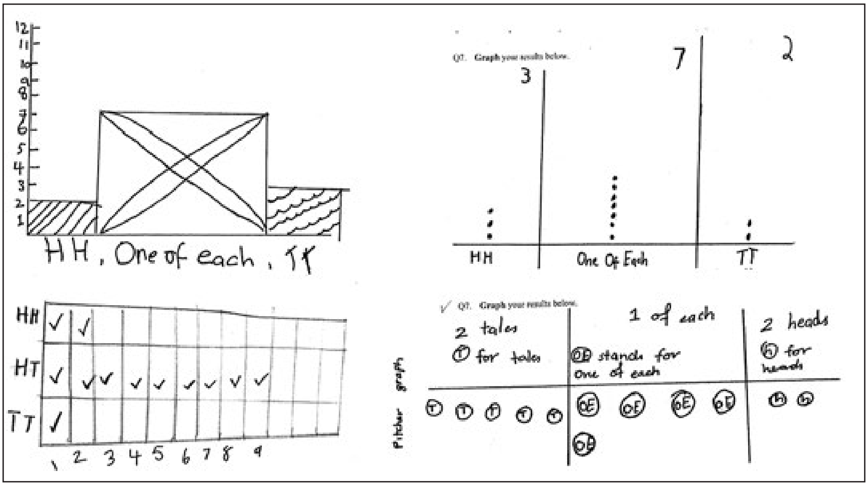

Figure 4. Four examples of graphs produced for outcomes of 12 tosses of two coins.

Research Question 3: How Do Students Relate the Observed Relative Frequency Estimates to the Theoretical Probabilities?

In exploring this question, we were interested in what the students had learned about the effect of a substantial increase in trials on the alignment of the experimental estimates and theoretical probabilities of tossing one and two coins (H and T for one coin; HH, TT, and one of each for two coins). The Sampler in TinkerPlots served to generate increasingly larger numbers of trials for the one-coin component and combined outcomes for the two coins. We were especially interested in students' learning with respect to the tossing of two coins, given the typical expected probabilities of 1/3 for HH, TT, and one of each, and the reported difficulties of both students and adults in appreciating why this is not the theoretical probability.

One coin. Using the Sampler to make repeated trials of 10 tosses of one coin, students recorded the number and percentage of heads generated each time and described the variation that they noticed. Generating yet more samples of trials of 10 tosses with the Sampler clearly enabled the students to see the variation in the outcomes, with 91% ($n = 88$) offering an explanation that corresponded with the data generated. Students' responses included "That the highest amount of heads were [*sic*] 7 and we never got zero," "It's mostly in the middle because it's equal. It's from

20% to 70%," and "The variation is between 20% and 60%. The difference is 40%."

The one-coin component concluded with the students using the Sampler to increase the number of tosses to multiples of 100 and 1,000 and noting changes in the percentages of heads as the number of repeats increased. Although the students had some difficulty in interpreting the percentages observed, in 59% of the responses (n = 88), students explained that the results approximated 50% more closely as the number of trials increased. For example, students explained, "Bigger number of repeats, less range," and "When you repeat the number by 1000 the % is closer to 50/50%." Interestingly, some students tested even larger numbers of trials, for example, reporting "We did 10000 there were [*sic*] no variation from 50 percent. 4996 (50%), 5004 (50%)."

Two coins. Before students used the Sampler for noting the effects of increased trials on the outcomes of tossing two coins, the group results for tossing two coins 12 times were combined to produce a whole-class plot. An example of one class's plot is displayed in Figure 5, in which students placed sticky notes on a white board. Similar plots were produced in each class.

Our aim here was to see if the students could explain why the shape of the

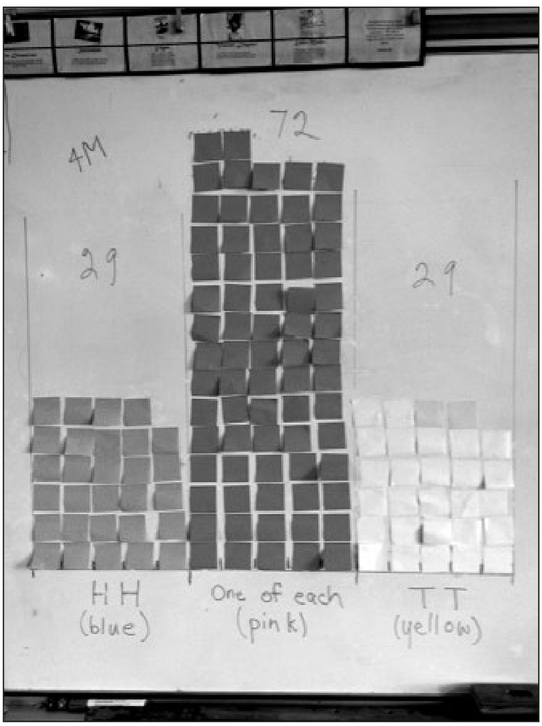

Figure 5. Example of combined class results for tossing two coins for one class.

combined class was about twice as high in the middle for HT and TH. Of interest were the levels of response and particularly those that could go beyond the Unistructural, single visual observation of "more" to explain the combination of outcomes in a Multistructural, sequential fashion. Table 2 presents a summary of the levels of response with examples and frequencies (cf. Figure 5). Sixty-five percent of students had at least an intuitive appreciation of the reason for the observed outcomes.

Table 2
Students' Responses to the Shape of the Class Graph of Outcomes Tossing Two Coins 12 Times

Level	Description	Examples	Frequency
2	Explanation of more possibilities/ combinations for {H, T}	• Because TH and HT are the same [s]o it has a double. • Because that has a better chance because it is 2/4 chance and the others are 1/4. • Because HT has a turnaround of TH, so the chance doubles.	25 (27%)
1	Observation of "more" outcomes, likely, chance	• Because it's more likely to have one of each. • The centre column was bigger than the other columns because most people got more one of each. • Its because it would be easy chance of getting a H, T together.	35 (38%)
0	NR, inappropriate or idiosyncratic reason	• Maybe because of the way the person shakes might make an influence on the way it outcomes. • I think it is just by chance because they are all 1/3 of a chance. • It is bigger because when you shake the cup it changes.	31 (34%)

To investigate whether the students would focus on the overall shape (in a proportional sense) of the class plot and their individual group plots rather than on the numerical details, we asked the students to describe the difference they noticed between the two. In general, the students recognized the similarity, with more than 60% demonstrating an intuitive awareness of the difference in proportions. Their explanations included "Our outcomes matched the class outcome

where one of each were most, H & H in the middle and the lowest was T & T," "They were similar, we did have one of each way larger but the actual answers were way way smaller," and "My outcomes for one of each is biggest and in the class is the biggest, but I got 6 and the class got 66."

After learning how to use the Sampler in TinkerPlots to conduct large numbers of simulations of tossing two coins, the students conducted 500 trials and kept track of the number and percentage of each outcome produced. The purpose of this part of the activity was twofold: first, to support the understanding that for a much larger number of trials, the results would approach 25% for each of two heads and two tails and 50% for one of each, and second, to consolidate the understanding that the HT and TH outcomes from the Sampler would be combined.

To develop the above understanding, the workbook guided the students through a number of steps asking them to explain what they observed to be the difference between the TinkerPlots graph and the class results graph (four columns in TinkerPlots and only three in the class graph). Seventy-five percent of students responded appropriately. The other 25% of students replied with other characteristics of the graph or process, such as "Yes, it's much bigger," "Yes because the data in each computer[s] are different," or "It's different because we have 2 coins." Next, when asked why the software-generated graph was more like the class graph after combining two bins for HT and TH, again 75% of students suggested an appropriate reason, with 1/3 of these specifically noting the combining of the two outcomes HT and TH. Nevertheless, 25% still struggled, providing no response or responses such as "No TinkerPlots are much bigger" or "Yes but the TinkerPlots still has % at the top." To provide students the opportunity to mention specifically that the middle column was twice the height of the other two, they were asked, "What do you observe about the size of the combined HT/TH column compared to the HH and TT columns?" Forty-four percent of students did note this difference about the height, whereas 44% recorded more general comments, such as "The HT/TH is bigger than the other ones." Only 12% of students did not respond or provided comments that were uninterpretable in the context, such as "They have combined 2 colors."

Research Question 4: How Do Students Construct and Interpret Formal Models of the Theoretical Probabilities for the Coin Outcomes?

Both the one- and two-coin components ended with the students representing their conclusions as formal probability models. For the one-coin representation, the students were introduced to a modified tree diagram (Figure 2) because any such representation was new to them. They were to respond to the question "What does our model tell you?" For the two-coin models, however, no initial model was presented, and the students created their own model in their workbooks and were asked to explain how their model "works."

Students' interpretation of the one-coin model. Four levels of interpretation were identified from the student workbook responses (n = 85). Five percent of

students gave an idiosyncratic response or a response that simply described the diagram, such as “The 10 cent coin has a H and a T with 2 sides.” An additional 9% simply stated that there are “two possible outcomes” or “a coin has 2 chances.” The bulk of the students (59%) assigned a numerical value to the chances but gave no explanation of their value. Examples here included responses such as “50/50 chance,” “equal chance,” or “half a chance” without any accompanying explanation. Only 27% could offer an explanation, such as “half a chance to get heads and half a chance to get tails is another way of saying 1/2” or “The model is telling us that we have a 50/50 chance of getting heads and tails out of 100.” Not surprisingly, the students’ creations of their two-coin models provided richer data and gave an indication of the students’ understanding of the theoretical probabilities of obtaining four outcomes.

Students’ two-coin models. A wide range of models representing the theoretical probabilities of obtaining the four outcomes of TT, HH, TH, or HT was evident in the students’ responses (*n* = 90). Analysis of their workbook models revealed five levels of increasing sophistication in visually representing the probabilities, as described in Table 3. Figure 6 illustrates models drawn in the workbooks at each level. The Level 0 model in the figure is idiosyncratic in focusing on the size of the coins and apparently reflecting this in the suggested fractions for probabilities. At Level 1, the model presented in the figure has not progressed passed viewing the two coins separately with the one-coin model. Levels 0 and 1 are considered Prestructural with respect to the SOLO model because students do not engage with the interaction of the two coins in their models. The example of a Level 2 model progresses to consider repeated outcomes but does not link them to appropriate combinations. At this level, the necessity to recognize the repeated outcomes (an essential element of the model) results in a Unistructural response. The two Level 3 models presented in the figure draw attention either to the illustration with two coins and potential combinations or to probabilities but not to both in the same model. At Level 3, this linking of two of the three essential elements of the model creates a Multistructural response. The two examples of Level 4 models include both an illustration of the combinations and their associated probabilities. Finally, at the Relational level, the three elements are combined for a complete model.

Although asked to do so, not all students provided an accompanying written explanation of their models. The oral presentations from students across the classes provided further insights into their understanding. Figure 7 shows a Level 0 model response in which the student continued to represent three equally likely outcomes and did not progress to the two-coin part of the activity. Rachel explained her model without suggesting probabilities:

> Well I’ve drawn the name of what you can get, of what options you can get for each, and I’ve drawn an example for each of them and then I’ve written there is an equal chance to land on two heads, two tails or um, or a one H and one Tail because there are 3 options it can land on, 2H, 2T or 1H 1T.

Table 3
Levels of Sophistication in Students' Representations of the Probabilities of a Two-Coin Toss

Level	Description	Frequency
4	Model correctly matches heads–tails outcomes with a visual model (or with probabilities) and displays combining of HT and TH probabilities, indicating evidence of understanding of the probabilities of 1/4 (HH), 1/2 (or 2/4; HT, TH), 1/4 (TT)	12 (13%)
3	Model acknowledges repeated outcomes for HT, HH, TH, TT, often in a list form, but only links these *either* to numerical probabilities (1/4 or 1/2) *or* to a visual model not both	28 (31%)
2	Displays or acknowledges repeated outcomes, perhaps as HT, HH, TH, TT, but does not link meaningfully to combinations of two coins (usually to single coins)	12 (13%)
1	Draws two coins but only produces a model for one coin (repeated) and cannot combine the probabilities as either 1/2 or 1/4	26 (29%)
0	Draws process rather than outcomes, retains 3 outcomes model, provides an idiosyncratic response (e.g., size of coin), or No Response	12 (13%)
Total		90 (1 absent)

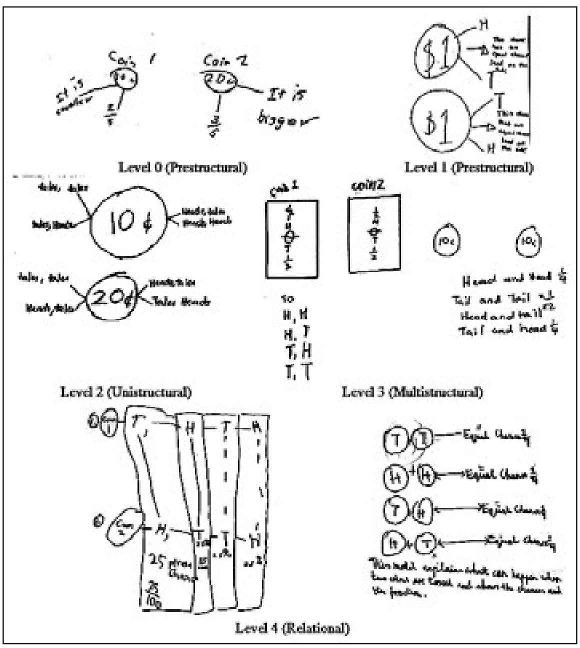

Figure 6. Workbook models for tossing two coins.

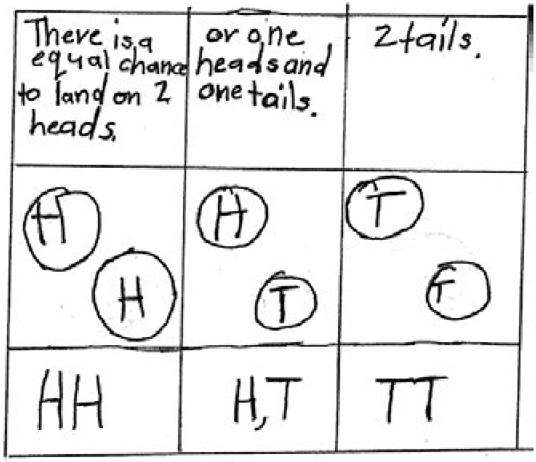

Figure 7. Rachel's Level 0 model for tossing two coins.

It is interesting that Rachel also suggested the three outcomes at the beginning of the two-coin part of the activity with fractions: "two heads 2/2, two tails 2/2, 1 head and one tail 1/2." Although suggesting three outcomes again, she did not suggest specific fractions (e.g., 1/3 representing equal parts).

Sarah produced the Level 1 response illustrated in Figure 8. It shows the single coins with outcomes (head labeled 1/2). Although she attempted to combine head and tail in the middle of the model, her explanation indicated that she had not moved meaningfully past the single coins.

> Well, I put Coin 1 and Coin 2 and then I put a coin that says together on it because for the, when Coin 1 and Coin 2 have drawn together. And I, for Coin 1 I did 1H and 1T and 1/2 and the same for Coin 2 and then when I, when they join together I put HT and then 1/2 and 2/2 because there's like, when a HH go together it's like 2 wholes, 2 halves, and that makes a whole.

Her initial listed outcomes were similar to those of Rachel and several other students in her class.

At Level 2, Mary explained the model in Figure 9.

> Well my model explains the possibility of the coin being rolled, 2 coins being rolled and the possibilities of being rolled together. So first there's like 2Ts which means Ts being rolled together, and I've done a little [key?] [Mary points to box in upper right corner.] so you understand, um T&H and H&H and H&T. Now originally I thought there was only 3 but there's 4. [*R: 4 what?*] 4 like options of rolling. [*R: What do we call those?*] Possibilities. [*R: Possibilities or possible outcomes.*] Yeah or outcomes, yep, more sciency and I've wrote a little thing, do you want me to read it? [*R: Yes please.*] Up the top you will see 2 coins, one's a 10¢, one's a 10¢, one's a 20¢ piece, if you see, up the top you will see, down the bottom I mean, you will see some combinations, these are the possibilities of being put down.

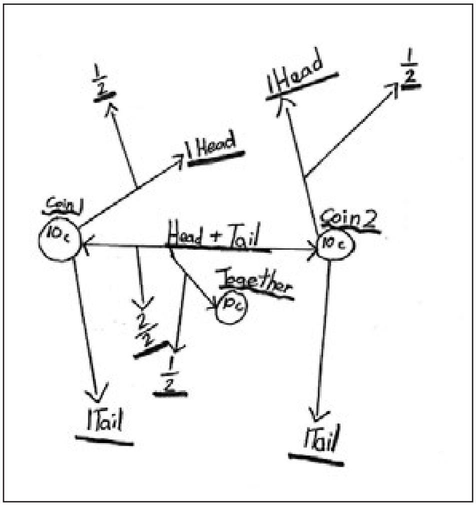

Figure 8. Sarah's Level 1 model for tossing two coins.

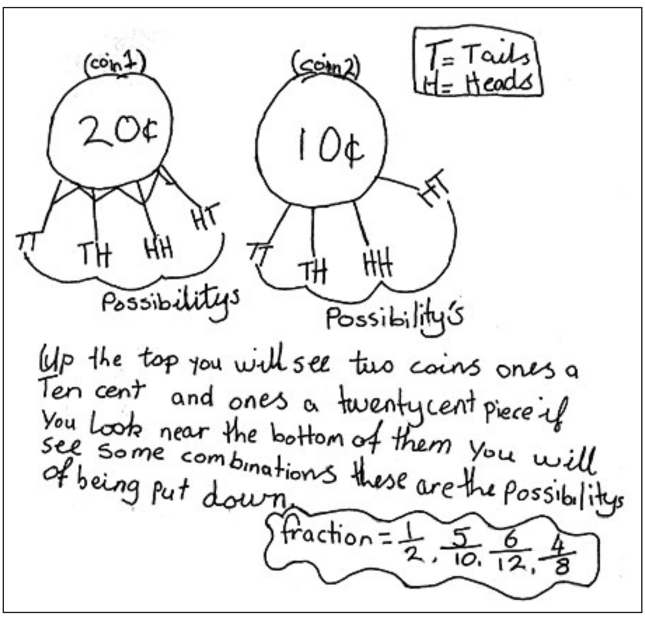

Figure 9. Mary's Level 2 model for tossing two coins.

Mary was then asked to write some fractions next to her possible outcomes and was asked to recall the fraction for the single-coin model. She recalled 1/2 but made the list of equivalent fractions at the bottom of the figure rather than labeling the four outcomes.

An emerging understanding of the derivation of the theoretical probabilities at Level 3 can be seen in Leanne's model in which she constructed a tree diagram and recorded, "This model shows the 4 possible outcomes if you toss 2 coins" (see Figure 10). However, she displayed the outcomes from each coin separately and did not make the appropriate links to combined outcomes to yield the correct probabilities. Although Leanne was able to represent the probabilities of the H and T outcomes for each coin and could list the four possible outcomes (in word form) at the end of her model, she did not list the final probabilities or display their links to the outcomes of both coins. Leanne's oral explanation illustrates the apparent difficulty she had in producing a Level 4 model.

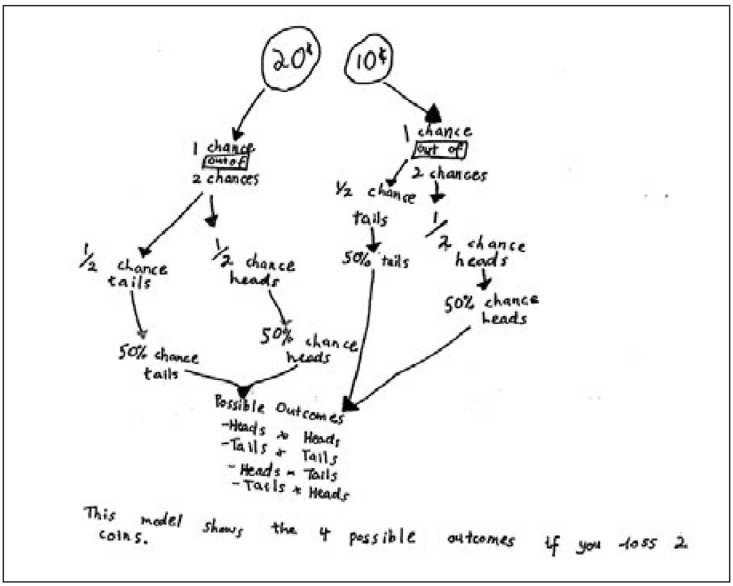

Figure 10. Leanne's Level 3 model for tossing two coins.

> For my model I used a 20¢ and a 10¢ coin and there's one chance out of two chances which is half a chance Tails and half a chance Heads for both coins so that's also a 50% chance. So the possible outcomes are Heads and Tails, Tails and Heads, Tails and Tails, wait a sec, Heads and Heads, Tails and Tails, Heads and Tails and Tails and Heads so this model shows the 4 possible outcomes if you tossed two coins.

As shown in Figure 11, Dagmar's Level 4 model listed eight possible outcomes and linked these to the probabilities of 1/4, 1/4, and 1/2. She explained her thinking in the following way.

> Well, there are two heads. One of the heads is in coin 1 and the other one is in coin 2. Then there are two tails, one of them is in coin 1 and the other one is in coin 2. Then there is one head and one tail, so um, the heads is in coin 1 and the tails are in coin 2 and then there's the same one except they are rotated. The heads are in coin 2 and the tails are in coin 1. [*R: And the fractions?*] And then the fractions is, the first one, two heads is 1/4, the second one is 1/4 as well, two tails, and then both of them, the last one equals 1/2. [*R: How did you work it out?*] Well, I said, um . . . that's 1/4 and that's 1/4, and then I, um, that's an equivalent fraction to 1/2 so yeah, so I just did 1/2.

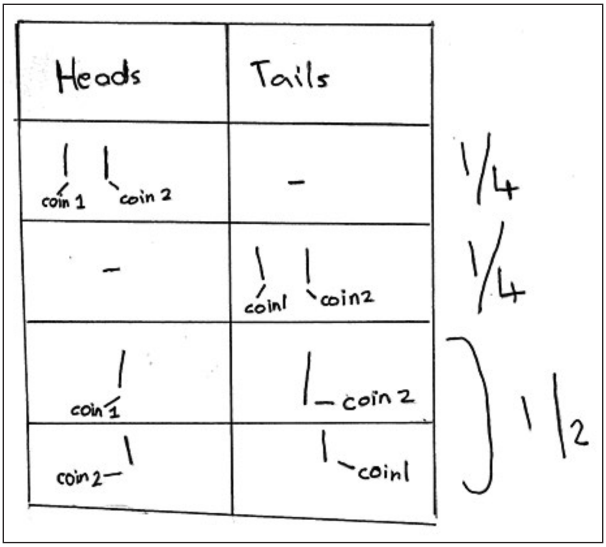

Figure 11. Dagmar's Level 4 model for tossing two coins.

Ruby produced two Level 4 models, one somewhat similar to Dagmar's and another that is shown in Figure 12. In using the word *independent*, it is assumed that she refers to the 10¢ and 20¢ coins.

> I also did the independent one just over here, and this is the combinations and these are all the three things and this Tails and Heads, Heads and Tails, the same thing, 50% or two out of four chance. And Heads and Heads is 25% because it's only one and one out of four chance. And Tails and Tails it's also 25% because it's just one, and 1/4.

These examples illustrate the wide range of approaches taken by students. Although over half of the students appreciated the need to combine the outcomes in a model representing the activities that they had undertaken, many had difficulty in producing a theoretical model that included both diagrammatic and numerical components.

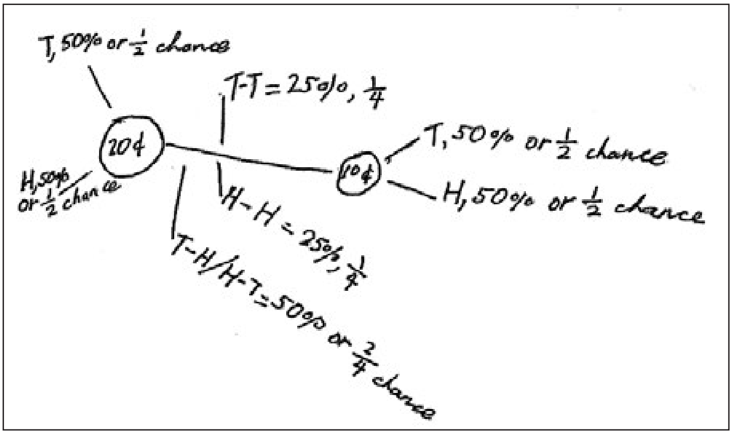

Figure 12. Ruby's Level 4 model for tossing two coins.

Discussion

In developing fourth-grade students' understanding of probability, we focused on the statistical concepts of variation and expectation as foundational learning components. We explored students' initial expectations of the outcomes of tossing one and two coins, how their expectations changed with repeated numbers of trials, and how computer simulation led to students' understanding of the closer relationship between experimental estimates of probabilities and theoretical probabilities as the number of tosses increased. Given that students' informal ideas about probability, often reflecting cultural experiences, can impede appreciation of this relationship, we incorporated an acknowledgement of intuitive probability. The culmination of the one- and two-coin investigations was the creation of formal probability models, considered here as a core link between the initial experimentation phase and the abstract numerical probabilities of the traditional approach to probability. We viewed these culminating models as the third phase within a three-phase development of probability understanding.

In reflecting on this development in the present study, we address Research Question 1 within the first phase, that of concrete experimentation. We address Research Questions 2 and 3 within the second phase, which involves organizing, structuring, visualizing, and representing the data. It is during this second phase that students' initial predictions or expectations are refuted, confirmed, or modified. Through observing the outcomes of increasing numbers of trials through simulation, students are beginning to develop an appreciation of the relationship between relative experimental frequency and theoretical probability. Research Question 4 aligns with the third phase, in which students represent the theoretical probabilities of their investigations as formal models. We review our findings within each phase in turn.

Phase 1: Concrete Experimentation (Research Question 1)

Children's initial expectations of the outcomes of tossing one coin suggested a basic awareness of the uncertainty of chance events and some intuitive appreciation of probability with respect to independence of outcomes and coin type. However, the students were less able to explain in terms of chance how their outcomes of tossing a coin once compared with their predictions. Although there was little evidence of an equiprobability bias (Lecoutre et al., 1990) in the students' expectations of tossing a coin 10 times, with nearly half expecting an equal number of heads and tails, this could also reflect a strict focus on theoretical expectation while ignoring the potential for experimental variation. Nevertheless, there was evidence of some students' awareness of the uncertainty of making such predictions, and overall, the students did not expect identical outcomes if the coin were tossed a further 10 times. Finally, many students appeared reluctant or unable to predict their outcomes prior to conducting the experiment of tossing their coin 10 times. This result could have reflected a lack of experience in making such predictions or could have been due to their uncertainty in actually doing so. Konold et al. (2011) might suggest that these students were reluctant to predict because they were thinking of the "true" probability rather than the classical, theoretical probability, which assumes equal likelihood of outcomes, or the experimental relative frequency, which is only an approximation based on a given number of trials. We cannot speculate on this possibility because we did not introduce the added complexity of true probability to the teachers and students in the study. Nevertheless, this finding points to the need to encourage young students to contemplate initially the possible outcomes when addressing an investigative question.

Not surprisingly, many students' predictions of the outcomes of tossing two coins were in accord with the common equiprobability response of three equal outcomes, each with a probability of 1/3 (Hawkins & Kapadia, 1984). There were some students, however, who predicted four possible outcomes, but only a couple could offer the associated correct probabilities. The students' difficulties here reinforce the importance of this initial phase in laying the foundations in terms of experiencing the variation in concrete trials and the raising of expectation about a theoretical probability.

Phase 2: Transitional, Representational Phase (Research Questions 2 and 3)

Collating, representing, and interpreting the data gathered from tossing one coin 10 times, both within groups and across class groups, indicated that students were aware of the trends displayed in the representations (in terms of rises and falls in the shape of the data distribution or likenesses to geometric figures). Comparatively few students, however, mentioned a middle value or a clustering of values on the plots until specifically asked about this; they also made limited reference to chance notions in their reasons for this clustering. Teacher questioning plays a vital role in nurturing these emerging conceptions of distribution, as Lehrer, Kim, and Jones (2011) demonstrated in their study of 10- and 11-year-olds designing measures of center and variability of distributed data. It is through

such support that young students' "reach comes to exceed their initial grasp" (Lehrer, Kim, & Jones, 2011, p. 735).

Students were able to create a range of representations of the data obtained from tossing their two coins 12 times. Although bar graphs were the most common, many used stacked plots, with which they had become familiar through TinkerPlots (cf. Figure 4, upper right). Displaying their data from their group experiments provided students with an important grounding for appreciating how the observed experimental frequencies approach the theoretical when increasing the number of trials. Progression to collating and representing combined class data for the 12 tosses with a concrete display further cemented this foundation and may be regarded as a means of connecting the physical actions with data to the more abstract computer-generated representations with the Sampler (cf. the embodiment notion of Hegedus & Tall, 2016; Tall, 2013). This enabled the students to see clearly how the combined class data for the HT and TH outcomes were about twice as high as those for the 2H and 2T outcomes.

Using the TinkerPlots Sampler to conduct large numbers of simulations and recording the outcomes served to develop the understanding that the results would approach 25% for each of two heads and two tails and 50% for one of each. Although the approach is not as sophisticated as Konold and Kazak's (2008) intervention with older students using an early version of the Sampler, the results here suggest that it has promise for helping students make the link between experimental relative frequencies and theoretical probability.

Phase 3: Formal Model Construction Research

The final phase, that of formal model construction, engaged students in demonstrating their experimental understanding symbolically and diagrammatically. The actual model construction, however, is only part of the process; students must be able to interpret their model, explain what it is conveying, and relate it back to their initial investigative question (cf. Batanero et al., 2005; Lehrer et al., 2011). Students' progression from the one-coin to the two-coin models showed substantial development in their understanding of theoretical probability. Because the creation of probability models was new to the students, sophisticated explanations of the values represented by the one-coin model were limited, perhaps reflecting few previous experiences in interpreting chance concepts in terms of fractions and percentages.

In contrast, the two-coin experiments led to students creating a wide array of models with more opportunity to display understanding of the theoretical probability evident. In fact, 44% of students demonstrated the identified sophistication of Levels 3 or 4, displaying the four repeated outcomes as HH, HT, TH, and TT and linking these appropriately to the combinations of the two coins or to the probabilities of the corresponding outcomes. The most sophisticated models provided evidence of an advanced understanding of the probabilities, as was evident in Dagmar's case. As noted previously, when students had the opportunity also to explain verbally their model constructions, greater insights into their levels of understanding were gained.

The importance of students communicating orally to their class peers, sharing their model creations, and explaining what they convey has been long emphasized in the literature (e.g., English, 2012; Lesh & Zawojewski, 2007).

By immersing young students in all phases of the probability modeling processes proposed here, we can establish the fundamental understanding for linking experimental estimates and theoretical probabilities, including the important role of variation and expectation. Variation is the concept underpinning the message that arises from experimental estimates of probability. Here, however, it is a metaconcept compared to how it is usually introduced to school students. When variation is first introduced (long before standard deviation), it is seen as a property of values in a data set and likely to be described by looking at a graphical representation showing spread (e.g., Watson & English, 2013). In the context of this study, the expectations, or estimated probabilities from the various samples, are the values whose variation is considered. Reduced variation from the proposed theoretical probability can be observed in two ways as the sample size increases: either comparing the variation in proportions across several samples of the same size or looking at single samples and at the reducing difference between the estimated probability and the proposed value. The confidence one has in the estimate rests on the confidence one has in the random sampling procedure and the sample size chosen to represent the given phenomena.

The culmination of learning expressed as models in the third phase shifts the focus from traditional, formal computation towards probability as conceptualization, description, and explanation (cf. Lesh & Zawojewski, 2007). In this way, young students make sense of a chance situation so that they can mathematize it themselves in ways that are meaningful to them. As Niss (2010) argued, mathematization of the situation in question is the key aspect of the modeling process, but it presents significant challenges for students. The present study has provided one approach to addressing such challenges in nurturing young students' emerging conceptions of probability.

Limitations

The ability to generalize from this study is limited, as it was based on the Grade 4 students in one government school in Australia. We acknowledge that English was a second language for 43% of the students. And although it was not possible to control for language, there was no indication in the classroom that any student did not understand the tasks that were set. Because students worked in pairs, including pairs in which one student was not a native English speaker, explanations of terms often occurred between students.

The fact that the activity was carried out over one school day may be considered a limitation, but it was necessary to fit in with the requirements of both the school and the research team. The constraints of the study within the school environment meant that it was not possible for the research team to introduce further contexts to reinforce the concepts developed with coins. Dice or spinners provide other contexts in which it is possible to suggest a theoretical model based on computer

simulations, but extending to a context where this is not possible, such as tossing drawing pins or bottle tops, creates complications in completing a large number of trials. The Sampler in TinkerPlots, however, can be set up with a "mystery" model of a phenomenon that students test by sampling to watch outcomes converge and then guess the model (e.g., Konold & Lehrer, 2008). Konold et al. (2011) may argue that we should have introduced the idea of "true" probability to the students. Further suggestions for dealing with this perspective on probability for older students than those in this study are found in Pfannkuch and Ziedins (2014).

Conclusion

Although not all of our Grade 4 students achieved total understanding of our aims in the activity, many did. We agree with Greer (2014), who emphasized the need for educational policy makers to "take seriously the point made by many experimenters and mathematical educators that laying the foundations for probabilistic thinking needs to start early (Meder and Gigerenzer 2014)" (p. 301). Some countries (e.g., Australia and New Zealand) have heeded this message, whereas others (e.g., the United States) are still debating the importance of early introduction to probabilistic thinking.

As noted previously, much of the research based on coin tossing has focused on more than two coins, runs of outcomes, or independent outcomes. This may have resulted from the perception that considering one or two coins was "too easy." This study has demonstrated that when introduced to young students, who are beginning to appreciate chance and probability, activities with one and two coins are not too easy. In fact, they provide the opportunity to experience variation and expectation in a concrete setting that can be extended from real coins to computer simulations, then to a theoretical model. Although not every child in the study created an appropriate theoretical model of the expectation for outcomes when two coins are tossed, every child did experience variation in outcomes and could write about it. This is an advance on the traditional textbook approach of introducing probability as pure mathematics with only the expectation aspect presented. Now that mathematics curricula around the world include statistics as well as probability (e.g., ACARA, 2015c; NGA & CCSSO, 2010), it is essential that the relative frequency estimation side of probability be introduced in the primary grades. Variation is the fundamental underpinning of statistics (e.g., Moore, 1990). Introducing variation through probability activities, such as those in this study, achieves two aims: It makes the concept vivid and memorable, and it forges an initial link between probability and statistics that will be critical throughout the years of schooling.

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Participatory and Anticipatory Stages of Mathematical Concept Learning: Further Empirical and Theoretical Development

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Tzur and Simon (2004) postulated 2 stages of development in learning a mathematical concept: participatory and anticipatory. In this article, we discuss the affordances for research of this stage distinction related to data analysis, task design, and assessment as demonstrated in a 2-year teaching experiment. We describe our modifications to and further explicate and exemplify the theoretical underpinnings of these stage constructs. We introduce a representation scheme and use it to trace the development of a concept from initial activity, through the participatory stage, and to the anticipatory stage.

Key words: Anticipatory; Conceptual learning; Learning stages; Learning theory; Mathematical concept; Participatory; Reflective abstraction

In the context of a program of research on conceptual learning of mathematics, Tzur and Simon (2004) postulated two stages of development in learning a mathematical concept: participatory and anticipatory. As with many theoretical constructs in the mathematics education research literature, the participatory–anticipatory stage distinction cannot be empirically “proved” or “falsified.” It is a way of conceptualizing aspects of mathematical concept development, and, as such, it must be evaluated in terms of its ongoing usefulness. In the first part of this article, we demonstrate the explanatory power of this stage distinction and expand the uses to which it can be put. In the second part of the article, we report on, explicate, and exemplify modifications that we have made to the evolving theoretical basis of the stage distinction. We then use these modifications to provide further theoretical explanation of the stages and representation of their development.

In both the first and the second parts of the article, empirical data are employed as examples to illustrate the theoretical work being discussed not to validate it. None of these examples reflect the level of analysis that would be required in an empirical research report. Therefore, although the data sets from which the examples are drawn deal with the learning of concepts of fractions and ratio,

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there is no attempt here to make any claims about fraction and ratio learning nor to posit learning trajectories with respect to these concepts. Our goal is to illustrate and elaborate frameworks for interpretation of data. In the first part of the article, the examples serve as contexts to demonstrate particular uses of the stage distinction. In the second part of the article, the examples are used to help clarify the elaborations and modifications to existing theory of concept development.

There is a lack of consensus in the research community regarding the meaning of particular terms (e.g., *concept, activity*). Rather than burden the reader with definitions at the beginning of the article, we insert definitions where we believe they are most helpful and at the point where they represent ideas that have been developed up to that point. Our goal is not to persuade readers to adopt our meanings for the terms. Rather, we try to keep the focus on distinctions that we have noted in the development of a mathematical concept.

Participatory and Anticipatory Stages of Concept Development

To illustrate the distinction between the participatory and anticipatory stages of concept development, Tzur and Simon (2004) gave the following example of what they called the *next-day phenomenon*.

> Consider a teacher who engaged students for a few lessons in partitioning paper strips to create unit fractions. Toward the end of this hands-on activity, the students were able to answer questions such as, "Which is larger, 1/6 or 1/8?" The teacher required the students to explain their answers and most students could clearly demonstrate with their strips and argue that 1/8 must be smaller than 1/6, because the strip showing eighths was cut into more pieces; so each piece had to be smaller. The next day, having completed the hands-on portion of the lesson sequence, the teacher begins the lesson by attempting to review the ideas generated by the students during the paper-strip activity. The teacher writes two fractions on the board, '1/7' and '1/5,' and asks which one is larger. To the teacher's surprise, most of the students claim that 1/7 is larger because 7 is larger than 5. The teacher wonders how students can "lose" overnight what they learned the day before. Intending to revisit the hands-on experience, the teacher asks the students to take out their paper strips to set up the problem. Soon after the students begin manipulating the paper strips, and *without completing* a paper-strip enactment of the problem posed, many students, who had earlier claimed that 1/7 was larger, raise their hands to explain how they know that 1/5 is larger than 1/7. (pp. 288–289)

Tzur and Simon (2004) argued that this phenomenon is not simply a case of learners forgetting what they learned the day before. They explained it by postulating two distinct stages that occur as learners develop a new mathematical concept. In the first stage, labeled *participatory*, learners develop an abstraction based on engagement in a particular activity (mental or physical actions, in this case, paper folding to partition paper strips). That is, through engaging in the activity, they develop knowledge of a mathematical relationship and no longer need to carry out the activity to determine the result; they can anticipate the result. Furthermore, the learners can justify and explain the logical necessity of the result. However, at this first (participatory) stage, use of this abstraction is limited. Learners have not yet learned to call upon the abstraction when they are not involved with or thinking

about the activity through which it was learned. If the learners are presented with a seemingly similar task the next day, outside of the context of the activity through which the abstraction was learned, they are not able to call on the relevant (from the observer's perspective) abstraction because they are not necessarily thinking about the activity for which they have a learned abstraction. However, in the second stage, labeled *anticipatory*, the learner is able to call upon the abstraction even when not engaged in the activity through which it was learned.

The distinction between these stages of understanding implies that, for the learners, the next day's task was not the same as the prior day's task, which the learners were able to solve, even if the tasks were word for word the same. A question posed in the context of the paper-folding activity was not the same task as that question posed unconnected to the paper-folding activity. Essentially, the first asked for an anticipation of the results of paper folding (partitioning), whereas the second was a more general question about fractions with no hint of how to approach it. The distinction also suggests that we define *task* not as just the written or oral articulation of the task. Rather, the task is in part defined by its place in a sequence of tasks and by the tools available or given to the learner with which to work.

Since 2004, Tzur and Simon have worked separately on the stage distinction. Tzur has published two articles using the participatory–anticipatory distinction. Tzur (2007) analyzed the stage distinction in a classroom teaching experiment and related the construct to the Vygotskyian construct of *zone of proximal development* (ZPD). Tzur and Lambert (2011) enlisted the term *prompt* to describe the feature of tasks that allows them to be solved at the participatory stage.

Our Research Program

Before we discuss and exemplify the uses of the stage distinction in a long-term teaching experiment, we provide some background on the research program and current project from which these examples are drawn.

Learning Through Activity

We have been engaged in a research program, Learning Through Activity (LTA; Simon, 2013a; Simon et al., 2010),[1] that builds on Piaget's (2001) theoretical construct of reflective abstraction. The aims of the program are to explain learners' development of mathematical concepts through a detailed examination of reflective abstraction in mathematics learning and to develop principles for promoting mathematical concept learning. The LTA research program is intended to develop integrated theory on aspects of mathematics learning and pedagogy.

LTA started from Piaget's (1980) claim that logico-mathematical knowledge is the

[1] LTA grew out of Tzur's (1995) dissertation research, Simon's (1995) early work on hypothetical learning trajectories, and the collaboration of Simon and Tzur (1996–2001). Since 2001, Tzur has worked separately using the *activity–effect relationship* framework (Simon, Tzur, Heinz, & Kinzel, 2004) to investigate learners' conceptualizations and learning in particular areas of mathematics (e.g., Tzur et al., 2013).

result of reflective abstraction. That is, a new abstraction is developed based on learners' reflections on their activity. After the abstraction has been made, the learner no longer needs to carry out the activity through which the learning occurred (as in the paper-strip activity). That which was produced through the activity is now anticipated. It is important to note that the mental activity that leads to a new abstraction, the process of reflection, and the new abstraction may not be conscious to the learner.

Central to the LTA research program is a specific adaptation of a one-on-one teaching-experiment methodology (Simon et al., 2010) that foregrounds learners' activity and eliminates or minimizes other factors that can influence learners' thinking, such as ideas offered by other learners and leading questions and hints offered by the researcher. The "teaching" consists (for the most part) of posing tasks, changing or modifying tasks, and asking for explanation and justification.[2] These adaptations allow for a concentrated focus on learners' activity and the resulting abstractions. As such, LTA research conducted with individual students is basic in nature.

The LTA research program employs a particular (and evolving) approach to designing instruction for the teaching experiments (Simon, 2013a, 2013b). Before explaining the approach to instruction, we examine briefly what we refer to as a problem-solving approach to instruction in order to provide a contrast with the LTA approach.[3] The LTA approach can be the basis of an approach to instruction that complements a problem-solving approach. In a problem-solving approach, learners are given a problem whose solution is beyond their current knowledge. They often work in small groups to try to solve the problem. A problem-solving task is, by definition, a task for which learners do not have a solution at the outset. As such, it is a task on which learners' success is uncertain. Typically, if it is a real problem for all of the learners (i.e., it is not the case that some of the learners already have the knowledge), a few of the learners solve the problem or solve it in a way that is consistent with the teacher's goals for the lesson. Two observations are in order here. First, success in solving the problem is inherently uncertain, and often, few learners solve the problem. The rest must learn from hearing the ideas of the few and participating in a discussion around those ideas. Second, research focused on such classroom lessons tends to be mute on the mechanism (or mechanisms) by which the successful solvers progressed from not knowing to knowing.

The LTA approach, on the other hand, is an explicit attempt to investigate the transition from the learners' current understanding to the new understanding. Toward this end, we create specific task sequences intended to bring about a

[2] This describes the ideal role of the researcher in the teaching sessions. However, the data demonstrate that, at times, the ideal role was not maintained.

[3] Although a problem-solving approach does not serve our research objectives, we strongly support regular use of problem-solving situations in mathematics classes. We have not yet specified particular ways of integrating the problem-solving and the LTA approaches to lesson design.

particular understanding (abstraction). We design LTA task sequences for our one-on-one teaching experiments in the following way:

1. Assess the relevant understanding of the learner.
2. Specify the learning goal (intended abstraction).
3. Identify an activity or activity sequence that the learner already has available that could be the basis for the new abstraction.
4. Design a sequence of tasks that is likely to bring forth the learners' use of this activity and lead to the intended abstraction.

We emphasize two key points regarding the LTA approach. First, learners come to the new abstraction in the course of solving tasks that they are already able to solve. There is no leap through problem solving that must be made. This can be seen in the next-day example above. Learners were engaged in a paper-folding activity that they knew how to do and came to an abstraction about the relative size of unit fractions. Second, in the context of our one-on-one teaching experiments, if learners are not able to solve a task, there is no help from other learners or the researcher available. It is the responsibility of the research team to modify the task sequence in such a way that the learners can build the new abstraction based on their current knowledge.

The Measurement Approach to Rational Number Project

This article is based on research conducted during the second phase (Years 2 and 3) of the 5-year Measurement Approach to Rational Number (MARN) project. The project is focused on two goals: (a) increasing understanding of how learners learn through their mathematical activity (i.e., advancing the LTA research program described above) and (b) understanding how learners can effectively develop fraction and ratio concepts through activities grounded in measurement.

We present data from the one-on-one teaching experiments on fraction and ratio concepts conducted with five different students. This phase involved developing and implementing task sequences for fraction and ratio learning and modifying those trajectories based on ongoing analyses. More in-depth retrospective analyses followed. The data in this article come from a one-on-one teaching experiment with Kylie during her fourth- and fifth-grade years. The teaching experiment with Kylie was the most extensive of the five we conducted; we worked with her for the academic years 2011–2012 and 2012–2013 in two 1-hour sessions per week. Prior to our work with Kylie, she was successful in school mathematics but generally lacked strong conceptual understanding.

In the MARN research project, we decided to use the computer application JavaBars (Biddlecomb & Olive, 2000) as the context for students' activity with fractions. In JavaBars, quantities were represented by bars (rectangles) of different lengths. The bars could be partitioned, and bars and parts of bars could be iterated (see Figure 1). In addition, the researchers designed a simple application for introducing the concept of a fraction as a unit of measure. These applications are used in the examples discussed below.

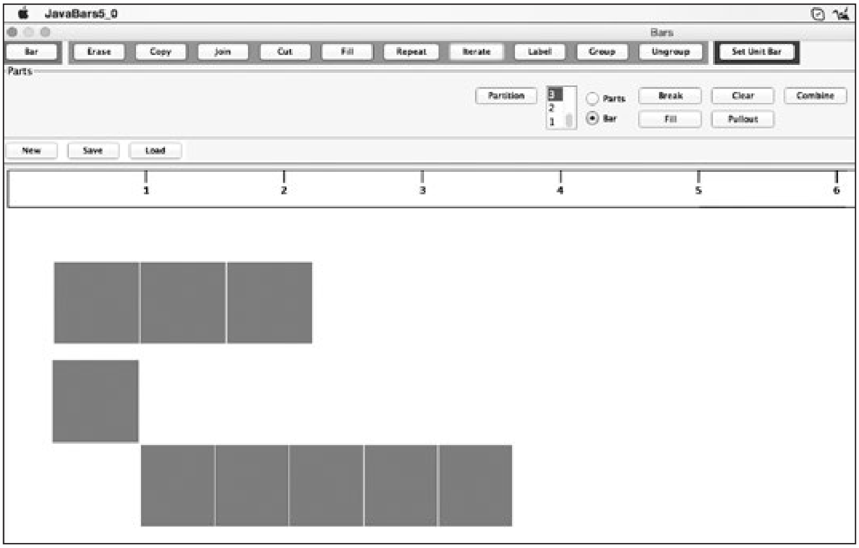

Figure 1. JavaBars computer application showing partition into thirds, pullout one third, and iterate the third 5 times.

Part 1: Usefulness of the Participatory–Anticipatory Stage Distinction

We discuss the usefulness of the stage distinction in the three key aspects of our MARN research: assessment, data analysis, and task design. The first, assessment, is included not because it is a new idea (see Tzur, 2007) but rather because the usefulness of the constructs for assessment has proven to be one of the most important uses of the constructs in our teaching experiments. The subsections on data analysis and task design demonstrate uses of the constructs beyond those discussed and exemplified in the literature. For this reason, we present examples to illustrate only the uses in data analysis and task design.

Usefulness for Assessment

We use the participatory–anticipatory distinction in assessing every abstraction that we foster. The goal in each segment of our teaching experiments is to foster an anticipatory stage of the concept. When a learner demonstrates an abstraction as the result of her engagement with a task sequence, we only have evidence for a participatory stage. Although it is possible that the learner has an anticipatory-stage concept, evidence of an abstraction during or immediately following the activity through which it was learned cannot provide evidence of an anticipatory stage. As a result, we always pose an anticipatory task for assessment purposes. We use *anticipatory task* and *participatory task* as shorthand indicating the stage of the concept that is considered necessary to solve the task. Generally, we pose the anticipatory task in the next teaching session to see if the learner can call on the new abstraction in the absence of the activity through which it was developed.

Tzur (2007) pointed out the importance of sequencing assessment tasks so that prior tasks do not cue the learner for a particular activity. The next-day example provided by Tzur and Simon (2004) can be taken as an example of beginning the next lesson by assessing the anticipatory stage.

Usefulness for Analysis

The participatory–anticipatory distinction is a key theoretical construct in our analysis of data. As demonstrated in the scenario from Tzur and Simon (2004), it explains the seemingly inconsistent performance from one session to another (the next-day phenomenon). Because of the frequency of data of this type, this alone is an important function of the construct. However, we have found the constructs to be useful tools for interpreting data that are not in the form of the next-day phenomenon. The following two examples demonstrate how the stage distinction allowed us to make sense of puzzling data. With respect to these examples, our claim is not that we have the definitive interpretation of the data. Rather, our claim is that these examples demonstrate the explanatory potential of the stage constructs.[4]

Example 1: Assessment of iterating a non-unit fraction. After not working with Kylie during the summer months, we started our second year of work by doing an assessment. We discuss here two of the questions that emerged from the analysis of the data generated and how the participatory–anticipatory distinction was useful in postulating answers to those questions. Midway through the assessment, Kylie was given the following tasks in succession:

• ***Task 1.1: This bar is three sevenths of a unit long. If I repeat it one hundred times, how long is my new bar?*** Kylie said, "Three seven-hundredths," and then changed her answer to "three-hundred sevenths." When she was asked for justification, she changed her answer to "three hundred seven-hundredths." She was unable to provide a justification for any of the calculations.

• ***Task 1.2: This bar is two fifths of a unit long. If I repeat it four times, how long is my new bar?*** Kylie once again multiplied both the numerator and denominator by four resulting in eight twentieths. She was not able to justify her solution and expressed a lack of confidence in her answer.

• ***Task 1.3: This bar is one sixth of a unit long. If I repeat it eleven times, how long is my new bar?***

[4] We numbered the examples in their order of inclusion in the article. The order does not reflect their chronological order; however, tasks and transcripts within each example do reflect chronological order. For conciseness, tasks within an example were sometimes omitted when the tasks were essentially the same and when they elicited essentially the same activity from the learner. In transcripts, we have used ellipses to omit words or phrases that are not essential to the meaning of the statements or make the excerpts difficult to read. Additionally, K stands for Kylie, and R stands for researcher (Simon).

K: Eleven-sixths.
R: Eleven-sixths?
K: Yeah.
R: Okay, convince me.
K: Well, I repeated it that many . . . Oh I know what the other one is [referring to the previous task].
R: Yeah, what?
K: It's eight fifths.
R: Okay, are you sure?
K: Yes!

• *Task 1.4: This bar is four ninths of a unit long. If I repeat it twenty-five times, how long is my new bar?*

R: What's that one?
K: It's uh . . . I know, four times twenty-five is a hundred-ninths.

In our analysis, we were initially puzzled by the data. Why could she do Task 1.3 correctly but not 1.1 and 1.2? Why was she able to do Task 1.2 (and 1.4) after solving 1.3, but not before? We settled on the following explanation: For Kylie to be able to figure out the bar that would be produced by repeating a two-fifths-unit bar four times, she would have to think about the two-fifths-unit bar as being the result of iterating two one-fifth-unit bars.[5] Kylie had previously demonstrated that she could make 2/5 by partitioning a unit into five parts, pulling out one part, and iterating it twice. However, the data seem to show that Kylie did not have an anticipatory-stage concept of a non-unit fraction as the result of an iteration of a unit fraction. That is, she did not think to call on that idea in the context of Tasks 1.1 and 1.2. However, when she was asked the result of stringing together unit fractions in Task 1.3, it caused her to engage in the activity of creating a non-unit fraction through iteration of unit fractions. Following that task, she was able to solve Task 1.2 and Task 1.4. These tasks were now participatory tasks; that is, in the context of thinking about iteration of a unit fraction to make a non-unit fraction, she was able to think about the non-unit fraction (two fifths) as the result of iterating a unit fraction (one fifth). That allowed her to solve the tasks involving the iteration of a non-unit fraction (Tasks 1.2 and 1.4).

Example 2: Working with ratio. In the session below, Kylie had been learning concepts of ratio. The tasks were designed to help her develop an abstraction of the multiplicative (functional ratio) relationship between the two quantities and to understand the invariance of that relationship.

[5] This is analogous to a composite unit concept in which 5 can be seen both as a single unit of 5 and as 5 units of 1.

• ***Task 2.1: One of the giant's steps is equal to six of Kylie's steps. If the giant walks 84 miles, how far would Kylie go in the same number of steps?*** Kylie needed to anticipate that the relationship between the giant's steps and her steps was multiplicative and that she could use this relationship to determine the number of miles she walked (make use of the invariance of that relationship). Note that the use of two different units of length in the task, steps and miles, with no conversion factor provided, was meant to preempt Kylie solving the task with a simple application of a per-one strategy or a buildup strategy.

K: What's eighty-four divided by six?
R: Fourteen.
K: I walk fourteen miles.
R: Why did you divide?
K: I know for each step the giant takes, I take six. . . . Every time the giant walks eighty-four miles, I walk fourteen. It's like one sixth of eighty-four.
R: How is that related to you and the giant?
K: Not sure. Oh yeah, every time he takes a step, I have to take six, so if I only take one step that's only one sixth of his step.

Kylie seemed to show that she had an abstraction of the invariance of the multiplicative relationship between how far she walks and how far the giant walks. One can think about this abstraction as having two interrelated parts, comparing the two quantities multiplicatively and knowing that that relationship is invariant across any distance traveled. Later in the session, we give her a slightly different task for which Kylie does not seem to call on the same abstraction.

• ***Task 2.2: Forty-two of Kylie's steps are equal to twenty-one of Max's steps. If Kylie walks twenty-five miles, how many miles does Max walk, if he takes the same number of steps?***

K: One hundred miles. Wait! I have to find out how many steps I take when Max takes one step.
[If the researcher had allowed Kylie to proceed in this manner, she would likely have set herself up for a similar solution to Task 2.1. However, the next question is what changed the task for Kylie.]
R: Can you look at these numbers and tell me the answer?
K: Seven? I thought forty-two, six times seven, and forty-two divided by six is seven.
R: And the twenty-one doesn't matter?
K: Ohhh, yes it does.
R: If I tell you twenty-one of my steps is forty-two of your steps, what do you know about our steps?
K: Yours is bigger.
R: How much bigger?
K: Twenty-one steps bigger.

The data presented in this example were puzzling. In the first task, Kylie called on a multiplicative comparison and seemed to know that it was invariant as the distances traveled varied. In the second task, she did not make use of a multiplicative comparison. How could we account for the difference in Kylie's thinking in these two situations? The participatory–anticipatory distinction proved to be useful in allowing us to generate a hypothesis. Based on prior analyses of Kylie's learning and the data described here, we made the following inferences. Kylie had significant experience considering unit fractions as smaller units that can be iterated to create the designated unit. In Task 2.1, the given information that six of Kylie's steps are equivalent to one giant step likely cued the activity of iterating Kylie's step (taken as a unit fraction) to make one giant step (taken as the unit).[6] Because Kylie interpreted the relationship between a unit fraction and the unit as a multiplicative relationship, she was able to anticipate the multiplicative relationship between her steps and the giant's steps and then use her understanding of the invariance of this relationship to determine the number of miles she would walk.

In Task 2.2, Kylie was about to convert this task to the form of the previous task (i.e., the number of Kylie's steps in one of the larger steps), which probably would have allowed her to solve it in the same way as she had for the previous task. However, the researcher prevented that approach because he wanted to see if Kylie could use the ratio between the two measures. As a result, Kylie was not able to mentally iterate one of her steps to make the larger step, so she did not think to use the relationship between unit fractions and units and thus did not consider the multiplicative relationship between the quantities. Once again, we see evidence of knowledge that is only at the participatory stage. The task as constrained by the researcher's follow-up question was an anticipatory task. Because it did not cue the activity of iterating a unit fraction for Kylie, she was unable to use her abstraction that was tied to that activity (i.e., iterating her step to make a giant step). In lieu of thinking about iteration of a unit fraction, she thought only about the additive comparison.

In these two examples of using the participatory–anticipatory distinction to explain puzzling data, we have demonstrated its usefulness in situations that are not of the form of the next-day phenomenon. In the next-day phenomenon, the same task can be either a participatory task or an anticipatory task, depending on what came before it (i.e., whether the learner is thinking about the key activity). In this last example, it was not the order of the tasks that was crucial but rather the extent to which the task evoked thinking about the key activity. The participatory–anticipatory distinction structured our examination of the data to focus on how Task 2.1 might have prompted an activity and the associated abstraction and how Task 2.2 might have not evoked that activity and the related abstraction.

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[6] Analyses of data from later in the teaching experiment on ratio were consistent with this interpretation. The analysis also led to the conclusion that an interpretation of Kylie's step as a part of the giant's step converted a task intended to evoke part-part reasoning into a task solved through part-whole reasoning.

Usefulness for Task Design

One affordance of the participatory–anticipatory distinction is that it specifies a need to design for a transition from the participatory to the anticipatory stage. Thus, our instructional planning takes this additional transition into account. Example 3 demonstrates this aspect of our instructional design.

Example 3: Developing a concept of a unit fraction. A key aspect of our teaching experiments on learning fractions was the idea of developing a fraction as a measure. This involves understanding a unit fraction as a measurement unit that is related to the unit multiplicatively. Its size is determined by the number of times it measures the unit (i.e., the number of times it iterates to create the unit). In designing our task sequences to develop a concept of a fraction as a measure, we began with a plan to engage learners in tasks that involved a computer application that was developed for the project. In this application, the learner is asked to measure a length of a wooden beam that they need to buy. They measure it with an A unit (which varies in size from task to task). The storekeeper, however, uses B units (which vary in size from task to task) to measure the beams that they sell. The learner is asked to measure the beam with the A unit and then to measure the B unit with the A unit. The application does not allow the learner to measure the beam with the B unit directly. To complete the task, the learner must communicate to the shopkeeper the needed length in B units. When the beam is not a whole number of B units, the learner must instruct the shopkeeper as to how many parts he must cut a B unit into and how many of those parts must be added onto the number of whole units. For example, if the beam is 11 A units long and a B unit is four A units, the beam must be "two B units and cut a B unit into four parts use three of them. So the length of the beam is two units and three of those parts." Tasks were also posed in which the beam was less than one B unit (the basis for proper fractions).

The next type of task that is posed involves a quantity represented in JavaBars as a bar. Learners are given (on the screen) a unit bar (unpartitioned) and several partitioned unit bars (e.g., 3 thirds, 5 fifths, 7 sevenths). The learner is asked to measure the quantity and give the answer in units. The learner measures the quantity with the unpartitioned unit then finds which of the partitioned units allows the extra bit to be measured accurately and gives the answer in terms of the number of whole units and the number of parts (described in terms of how a unit is partitioned and how many parts are used to measure the extra bit).

One could claim that learners who become adept at describing the length of the quantity in this way are using the concept of a unit fraction as a measure, albeit without the appropriate language and symbolization. However, our use of the participatory–anticipatory distinction allowed us to foresee that the learners would only have a participatory-stage concept after these tasks, which led to the design of an additional set of tasks. Critical to understanding unit fractions is the understanding that unit fractions measure the unit a particular number of times and that this number determines the size of the unit fraction relative to the unit.

In the application tasks (see Figure 2) and the partitioned-unit tasks that followed, the activity of relating the part to the unit (e.g., measuring the unit with the part) was cued by the task. The learners did not need to call on that activity. In the application tasks, the learners were required to measure the B unit with the A unit. In the partitioned-unit tasks (see Figure 3), the relationship between the part and the unit was cued by giving single units partitioned into a certain number of parts. In contrast, an anticipatory task would require the students to think to determine the relationship between the part and the unit (i.e., calling on the activity). Therefore, we planned tasks to develop the anticipatory stage, the ability to think to call on the activity using the part to measure the unit.

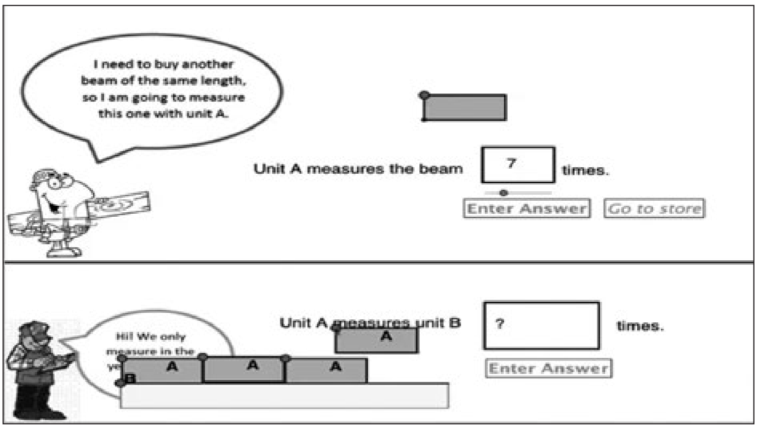

Figure 2. Measuring beam with A unit and ordering beam in B units.

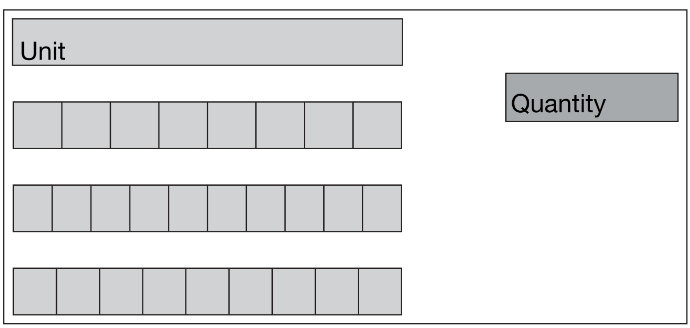

Figure 3. Measuring the quantity with partitioned units.

The set of tasks that we designed to move from the participatory stage to the anticipatory stage made use of JavaBars. The learners were given a bar to be measured (the quantity), a unit bar, and a set of partitioned long bars

(see Figure 4). The long bars were longer than both the quantity and the unit. Their partitions corresponded to different unit fractions of the unit, although this was not indicated to the learners. We anticipated that the learners would measure the quantity with the unit, realize that the unit did not measure the quantity exactly, and find one of the long bars to measure the part of the quantity that could not be measured with whole units. Only one of the long bars had parts that measured the quantity precisely. The requirement that the quantity's measurement be given in terms of the unit on the screen made it necessary for the learners to determine the "fraction" represented by the part of the long bar used. The learners, therefore, had to measure the fractional part of the quantity with the long bar and measure the unit with the long bar in order to determine the number and size of the parts in the piece of the quantity being measured by the long bar. However, in contrast to the participatory tasks, the second measurement was neither given nor prompted. The two types of learner-initiated measurements contributed to an anticipatory understanding of the size of a "fraction" as a result of both the number of parts and their size, as determined by the number of times they measure a unit.[7] In particular, the learners progressed to knowing to call on the second measurement in order to determine the relation of the part to the unit. It was the participatory–anticipatory distinction that framed our design of this task sequence.

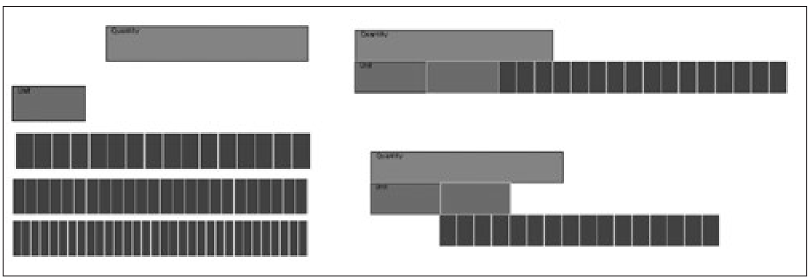

Figure 4. Measuring the quantity with a unit bar and with a long bar made up of fourths, sixths, or eights: task (left) and solution steps (right).

Part 2: Theoretical Contributions

Having discussed and exemplified the uses of the stage distinction for research, we now turn our attention to explication of the mechanisms underlying mathematical concept development, from extant knowledge through the participatory stage to the anticipatory stage. This is emerging theory; empirical and theoretical work in this area is ongoing.

First, we discuss our current formulation of reflective abstraction. Next, we describe how we use this formulation to explain the transition to the participatory

[7] Our empirical evidence suggests that another activity can contribute in important ways to a strong understanding of this idea. However, that discussion is beyond the scope of this article.

stage of a concept. Finally, we consider the transition from the participatory to the anticipatory stage.

Change in the Theoretical Basis for the Anticipatory–Participatory Stage Distinction

Reflective abstraction. In the LTA research program, we work on explaining mathematical conceptual learning from learners' extant knowledge to a participatory stage of a new concept to an anticipatory stage of that concept. It is our understanding and adaptation of Piaget's (2001) reflective abstraction[8] that frames our thinking. Our work involves a spiral in which our theoretical framework informs our empirical work: The results of our empirical work cause modifications in our theoretical framework and so on. In developing our formulation of reflective abstraction, the transition to the participatory stage, and the transition from the participatory to the anticipatory stage, three ideas were critical:

1. Grounded in our prior empirical work and interpretation of Piaget's (2001) reflective abstraction, the LTA program began with the idea that learners engaged in goal-directed activity, based on their extant knowledge, could make new abstractions. Further, these new abstractions can be understood as a *coordination of actions* (Piaget, 1980).
2. Attention to the learners' goals, as they work through mathematical tasks, is critical to accounting for conceptual development. First, the learners' goals determine to a great extent what prior knowledge they call on and what they attend to as they engage in a task. Second, Tzur and Simon (2004) pointed out that an important distinction between the participatory and anticipatory stages is the goal for which the new abstraction can be called upon.
3. Any formulation of the development of new mathematical concepts must allow for explanation of the building up of more advanced concepts from prior concepts. This suggests a need for a recursive structure that allows the result of conceptual development at one level to be a building block of a concept at the next level. Montangero and Maurice-Naville (1997) wrote, "The concept of reflective abstraction . . . enabled Piaget to show the continuity underlying the formation of knowledge, even when completely new forms appear" (p. 62).

We now discuss, based on these three ideas, how we characterize the development of the participatory stage of a new concept. Up to this point, we have used several terms without definition. This has been necessary because their usage in the literature we were citing was not always the same as the usage we want to establish from here on. We define a *concept* as made up of a goal and an action to achieve that goal. A concept is the result of a single reflective abstraction. This is a double claim. First, a concept is the result of reflective abstraction. Second, in

[8] Translated from French also as "reflecting abstraction."

analyzing the development of a concept, we are focused on the result of a single coordination of actions. A concept is a researcher construct and not a claim about what is true for the learner. *Goal* refers to the intention of the learner and should not be confused with the pedagogical goals of the teacher or researcher. An *action* (term taken from Piaget's coordination of action) is the mental and or physical acts that are called on to achieve a particular goal. The action component of a concept is constituted through reflective abstraction. These meanings are further developed in the discussion and example that follow as well as in the next section.

We now can discuss reflective abstraction. Piaget's (2001) explanation of reflective abstraction emphasized that a new concept results from a coordination of actions. However, we note that actions do not exist in isolation. They are connected to goals making up concepts. When learners call on available actions, they do so because (and only if) the learners' goals or subgoals are compatible with the goal of the available concept, of which the action is a part. Therefore, we understand a coordination of actions as a *coordination of existing concepts*. Calling on an action is always done to accomplish a goal, so we postulate that what gets built from that action must incorporate the goal for that action. It is this coordination of concepts to make a more advanced concept that we use to explain building more advanced knowledge from existing knowledge. From here on, when we talk about a learner calling on an action, it should be understood that she is calling on a concept of which the action is a part.

Example 4: A unit fraction of a unit fraction. We use an example from a MARN teaching experiment to demonstrate these ideas about reflective abstraction. In this example, Kylie is developing an abstraction of recursive partitioning (i.e., a unit fraction of a unit fraction). Hackenberg and Tillema (2009) defined recursive partitioning as "partitioning a partition in service of a non-partitioning goal, such as determining the size of 1/3 of 1/5 of one yard in relation to the whole yard" (p. 2).

• ***Task 4.1:* [*Using JavaBars, R draws a bar on the screen.*] *This is one third of a unit. Make a bar that is one sixth of a unit.*** Kylie made it clear that she did not know how to just "cut up" the bar on the screen. She made the whole by iterating the third three times and then cut the first third in half. She indicated immediately that one of the small pieces is one sixth.

• ***Task 4.2: This is one fifth of a unit. Make one tenth of a unit.*** Kylie used the same process. She iterated the one fifth 5 times to make the whole and then partitioned the first fifth into two subparts. She reported, "Here, you have one tenth of a unit."

• ***Task 4.3: This is one third of a unit. Make one ninth of a unit.*** This time Kylie immediately divided the one-third bar into three pieces (without iterating to make the whole).

K: One of those is one ninth.
R: How do you know?
K: Because, um. How many times does three go into nine? . . . Three times. And it's one third! So. Three times three is nine, and one of—if you cut it up into thirds again. That is, um. . . . And you take one, it would be . . . one third. . . . But that's really one ninth of a unit.

Kylie seemed to indicate that she thought about what number of parts would iterate three times to the whole. She therefore knew that one third of the one third would iterate nine times in the whole.

• ***Task 4.4: This is one fifth of a unit. Make one twentieth of a unit.*** She immediately cut the fifth into four pieces. She went on to complete two more tasks in this way.

In this example, Kylie learned that she could produce $1/mn$ from $1/n$ by partitioning $1/n$ into m parts. She developed an abstraction that partitioning $1/n$ into m parts creates a part that iterates mn times to the whole unit. Let us look more closely at this transition.

At the outset, Kylie had no way to think about partitioning a unit fraction $1/n$ to make the fraction $1/mn$. However, Kylie had knowledge that allowed her to make the requested fractional part. She understood that $1/n$ of a unit is a part that can be iterated n times to make a whole unit. She also knew that she could partition a unit to make any unit fraction. She used this knowledge to iterate the original part, one fifth, five times to make the whole (Task 4.2). After she had made the whole unit, she knew that she needed to partition it into 10 parts. Because she had a bar that was already partitioned into five parts, she was faced with a subtask involving partitive division with whole numbers: If I have 10 items distributed among 5 groups (10 subparts distributed among 5 parts), how many items will be in each group (subparts in each part)?

The activity Kylie used for Tasks 4.1 and 4.2 eventually led to the abstraction articulated above. Kylie used that activity in the next two tasks. The activity consisted of a sequence of actions, iterating the part to make the whole and using partitive division to determine the number of subparts per part. A coordination of these actions allowed her to know immediately that she could produce $1/mn$ from $1/n$ by partitioning $1/n$ into m parts.[9] She could justify this relationship when asked by discussing the activity through which it developed.

The important point here is that the new action was not an anticipation of the sequence of the existing actions.[10] In Task 4.3, she no longer employed the

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[9] Here, we are only claiming that an abstraction was made. We are not focusing on whether the concept was at the participatory or anticipatory stage.

[10] However, having an anticipation of the whole process (exemplified in Task 5.2) was a precursor to the development of the new concept.

sequence. Rather, she had developed a new action that was at a higher level than the sequence from which it was built and that allowed her to know the result at once. Kylie did not need to go through a sequence of mental actions; that is, she did not need to mentally generate the unit prior to partitioning the part. She was "seeing" the situation through a new concept, a concept that was built from the concepts that made up the activity sequence she had been using. One way to describe her new concept is that she now knew that taking $1/m$ of $1/n$ created a part that is n times smaller in relation to the unit than it is in relation to the part. She now had an abstraction about the relationship of a part of a part (subpart) to the unit. In her statement "How many times does three go into nine? . . . Three times. And it's one third! So. Three times three is nine, and one of—if you cut it up into thirds again," Kylie indicates that she has abstracted that the size of the original unit fraction ($1/n$) has a multiplier effect on the iterations of the subpart $1/m$. So, if one cuts $1/n$ into m subparts, the subpart created will be $1/mn$ of the unit. The example demonstrates how the abstraction is a new way of conceiving of the situation, not just an anticipation or abbreviation of the sequence of actions used formerly.

Justification for described changes in the theoretical basis for the anticipatory–participatory stage distinction. The theoretical basis for the stage distinction that we discussed in the last section represents a departure in two significant ways from the theoretical basis currently in the literature. The first is a move away from *scheme* as a way of characterizing existing and new conceptual knowledge. The second is a move away from *reflection of activity–effect relationships* as a mechanism for explicating reflective abstraction. We share our justification for each of these changes.

Focusing on concepts rather than schemes. To understand our choice not to tie discussion of reflective abstraction to the construct of scheme, some context for this choice is needed. In the LTA research program, we focus on inquiry into mathematical conceptual learning with the aim of postulating empirically based mechanisms to explain the development of new (for the learner) mathematical concepts. From the beginning, we have used Piaget's (2001) reflective abstraction to frame our inquiry. We work from the understanding that Piaget's postulation of reflective abstraction was important because the constructs of perturbation and accommodation were insufficient to account for conceptual learning (Vinner, 1990). Inhelder, Sinclair, and Bovet (1974) emphasized this point, discussing experiments in which children determined that their predictions were incorrect and emphasizing the importance of reflective abstraction.

> Such experiments . . . clearly showed that [the conflicting results] . . . do not ipso facto lead to the formation of operatory structures. As Piaget hypothesized, these structures, particularly in the case of logical and mathematical operations, appear to be the product of the subject's own coordination of actions, which is carried out by means of a process of reflective abstraction. (Inhelder, Sinclair, & Bovet, 1974, p. 13)

In our research, we use reflective abstraction to account for a building up of mathematical knowledge from prior knowledge, that is, the making of new abstractions using extant abstractions as the raw material.[11] Following other researchers (e.g., Vinner, 1990), we consider perturbations (disequilibrium) to be insufficient in accounting for learning. Further, in our LTA approach to promoting learning and our analyses of the data generated in our teaching experiments (cf. Simon et al., 2010), the construct of perturbation has proven not to be compelling in accounting for the learning process. Therefore, it does not figure in our accounts of learning.[12] This is discussed in depth in Simon (2013a).

Tzur and Simon's (2004) explication of the stage distinction was based on von Glasersfeld's (1995) explanation of a scheme. In an attempt to capture Piaget's construct of a scheme, von Glasersfeld (1995) described a scheme as having three parts:

1. Recognition of a certain situation;
2. A specific activity associated with that situation; and
3. The expectation that the activity produces a certain previously experienced result. (p. 65)

This tripartite model is tied theoretically to the construct of perturbation, particularly perturbations caused by unexpected results. Thus, the construct of scheme is part of an explanation of learning that emphasizes modifications of existing schemes due to perturbations that occur in the use of those schemes. This is not our focus, so we decided not to use the construct of scheme because of its origin in explaining learning as resulting from perturbation. We also found that the tripartite model of a scheme was less easily used to portray our understanding of the development of new, higher level concepts through reflective abstraction. That is, the model did not highlight one of the key components of our understanding of both reflective abstraction and the stage distinction, the learner's goal. For these reasons, we chose not to work with the construct of scheme and to focus instead on concepts using the specification of concepts described above.

Choice not to use reflection on activity–effect relationships. To begin to describe a mechanism to explain the participatory–anticipatory stage distinction, Tzur and Simon (2004) built on their framework for explicating reflective abstraction (Simon, Tzur, Heinz, & Kinzel, 2004).[13] According to that framework, a

[11] Not all conceptual learning is the result of reflective abstraction. Generalizing assimilation (Steffe & Olive, 2010) results in the modification of extant concepts. However, we are interested in the construction of new, higher level structures for which reflective abstraction seems to offer considerable explanatory power.

[12] We are aware that researchers, for whom perturbations is a key lens, will always be able to identify perturbations in our empirical examples. Our analysis of our data does not suggest an important role for the construct of perturbation.

[13] Tzur (i.e., Tzur, 2007; Tzur & Lambert, 2011) has continued to use this framework.

mathematical abstraction is the product of reflection on an activity–effect relationship. Simon, Tzur, Heinz, and Kinzel (2004) used effect to indicate a second activity that was coordinated with the first (in keeping with Piaget's explanation of reflective abstraction). They wrote, "Note the activity and the effect are conception-based mental activities, our interpretation of Piaget's (2001) notion of *coordination of actions*" (Simon et al., 2004, p. 320). However, we believe that the use of effect and activity–effect relationships introduced two problems that our current formulation potentially remedies.

The first problem is that the notion of activity–effect can suggest a chronologically sequenced cause and effect. Further, it can suggest a linkage of actions as opposed to the creation of a mental structure that is at a higher level than the actions from which it derives. Piaget (1950) wrote,

> [Reflecting abstraction] does not lead to a simple generalization. . . . Reflecting abstraction is constructive insofar as it is linked to the elaboration of a new action; this new action is on a higher level than the action from which the characteristic under consideration was abstracted. (as cited in Campbell, 2001, p. 11)

Although Simon et al. (2004) referred to Piaget's notion of a coordination of actions, they described the framework in ways that could be interpreted as an anticipation of a sequence of two activities:

> A conception can be thought of as the ability to anticipate the effect of one's activity without mentally or physically running that activity. . . . We suggest that the records of experience from which this abstraction derives are records of activity associated with the effects of that activity. (p. 319)

Tzur's subsequent work seemed to accentuate a sequential relationship between activity and effect.[14] Tzur and Lambert (2011) wrote, "Simon & Tzur (2004), Tzur (2007), and Tzur & Simon (2004) proposed to use *effect* in reference to what a learner's mental system monitors and notices to follow an activity while it is unfolding" (p. 420). Tzur (2007) wrote, "*Effect* refers to any chunk of learners' experience that they identify as that which follows the activity" (p. 276). Such description seems to invite interpretation of reflective abstraction as a learned sequence. We emphasize that reflective abstraction is not the abstraction of a sequence of actions, but rather it is the construction of a new structure based on lower level actions,[15] a structure that may no longer involve a sequential relationship. It is for this reason that we have moved away from reflection on activity–effect relationships and focus instead on explicating Piaget's coordination of actions.

A second problem with the use of effect is that it can be confused with the result

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[14] We make no claim that Tzur would argue this view of a mathematical concept (and reflective abstraction). Rather, our point is to demonstrate the implications that can be taken from an activity–effect formulation.

[15] This is not a claim derived from empirical data. Rather, it is part of the definition of reflective abstraction (Piaget, 2001) and, as such, a starting place for our work and that which we are trying to explicate.

of the use of an activity. Simon and Tzur (Simon et al., 2004; Tzur & Simon, 2004) used *effect* to refer to one of the actions that were part of the coordination of actions, whereas *result* has a different meaning. A result is what is produced by the activity. When an abstraction (coordination of actions) is made, the learner can anticipate the result without going through the activity from which the abstraction was made.[16] For example, Kylie could anticipate that taking one eighth of one seventh would produce one fifty-sixth. By anticipation, we mean that she came up with the answer, one fifty-sixth, without going through the original activity (that began with iterating to the whole).

To summarize our discussion, the learning process begins with the learner setting a goal (e.g., to complete a particular task) and calling on available actions (concepts) to accomplish the goal. Initially, the learner uses these actions in sequential fashion. However, through reflection on her activity, the learner may come to coordinate those actions into a single, higher level action (concept). As a result of this coordination, she no longer needs to go through the sequence of actions used previously. The result of the coordination is a structure that is at a higher level than the component concepts. It is knowledge that did not exist previously.

We have argued that one of the reasons why effect and activity–effect relationships do not well represent reflective abstraction is that the terms can imply a learned anticipation of a sequence of activities as opposed to the creation of a new, higher structure. We point out that learning the sequence of activities needed to solve a task is (an early) part of the learning process. Initially, the learner might be solving the task step by step without anticipation of the whole solution process. Reflection on those actions can result in the learner being able to anticipate the entire sequence of activities; Thompson (1994) referred to this as *internalization*. At that point, the learner knows the sequence of actions and can mentally review them without actually working on a task. However, whereas the coming to anticipate the action sequence of the solution is an important step, it does not represent the learning of a new concept and is not the focus of our theoretical development, nor was it the focus of Simon et al. (2004) in their attempt to explicate reflective abstraction. Our movement away from the activity–effect framework is, in part, an attempt to avoid conflation between the development of internalized action sequences and the coordination of actions that produces a new, higher level concept.

Representing the Process of Learning Through Activity: Reflective Abstraction in the Context of Mathematical Tasks

In this section, we will further specify reflective abstraction, which serves as the theoretical basis for the task distinction, and introduce how we represent the process of reflective abstraction. The biggest challenge in conceptualizing, discussing, and representing the learning process is its recursive nature: What is created at one level is a building block for what is created at the next level. Finding ways to communicate

[16] This definition is consistent with the use of the term result in Simon et al. (2004) and Tzur and Simon (2004); however, the term was not explicitly defined in those articles.

about entities that are the output at one level and the input at another is nontrivial.

We developed a representation for several purposes, including:

1. Articulating more precisely our theoretical constructs related to reflective abstraction, given the complexity caused by the recursive nature of conceptual learning;
2. Checking the logic of our constructs (use of the representation has already resulted in several modifications in our explanations); and
3. Providing a basis for explicating the difference between the participatory stage of a concept and the anticipatory stage and the transition process that occurs.

In keeping with our definition, we represent a concept as G_n–A_n. G_n is the goal, and A_n is an action that produces a result that accomplishes the goal. The complex of goal and action, represented by G connected to A by a dash, signals an available concept that was constructed through reflective abstraction. We use the subscripts to be able to talk about concepts and their goals and activities at different levels. We use subscript 1 for the concept whose development we are trying to explain and subscript 0 for the concepts that were already established and served as building blocks for the developing concept. Thus, the subscript number is used relative to the concept whose development we are trying to explain. To explain learning, therefore, we need to explain how concepts that can be represented as G_0–A_0 can be used as the material for constructing a concept represented as G_1–A_1.

The learning process that we aim to describe begins when the learner sets a new goal in response to a mathematical task and calls on a set of available actions to achieve that goal. We label this goal as G_T to signify the goal for the task at hand. The subscript T distinguishes the goal from those that we label G_0 and G_1. G_T is a 0-level goal, in that it is a goal that calls on available actions (no learning required),[17] but it is a new goal, not part of any one existing concept. We represent this first step as G_T ($A_{0a} \rightarrow A_{0b}$).[18] This is what we mean by an *activity*. That is, an activity is a goal and a sequence of available actions for solving a novel task. An activity is a precursor to a new concept. As noted earlier, each of the actions called upon is part of an existing concept. That is, G_T actually calls on the concepts G_{0a}–A_{0a} and G_{0b}–A_{0b},[19] thus a more complete representation is G_T (G_{0a}–$A_{0a} \rightarrow G_{0b}$–$A_{0b}$). Again, this last point is critical because it allows us to demonstrate how concepts build on concepts recursively. We take the complete representation as implied and, for ease of communication, continue to use the abridged version, G_T ($A_{0a} \rightarrow A_{0b}$), going forward.

[17] The learning that we are interested in here is the development of a new concept, which involves the construction of a new, higher level action. If a goal is achieved using available actions, we consider that learning has not taken place.

[18] The parentheses and arrow denote a sequence of actions. The portrayal of two component actions is arbitrary because the number can vary.

[19] The second subscript is used to distinguish different available (0-level) conceptions called on initially to achieve the task goal (G_T).

So far, the learner has set a goal for the task and called on a sequence of actions to accomplish the task, G_T ($A_{0a} \rightarrow A_{0b}$); that is, the learner has created an activity. In Example 4, Kylie set the goal, G_T, of creating a fractional bar of a particular length. She called on the action sequence that follows:

1. Iterate the given fractional bar (1/*n*) *n* times to make a unit.
2. Determine using partitive division how many times (*m*) to partition one of the parts.

In the course of engaging with the tasks, Kylie created a new abstraction, a coordination of actions (coordination of existing concepts). The circle (see Figure 5) containing the actions is meant to show both the building blocks of the new action and the fact that this is no longer a sequence but a higher level structure built from lower level ones (coordination of actions). Because the circle represents the creation of a higher level action, the circle representing the coordination of actions can be replaced by A_1 and the new abstraction represented as G_T–A_1.

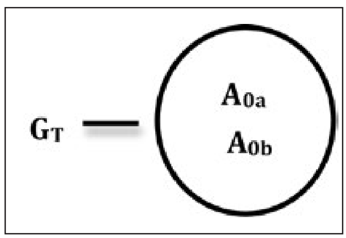

Figure 5. Coordination of actions: Actions are no longer related sequentially but are part of a higher level structure.

Representation of Stages

So far, we have represented the achievement of a coordination of actions, the creation of a new abstraction G_T–A_1. However, ultimately, we are trying to explain the development of a new concept at the anticipatory stage, G_1–A_1. The representation makes clear that the remaining explanation involves the connection of the coordination of actions, represented by A_1, with the anticipatory stage goal, G_1.

Before discussing representation of the transition to the anticipatory stage, we postulate an additional minor transition that is often required for achievement of the participatory stage. This is a shift in the students' goal, which is often provoked with a modification of the task. So far the student has developed a reflective abstraction that includes the task goal, G_T. For the abstraction to be at the participatory stage, the goal of the abstraction must correspond to the eventual anticipatory goal.

In Tzur and Simon's (2004) next-day example, the students' original goal, G_T, could be identified as *to subdivide a paper strip into the designated number of equal parts*. However, toward the end of the initial lesson, the teacher asked the students which was larger, 1/7 or 1/5? Setting a goal of determining which of two fractions is larger is different from G_T and an important step in arriving at the participatory

stage, given that the concept in question involves the relative size of two fractions.

So far, we have represented the learner's achievement of a coordination of actions, G_T–A_1. G_T is the goal that was set for the original tasks. Therefore, we postulate a transition from G_T–A_1 to G_{T1}–A_1. G_{T1} is a goal that is seemingly the same as G_1 but is specific to working within a focus on the original activity. We illustrate this further by building on Example 4.

In Task 4.4, Kylie was given the task "This is one fifth of a unit, make one twentieth of a unit." The goal that we attribute to Kylie, G_T, is directly linked to subdividing bars, the original activity. Tasks of this type could be directly followed by tasks of the type "What fraction of one seventh is one twenty-eighth?" This task invites a goal that we call G_{T1}. G_{T1} differs from G_T and G_1 in subtle but important ways. G_T is a goal that was achievable without new learning. That is, the student can call on available actions to produce an activity that achieves the goal. In our example, the task "This is one fifth of a unit, make one twentieth of a unit" provoked a goal that was achievable for Kylie with no new learning because she could use her activity involving iterating to create the unit. However, G_{T1} (e.g., goal set for "What fraction of one seventh is one twenty-eighth?") was only achievable after a new abstraction had been developed. Given this question initially, Kylie could not have called on the activity involving completing the unit and would not have known how to solve the task. The other important contrast is between G_{T1} and G_1. In our example, we can note that they would both be set in response to tasks that are worded identically. However, G_{T1} must be set in the context of the partitioning activity Kylie was engaged in, that is, in response to a task that followed the tasks that led to her new abstraction. In contrast, we could use a task with identical wording a few days later, when Kylie was not thinking about the particular partitioning activity, and it would be an anticipatory task. In this case, we would represent the goal as G_1. We use G_1–A_1 to represent the anticipatory stage of a new concept and G_{T1}–A_1 to represent the participatory stage of learning a new concept.

We have discussed how a concept at the participatory stage can only be called on when the learner is engaged in or thinking about the activity in which it was learned. We have also specified that the activity is the precursor to a new concept (reflective abstraction). Table 1 summarizes the steps leading to the anticipatory stage of a new concept and the relationship to the activity.

Next, we show how we currently represent the transition to the anticipatory stage. As we have indicated, what is missing from the participatory stage is a link between the new concept and the ultimate goal (G_1), which we refer to as the *anticipatory goal*. We represent this as follows:

G_{T1}–A_1 (participatory stage)

$\downarrow$

G_1–G_T–A_1 (transition from participatory to anticipatory)

$\downarrow$

G_1–A_1 (anticipatory stage)

Table 1
Relationship to Activity and Stage of Development of Concept

	Relationship to activity	Development of a new concept
G_T–($A_{0a} \rightarrow A_{0b}$)	New activity	No new abstraction
G_T–A_1	In context of activity	Result of reflective abstraction
G_{T1}–A_1	In context of activity	Concept at participatory stage
G_1–A_1	Independent of activity	Concept at anticipatory stage

In representing the transition in this way, we are postulating that the transition involves developing a connection between the anticipatory goal and the participatory stage of the concept. Ultimately, a new anticipatory-stage concept is developed in which the anticipatory goal is linked directly to the action, A_1. We will exemplify the use of this representation in the discussion of the transition from participatory to anticipatory that follows.

Transition from Participatory to Anticipatory

One of the ongoing problems in our research program is to understand the transition from participatory to anticipatory stage of a concept and specify design principles for promoting the transition. Our conceptualization of the two stages suggests that the instructional intervention for promoting the anticipatory stage must produce a link between the anticipatory-stage goal and the abstraction that underlies the participatory stage.

Although we do not have a satisfactory theoretical description of this transition or of an approach to promoting the anticipatory stage, we have had some success in promoting it. In this section, we provide an example and discuss our current analysis of this transition.

After an abstraction has been made, we typically pose an anticipatory task in a subsequent session to determine whether the concept is at a participatory or anticipatory stage. (Note that we have already discussed that two tasks that are word for word the same can be either participatory or anticipatory, depending on what they follow and the resources provided for their solutions.) When we pose the anticipatory task in the subsequent session, frequently, the learner provides either no solution or an invalid one (e.g., the next-day phenomenon), suggesting that the learning is at a participatory stage. It is at this point that an intervention is needed to promote an anticipatory stage of the concept. The example that follows provides a typical scenario.

Example 5: Fraction of a whole number. This example involves Kylie learning to reason about a fraction of a whole-number quantity, something she was unable to do in preassessment tasks. All tasks involved whole-number answers. The participatory stage of the concept was developed initially using

JavaBars. A bar was created, and Kylie was told that the bar was a certain number of units long. She was asked how long a particular fraction of the bar would be. She partitioned the bar, pulled out a unit fraction, and iterated the unit fraction to generate a new bar that represented the fraction of the bar specified in the task. She then evaluated the length of the fractional bar in units.

Having carried out such solutions, Kylie was then able to think about taking a fraction of a quantity without making a bar on the screen (see Task 5.1). We took this as evidence of a coordination of action, G_T–A_1.

• ***Task 5.1: If I had a bar that is eighteen units long, do you know how long a bar that is two thirds of that bar would be?***

K: . . . Twelve units.
R: Okay, how'd you get twelve?
K: Because of six times three . . .
R: Where did the six come from? . . .
K: The six came from the two thirds, and the thirds that I cut the eighteen up into. . . .
K: And I know . . . I know that six times three is eighteen.
R: Okay.
K: I figured six plus six equals twelve. And since two thirds, and you wanted to know how many units was in two thirds. I said twelve, because six units plus six units equals twelve units.

She followed this by solving tasks that did not explicitly refer to bar representations (e.g., Task 5.2). These solutions provided evidence of a participatory stage, G_{T1-A1}, because the inferred goal was now G_{T1}, a goal that corresponded to the anticipatory goal.

• ***Task 5.2: What is five sixths of twelve? . . .***

K: [pause, thinking] Oh, I know. It is ten units.

• ***Task 5.3: What is five fourths of twelve? . . .***

K: [pause, thinking] Fifteen.
R: Explain that to me and then we're done. . . .
K: I know that three times four . . . would be four fourths. Since it was five, I know that adding three to twelve equals fifteen.
R: Okay.
K: Yeah, I know you're going to say, "That's just a number, tell me what that means."
R: So, you said. So, first of all, you said for, five fourths of twelve, okay so you've said something about a three, what's that three?

K: The fourth. . . .
R: And then, so five fourths is . . .
K: Is fifteen.
R: Because . . . ?
K: Because . . . The, if there's four fourths that equals twelve. Add one more, would be five fourths. . . .

At this point, we concluded that Kylie had at least a participatory stage of understanding of fraction of a whole-number quantity. We represent this as G_{T1}–A_1. G_{T1} is the goal of taking a non-unit fraction of a whole number in the context of thinking about taking fractions of multiunit bars. The original activity can be thought of as involving the following actions:

1. Creating a non-unit fraction of the bar (the whole-number quantity) through partitioning, disembedding, and iterating.
2. Using whole-number, partitive division and multiplication to determine the size of the non-unit fraction of the quantity.

A_1 involves a coordination of actions, Steps 1 and 2 of the activity. This coordination resulted in Kylie's ability to compute the result without having to separately create the non-unit fraction. The computation was now at a higher level because the evaluation of the size was now coordinated with the creation of the fractional part. In a subsequent session, we posed an anticipatory-stage task and Kylie demonstrated that her concept was indeed only at the participatory stage.

• ***Task 5.4: What is five fourteenths of 322?***

R: [Kylie appears stumped.] Is it going to be more or less than three-hundred-twenty-two?
K: . . . It's going to be less.
R: Okay. You don't have any idea how to do it?
K: Nope.

The researcher used the next task both to confirm that Kylie did not have the concept at an anticipatory stage and to begin promoting the anticipatory stage.

• ***Task 5.5: What is three sevenths of 28?***

[In response to the task, Kylie tried some different meaningless manipulations of the numbers, but was unsure how to solve it.]

R: All right, so if I had a bar that was twenty-eight units long, do you know what . . .
K: Yeah . . .

R: Do you know how long three sevenths of that bar would be?
K: . . . Well if I broke it up into seven pieces and broke it up pulled out. . . . Oh, ohh, I get it. Okay. . . . Three sevenths would be . . . twelve.
R: Tell me how you got twelve.
K: I know that each seventh is worth four pieces.
R: Is four units long?
K: Each seventh is four units long, so three sevenths is twelve units long?

Note that as soon as the researcher verbalized representing the quantity as a bar, Kylie knew how to solve the task. His reference to the bar elicited the activity in which her abstraction was grounded. This use of the abstraction supported the idea that Kylie had not lost (forgotten) the abstraction; she needed to be thinking about the activity to access it. At this point, Kylie could solve similar tasks without difficulty and explain her thinking.

R: What is four fifths of forty? . . .
K: That would be, thirty-two.
R: Okay, can you explain to me how you got thirty-two?
K: . . . If I had a bar that was forty units long, and I had four fifths of that bar . . . well if I had five fifths and I had it broken up into five pieces, and each of those pieces was worth eight . . . units, and I had four of those, and they were worth eight units and that would be thirty-two.

• ***Task 5.6: What is five fourteenths of three hundred twenty-two?* (*Task 5.4 revisited*)**

K: What is three hundred twenty-two divided by fourteen?
R: Twenty-three.
K: Each of those pieces, each fourteenth is worth twenty-three [units], five pieces . . . What's five times twenty-three?
R: One-hundred-fifteen.
K: So, it would be one-hundred-fifteen.

However, once again, we only had evidence of a participatory stage of the concept. All of Kylie's solutions after the researcher's reference to a bar could have been accomplished with a participatory stage of the concept. An anticipatory task was posed several sessions later after no further work with the concept.

• ***Task 5.7: Brian has a collection of six-hundred-thirty marbles, and thirteen twenty-firsts of them are green. How many marbles are green?***

K: Well, we need to divide the six-hundred-thirty marbles by twenty-one.
R: [Pause] You want me to do that for you?
K: Yes, I do.

R: Thirty.
K: Okay, so that would be one twenty-first.
R: Okay.
K: We need to multiply thirty by thirteen.
R: Three-hundred-ninety.
K: Okay. So three-hundred-ninety marbles are green, . . . which would be thirteen twenty-firsts.

In Kylie's solution of this task, we see evidence of an anticipatory stage, G_1–A_1. Kylie could solve the task without being involved in or cued for the activity of partitioning a bar. Rather, she was able to call on her new, higher level action (A_1) in relation to the anticipatory goal (G_1), find *m*/*n* of a whole number.

We now offer an analysis of the example focusing on the transition from a participatory to an anticipatory stage. Originally, Kylie's task goal was to determine the length of a bar that was, for example, 2/3 of an 18-unit bar. She developed an abstraction and was able to do fraction-of-a-set (fraction-of-a-composite-unit) tasks when no bar was drawn or referenced (participatory stage, G_{T1}–A_1).

When Kylie was presented with the anticipatory task, her goal (G_1) was different than it was for the participatory tasks because the participatory goal, G_{T1}, was a goal grounded in thinking about finding a fraction of a bar. In the anticipatory task, her goal was to find a fraction of a whole number (without any temporal or expressed connection to the bars). Without a connection between the anticipatory goal and the activity through which she developed the (participatory) abstraction, she was not able to call on that abstraction. The question here is how that changed.

We understand the transition in the following way. When Kylie was unable to do the anticipatory task, the researcher prompted her to think about the quantity as a bar, which triggered her previously learned abstraction. However, this consideration of the quantity as a bar was not the same as the work through which she learned the abstraction. This time, she was operating with a different goal. She was trying to take a fraction of a whole number (thought of as numbers and not as bars). Considering the quantity as a bar in this situation was not a given of the task or the situation, it was a way of addressing her computational (activity-independent) goal. We hypothesize that as a result of this experience of using her activity-based abstraction in service of her activity-independent goal, she began the process of connecting the anticipatory goal with the participatory concept. This was the link that needed to be forged in order to transition from participatory to anticipatory.

Two comments are important relative to the example discussed here. First, it may be that only one task in a session can contribute to forging this link. In the first task, the learner sets the anticipatory goal (G_1). In each subsequent task, the learner can work from a participatory goal, (G_{T1}) because the original activity has already been called upon during the first task. The subsequent tasks do not require the learner to think to call on the coordinated action (A_1) based on the

anticipatory goal (G_1) because the learner is now thinking in terms of the participatory goal (G_{T1}). The second point follows from the first: The transition to the anticipatory can take several sessions of this type. Our corpus of data includes examples of more extended transitions.

We consider the explanation that we are offering in this section to be very preliminary. Whereas our explanation of coordination of actions leaves work to be done in order to better understand the mechanism, we do not yet understand the mechanism by which a learner makes a link between an anticipatory goal and a previously learned abstraction. That is, we are not able to describe the nature of the connection that is achieved between the anticipatory goal and the participatory concept (G_1 and G_T–A_1) or the process by which the connection is made.

Conclusion

The participatory–anticipatory stage distinction has proved useful in the instructional design, assessment, and data analysis phases of an extended (2-year) teaching experiment. In instructional design and assessment, our use of the stage distinction allowed us to devise task sequences and assessments that are focused on and sensitive to the limitations of the participatory stage and the transition to the anticipatory.

The stage distinction makes an important contribution to the analysis of qualitative data related to learners' thinking and learning. We have demonstrated how the stage distinction can provide the basis for making sense of puzzling data. Although the next-day phenomenon (Tzur & Simon, 2004) is a frequent aspect of such data, we have demonstrated how the stage distinction can be useful in accounting for puzzling data segments that take other forms. In particular, the stage distinction focuses researchers on inquiring into how the nature of particular tasks (as seen from the student's perspective) might explain their ability or inability to call on a concept that they have at only a participatory stage.

The stage distinction is based on a characterization of reflective abstraction for learning mathematical concepts. We have explicated, exemplified, and created a representation of an emerging characterization of reflective abstraction. In so doing, we have discussed why we have moved away from the reflection-on-activity-effect-relationships characterization used in Tzur and Simon (2004) and subsequent studies. We have described an elaboration and representation of what Piaget (2001) called a coordination of actions. An ongoing goal of our research program is to generate multiple empirically based images of coordinations of actions and to further analyze the processes by which coordinations of actions take place.

Our elaboration of the theoretical basis of the stage distinction has led to a description of how the development of a concept is linked to an evolution in the learners' goals. In the process of developing this description, we have identified four different types of goals relative to the learning of a concept, goals that we designate as G_0, G_T, G_{T1}, and G_1. These distinctions are particularly important in designing a task sequence aimed at producing the anticipatory stage of a

particular concept. Whereas G_0 indicates the goal of an existing concept that is being called upon, G_T, G_{T1}, and G_1 refer to goals of a concept in development. In designing a task sequence, a different type of task is used to elicit the setting of each type of goal. Further, the work that a student can do in response to G_{T1} is based on the learning that occurred in response to G_T, and the work that can be done in response to G_1 is based on the learning that occurred in response to G_{T1}. Thus, the hierarchy of tasks corresponds to a hierarchy of goals.

We have characterized and represented the transition from participatory to anticipatory and presented an analyzed data segment to demonstrate and explicate our limited understanding of how the transition from a participatory to an anticipatory stage occurs. A significant ongoing challenge is to generate data and analyses of the data that can be the basis for further theoretical elucidation of this transition.

Finally, the theoretical work on the stage constructs that is reported here and the empirical work on which it was based did not focus on application to classroom settings. Work is needed both on how these constructs can be used in the classroom (curriculum planning and teaching) and on the nature of understanding related to the constructs that might be promoted among prospective and practicing teachers.

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Book Review

Who Is Behind the Nationalization of Mathematics Education?

Math Education for America? Policy Networks, Big Business, and Pedagogy Wars. Mark Wolfmeyer. (2014). New York, NY: Routledge, xiii + 194 pp. ISBN: 978-0-415-71300-9 (hb) $160.00, ISBN: 978-1-315-88356-4 (e-book) $54.95.

Reviewed by Trevor Warburton and Ed Buendia, *University of Utah*

In the United States, there has been an increasing push toward what Mark Wolfmeyer, the author of *Math Education for America? Policy Networks, Big Business, and Pedagogy Wars*, refers to as the nationalization of mathematics education, which includes both mathematics education curricula and the intended purpose of mathematics education. The drive toward the nationalization of mathematics education recently culminated in the adoption of the *Common Core State Standards for Mathematics* (CCSSM; National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010). The potential effect of nationalization on what and how teachers teach and what and how students learn is significant. Because these standards have the potential to shape the future of mathematics education in the United States, educators have a right and a responsibility to know who is behind the nationalization of mathematics education. In contrast to other recent texts examining the adoption and integration of mathematics standards (Burke, 2013; Germain-McCarthy, 2013; Manley & Hawkins, 2012), Wolfmeyer takes an explicitly critical approach to this project by focusing on the networks of power and influence. This approach allows Wolfmeyer to address the intentions or interests of the parties spearheading the push toward nationalization.

Wolfmeyer takes on the tasks of identifying who is behind the move toward the nationalization of mathematics education as well as making transparent their vision of mathematics for the citizens of the United States. He argues that mathematics education is being skewed toward the interests of corporations by corporate proxies and academics. Wolfmeyer shows the intermingling of the corporate representatives, academics, and government agencies behind the nationalization of mathematics education through a critical analysis of the policy networks, or social network analysis (SNA).

In Chapter 1, Wolfmeyer opens with a discussion of the arguments that are addressed in the book and a description of his analytic approach. In contrast to the perception that "the overarching purpose of math education for America is to develop a talented workforce for corporate profit" (p. xi), Wolfmeyer takes the

position that the nationalization of mathematics education (what he calls "math education for America") works against the potential of mathematics education to create increased social equality.

In Chapter 2, Wolfmeyer describes SNA and its relevance to mathematics education policy research in greater detail. The focus of SNA is on the individual and corporate actors who interact to construct policy. This focus propels Wolfmeyer toward identifying and situating interactive corporate and academic actors as the pivotal architects of mathematics policy. This removes the exclusive focus on the federal and state government policy machinations that are typical of policy analyses. SNA is appropriate and effective, considering Wolfmeyer's aims. He employs this lens to provide readers with policy network maps of the interactions between people and agencies by identifying three key events and their culminating documents. These three events include the formation of the National Mathematics Advisory Panel (resulting in the 2008 release of the U.S. Department of Education's *Foundations for Success*), the publication of *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001), and the release of the CCSSM (NGA & CCSSO, 2010). The smaller network maps for each of these events are then combined into a single network map that illustrates the interconnectedness of these policies.

In Chapter 3, Wolfmeyer uses these maps to explain how network actors connect to the human capital interests of corporations. Although he explains how mathematics education could be used to further human capital development, he assumes that the reader understands the limits of the human capital model and does not offer a robust critique. In Chapter 4, however, he uses the global nature of human capital arguments to explain the apparent victory of traditional mathematics education (referring to emphasis on correct answers, assessments, and elite students) over progressive approaches (referring to an emphasis on process and the needs of all students). Wolfmeyer explains how education for human capital development subsumes both the traditionalists and progressives in mathematics education, even though he does not give details of the distinction between the two groups. He explains that although they have argued over pedagogy, both camps align with human capital interests, albeit in different ways. Interestingly, he cites "equity" as one of the key ways that progressives align with human capital interests. He argues that progressives have latched on to corporate proclamations of equity and, as a result, have ended up furthering human capital interests.

In Chapter 5, he furthers this argument by explaining how both traditionalists and progressives have aligned with human capital interests to position U.S. mathematics teachers within a global competition, specifically against the educational outcomes of China. In the process, progressives and traditionalists have portrayed U.S. mathematics teachers as lacking mathematical knowledge. This has led to mathematics teacher preparation that emphasizes content knowledge over pedagogical ability and the potential for education to promote social equity. As a result, teaching may become a less attractive career for those with commitments to equity and social justice. Furthermore, human capital interests have promoted hiring and firing policies that make it easier to remove teachers who oppose human capital

aims. In Chapter 6, Wolfmeyer explains the role of educational testing. This industry, he says, has integrated itself into policy networks, reaping enormous contracts to provide testing services by developing connections with the federal government. These tests, for teachers and students, have pushed the pedagogy wars in favor of traditionalists. Finally, in Chapter 7, Wolfmeyer brings together the ideas developed in previous chapters to explain how the conflicting interests in the nationalization of mathematics education have led to an ineffective system. He briefly proposes three alternative and expansive (rather than narrowing) views.

Although Wolfmeyer says that the book "is intended for those interested in math education and/or educational policy" (p. xi), researchers will most likely find the text useful, especially those who take a critical approach to mathematics education research. This volume makes two important contributions to our understanding of the insertion of corporate ties and interests. First, Wolfmeyer digs into the historical development of these corporate ties and makes explicit their connection to (through government and other organizations) and their interest in the nationalization of mathematics education. Second, he names the individuals and organizations involved and demonstrates how corporate interests have changed the nature of the support of these individuals and organizations. He convincingly explains how corporate and government interests may be at odds with other views of the purpose of mathematics education. He also argues that corporations are involved in mathematics education to create workers with the skills that businesses want because a larger pool of qualified employees can drive wages down. As explained by Wolfmeyer, governmental organizations facilitate the creation of these networks by appointing the individuals who serve both on quasi-governmental organizations and on corporate boards. Testing companies have an additional interest in the nationalization of mathematics education—getting and keeping megacontracts from federal and state governments. He explains how corporations recruit academics, in the role of "flexians" who represent multiple interests, including those of corporations and traditional scholarship. A mathematics education researcher may also work as a policymaker or consultant for organizations with contradictory views. Wolfmeyer cogently uses the role of flexians to illustrate the contradictions inherent in the nationalization of mathematics education.

One of the contradictions that Wolfmeyer presents is the apparent push by corporations for curricular emphasis on problem solving, even though curriculum materials continue to be mostly traditional. This tension is connected to contradictions within teacher preparation. Wolfmeyer explains that accrediting agencies and for-profit colleges, under the influence of human capital interests, push traditional teacher preparation to reduce classes on pedagogy and the social foundations of education. The effect is a narrowing of mathematics teacher preparation to advanced mathematical knowledge or, at most, as specialized mathematics knowledge for teaching. However, arguably, pedagogy classes provide teacher candidates with the skills and abilities to teach problem solving, and the social foundations courses prepare teacher candidates to teach a diverse student body.

Wolfmeyer's work helps explain this contradiction by noting how traditionalists and progressives have agreed on the deficiency of U.S. mathematics teachers' content knowledge. The result of the narrowing scope of teacher preparation will be a mathematics teaching force that is less able and less willing to challenge human capital development.

Despite the quality of the analysis and the significance of the topics and findings, we feel that *Math Education for America?* may attract fewer readers than it deserves. Wolfmeyer's political stance is clear throughout. Although we applaud this clarity, the lack of nuance in the equity argument may turn away potential readers. For example, readers may find it difficult to reconcile Wolfmeyer's explanation of equity in mathematics education with definitions of equity that explicitly reject corporate involvement (e.g., Frankenstein, 1990; Gutiérrez, 2002; Gutstein, 2006; Secada, 1989). Unfortunately, Wolfmeyer has collapsed the equity agenda into the simplistic "math for all." By narrowing the definition of equity, Wolfmeyer has potentially missed an opportunity to connect with a broader range of mathematics educators. As an alternative to Wolfmeyer's explanation of collusion between equity-minded mathematics educators and corporations, we propose that the language of equity may have been co-opted (Gutstein, 2009) by these corporate interests or, as he briefly mentions, corporate interest in equity may be a facade. Wolfmeyer does not provide a convincing enough argument about the involvement of equity-minded mathematics educators. The argument that corporate interest in equity may be a facade also leads us to question Wolfmeyer's argument that corporations have an interest in the development of problem-solving skills among mathematics students. For example, Apple (1992) noted that the majority of U.S. workers are in the service industry and do not require advanced mathematical skills. Thus, the corporate human capital interest in problem solving may be limited. Corporations want some to have advanced mathematical skills (in order to drive down wages) and many more without those skills to fill service sector jobs.

Much of Wolfmeyer's analysis makes reference to human capital theory and corporate interest in human capital development. Perhaps assuming that the reader takes the same critical view of human capital theory that Wolfmeyer does, he does not explain the negative effects of a human capital approach to education. An explanation would be essential to persuading a greater number of progressive mathematics educators of his views.

We close with some final suggestions and questions that we hope to see Wolfmeyer and others in the field take up in future work. The complexity and thoroughness of the sociograms are impressive. However, we sometimes skipped the maps to get to the text as we read. A book may not be the medium to view the entire network map. Rather, the book could do two things: first, present selections of the network map, as is done in other portions of the book, and second, use the network maps as a metaphor to explore the interconnectedness of policy development. The author might provide interactive versions of the network maps so that interested researchers could explore them in detail.

We are also left pondering the global connections. Wolfmeyer often mentions the "global economy" and "global elite," but the maps are mostly national. With the exception of the chapter on teachers' content knowledge, there is minimal exploration of global contexts. We wonder how global interests connect to a national (U.S.) mathematics education as well as how those interests connect to the U.S. network maps. There is a significant analysis of the role of educational testing companies, but what global connections do they have? How do international organizations influence these networks?

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Informing Practice

The Editorial Panel of *Mathematics Teaching in the Middle School* is seeking submissions for the department, Informing Practice. The articles written for this department should entice and invite classroom teachers to learn about aspects of research that are closely related to their classroom practice.

Topics that may be of interest can include—but are by no means limited to—teaching fractions, learning through problem solving, and using representations of linear relationships. Recent topics have included such areas as productive struggle, journaling, and professional noticing.

The article should do the following:

- Set up a classroom problem, issue, or question that will entice readers into the research
- Describe relevant research in a teacher's voice
- Incorporate examples, illustrations, and diagrams that will bring the research alive
- Provide specific recommendations or tips for classroom teachers.

The manuscript should be no more than 2000 words, and figures and photographs should be included at the end. Send manuscripts by accessing **mtms.msubmit.net**. On the tab titled "Keywords, Categories, Special Sections," select *Informing Practice* from the Departments/Calls section. For any questions, please contact **mtms@nctm.org**.

(*Ed. note:* For practical information about how to report on research that can be applied to the classroom, see the NCTM Research Committee's offering in the March 2012 issue of *JRME,* "Reporting Research for Practitioners: Proposed Guidelines," pp. 126–143.)