Lectures in Advanced Mathematics: Why Students Might Not Understand What the Mathematics Professor Is Trying to Convey

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SUPPLEMENTARY MATERIALS

S1. Our Research Team and a Real Analysis Teacher's Interpretation of the Lecture

The goal of this supplementary material is to provide a thorough review of our data analysis process. We discussed only two phenomena in the main paper to increase brevity and clarity. Here, for the sake of transparency, we present our analysis.

S1.1. Our Analysis of the Lecture

S1.1.1. General comments. Before discussing what mathematics we thought were the main ideas in the lecture proof, we observe that this lecture proof had characteristics that were typical of many lectures in advanced mathematics. Dr. A spent his entire time between the students and the blackboard; for the majority (61%) of the time, his back was to the class as he was writing the proof on his blackboard. In our judgment, students only had minimal participation in the proof. At five points during the presentation of the proof, Dr. A asked questions of the class, but two of those questions seemed rhetorical since Dr. A quickly provided the answer to them before students had the chance to respond. The other three questions used the classical initiate-response-evaluate format; in each case, Dr. A asked students to supply the next step in the proof that was being worked on. Twice the students did not immediately supply an answer to Dr. A's question so Dr. A provided a hint. For instance, in Lines 54–59 (the entire transcript of the proof appears in the Appendix), the following exchange occurred:

Dr. A: Now we know this is small [*circles one mathematical expression*], now what can we say about this expression right here [*underlines the geometric series* $1 + r + r^2 + \ldots + r^{m-n}$]? [*pause*] Anybody have a vague idea? I'll give you a hint: Calculus II.

Student: Geometric series?

Dr. A: Thirty or forty years ago. [*Points to the student who spoke.*]

Student: Geometric series.

Dr. A: Geometric series! You have to always keep geometric series in your toolbox.

We also noted that Dr. A's lecture proof was significantly more detailed than his blackboard proof. The blackboard proof consisted of a polished proof that might appear in a textbook. However, in the lecture proof, he supplemented the blackboard proof with many oral comments about how the proof was proceeding and why he proceeded in the way that he did.

S1.1.2. Main ideas from the lecture. One can show a sequence is convergent by showing it is a Cauchy sequence. This is especially useful in situations in which you do not know what the limit of the sequence will be. In real analysis courses, a sequence $\{x_n\}$ is defined as convergent with a definition similar to the following: A sequence $\{x_n\}$ converges to a limit L if and only if for all $\varepsilon > 0$, there exists a natural number N, such that if $n > N$, then $|x_n - L| < \varepsilon$. If one wants to prove a specific sequence is convergent by applying this definition, he or she usually needs to first propose a specific value for the term L, defined as the limit of the sequence, to which the sequence $\{x_n\}$ converges. In the theorem that Dr. A is proving, a specific sequence is not given. Rather, the theorem only states that the sequence has the property that the distance between any two consecutive elements x_n and x_{n-1} is less than r^n, where r is a constant with $0 < r < 1$. Since no specific sequence is given, it is impossible to specify what the limit of the sequence would be. One cannot prove that such a sequence converges by applying the definition of convergence, so another approach is needed.

In the proof, the sequence is shown to be convergent by demonstrating that it is a special type of sequence called a Cauchy sequence. In a previous class, Dr. A had proven that all Cauchy sequences are convergent. A main idea stressed in Dr. A's proof is that this theorem was useful to apply when one wanted to prove a sequence was convergent but could not determine what the

limit of the sequence was. Dr. A stressed this idea several times in the lecture. In Lines 9–17, Dr. A asks the students what types of sequences converge even if the limit cannot be determined. In these excerpts, Dr. A said,

> There's no mention of what the definition is of the sequence, so there's no way we're going to be able to verify the definition limit of a convergent sequence, where we have to produce the limit. So what do we do? […] What kind of sequences do we know converge even if we don't know what their limits are? It begins with a "c."

In Lines 25–28, after writing the definition of a Cauchy sequence, Dr. A reiterates this:

> This is how we prove it is a Cauchy sequence. See there is no mention of how the terms of the sequence are defined. There is no way in which we would be able to propose a limit L. So we have no way of proceeding except for showing that it is a Cauchy sequence or a contractive sequence.

From our perspective, illustrating that one can prove a sequence is convergent without knowing its limit by showing it is Cauchy is the primary reason Dr. A presented this proof. As Dr. A was stressing how to approach proving certain types of theorems, this idea was coded as method.

There is a common structure for writing proofs that show a sequence is a Cauchy sequence. Most proofs about convergence in real analysis have a common structure. A sequence is defined as being Cauchy if for all $\varepsilon > 0$, there exists a natural number N such that for all natural numbers m and n where $m, n > N$, $|x_n - x_m| < \varepsilon$. To prove that a sequence satisfies a "for all-there exists" statement, one begins the written proof by letting the universally quantified variable be given. Next, one defines the existentially quantified variable (in this case N) usually in terms of the universally quantified variable, and then justifies that the existentially quantified variable has the property contained in the "for all-there exists statement" (in this case, for all m and n greater than N, $|x_n - x_m| < \varepsilon$). Setting up these types of proofs is difficult for students, but it is also a procedural skill that is repeated for most proofs of this type. In Lines 21–25, Dr. A makes it clear that the setup he is using for this proof is common for all proofs about Cauchy sequences:

> How do we start our proofs about convergence, or Cauchy sequence? [*Faces board*.] Let epsilon be greater than zero be given. [*Writes this sentence on the blackboard*]. And now we'll state what it is we have to show. We will show that there is an $N(\varepsilon)$ for which $x_m - x_n$ would be less than epsilon when m and n are greater than this number $N(\varepsilon)$. [*Writes this sentence on the blackboard*.] This is how we prove it is a Cauchy sequence.

We coded this utterance as simultaneously conveying both verification and method. Dr. A is both explaining how the proof structure enables one to establish that the sequence in question was a Cauchy sequence (indeed, at the end of the proof Dr. A said that "verifying that a given sequence is Cauchy" was "one of the important results" of that part of the course) and indicating that proofs about Cauchy sequence usually begin similarly.

The triangle inequality is useful for showing that the sum of small terms is small. Like many proofs in real analysis, after the initial assumptions of the proof are made, the prover is often permitted to assume some quantities are small and is required to use this to demonstrate that another specified quantity is small. In the case of this particular proof, from the theorem statement, Dr. A assumes that $|x_n - x_{n-1}| < r^n$ for some number r between 0 and 1 (ensuring that r^n will become arbitrarily small as n becomes arbitrarily large) and desires to prove that for any ε, he can find an N such that for any m, $n > N$, it is the case that $|x_n - x_m| < \varepsilon$. Trying to show that some quantity can be made small often involves the use of a heuristic called the triangle inequality. The triangle inequality states that $|a + b| \leq |a| + |b|$, or more generally that the absolute value of a sum of terms is less than or equal to the sum of the absolute value of those terms. When Dr. A uses the triangle inequality in Lines 39–46, he emphasizes this idea.

> Now what would you do next? The triangle inequality. Over and over we have to use the triangle inequality. I should point out that in the homework I just passed out. Again, a number of you still are not comfortable using the triangle inequality. You have inside an absolute value sign terms with minus signs. And you drop the absolute value signs and still have terms with minus signs. And somehow, you have to be able to get rid of those using the absolute value given by the triangle inequality.

As Dr. A emphasized a heuristic for constructing proofs, this idea was coded as method.

The formula for geometric series is a useful technique for working with inequalities in real analysis. It should be in a student's mathematical toolbox for keeping quantities small. Many proofs in real analysis involve working with inequalities. In this specific proof, Dr. A uses the formula for a geometric series to simplify an inequality that he is considering. When he applies this formula in Line 59, he indicates this should be part of students' mathematical toolbox, saying, "Geometric series! You have to always keep a geometric series in your toolbox."

When mathematicians refer to a mathematical toolbox in this setting—presenting a proof in a real analysis course—we interpreted the mathematical toolbox as comprising techniques of working with inequalities to keep desired quantities small. However, it might be possible that Dr. A was speaking of any technique to keep quantities small. We corroborate in the next

section that Dr. A's interpretation of the term *toolbox* at least includes techniques for keeping quantities small. As a mathematical toolbox involved techniques that would be useful for the construction of other proofs, we coded this idea as method.

S1.2. Another Lecturer's Understanding of the Lecture

To corroborate that other mathematical acculturated individuals would share our interpretation of the main idea conveyed in Dr. A's presentation of the proof, we showed this lecture to an instructor of the course who was teaching a different section. The instructor identified three themes that he believed Dr. A was trying to convey. First, like us, he believed the main idea of presenting this proof was to show how one can prove a sequence is convergent when the limit of the sequence is unknown by showing it is a Cauchy sequence. The instructor said,

> So I think that's his main objective here. Because I'm assuming that up until this point, they've been showing convergence by—well, by definition you need to find the limit first. So he's saying okay, but what if you don't have a way to find the limit? Can you still show something converges? Well yeah, if you know it's Cauchy. I think that's one thing he's going to be trying to do here. I mean, you know, when you want to do different strategies for convergent, to prove a sequence converges, how are you going to do that? Well, if you can do it directly—as in, do you know the limit? Can you do it directly? Go for that. If you know it's monotone and bounded, that might work. But if you have no idea what the limit could be and it's not monotone, I think usually they're left with the option: Show it's Cauchy. I think that's probably one thing he's trying to show.

Second, the instructor noted that Dr. A was illustrating how to set up proofs showing a sequence is Cauchy. Third, the instructor noted that Dr. A was trying to show how the proofs were centered on the idea of trying to show one desired quantity must be small using the fact that other quantities are known to be small.

> He said, we have this thing $x_m - x_n$ we want to be small. […] And then he said, but we don't know about this term. What do we know about? We know about the difference of consecutive terms. How can you write this in terms of what we know about? And that's the difference they need to see. They say well, what do we actually know about it, what's my, how can I change this into something that I know about. And it's perfect.

This is related to our reference to the mathematical toolbox. Unlike us, the instructor did not comment on the importance of Dr. A's use of the triangle inequality of this proof.

S1.3. Summary

Our summary of the ideas we perceived in Dr. A's lecture is presented in Table S1.

Table S1
Summary of Our Perceptions of the Main Ideas of Dr. A's Lecture

Ideas	Type of idea	Coding
One can prove a sequence with an unknown limit is convergent by showing it is Cauchy.	Discovery[a]	Lines 9–17, Lines 25–28. Dr. A stated orally why Cauchy's theorem is both useful and necessary to prove this theorem.
How one sets up a proof that shows a sequence is Cauchy	Verification, Discovery	Lines 21–28. Dr. A wrote out the structure of the proof, explaining what needs to be shown to prove a sequence is Cauchy.
The triangle inequality is useful in proving series in absolute value formulae are small.	Discovery	Lines 39–46. Dr. A stated orally that the triangle inequality is used "over and over again" in these proofs.
The geometric series formula is part of the mathematical toolbox to keep some desired quantities small.	Discovery	Line 59. Dr. A stated orally that the geometric series formula needs to be in the students' mathematical toolbox.

[a] By discovery, we are referring to de Villiers' (1990) category that one purpose of a proof is to foster discovery by showing new ideas or methods to prove other theorems.

We note that three of the four main ideas that we recognized concerned discovery exclusively. Further, these ideas were only expressed in the lecture proof in the form of oral comments, not in the blackboard proof.

S2. Dr. A's Interpretation of the Lecture

S2.1. General Comments

The interview with Dr. A was surprisingly long. He spent 75 minutes discussing the proof that took him 10 minutes to present. Several things about this interview are worth noting. First, Dr. A repeated several main themes throughout the interview and rarely contradicted himself. He used these main themes as a justification for his learning goals for students and his methods of instruction. When asked why he chose to present this proof to his students, Dr. A gave an 11-minute response, situating Cauchy sequences along students' mathematical progression starting with calculus and concluding with the study of measurable functions in graduate school. These tendencies all support the notion that Dr. A's instruction was based on a coherent belief system and a good deal of thought.

A recurring theme in Dr. A's interviews was the importance of using repetition in lectures. Dr. A does not expect students to gain intuition and understanding the first time they view a proof. Rather, he hoped that by repeated exposure to the same idea, students would come to grasp the idea. He discussed repetition as critical for mathematicians' learning and hence should be for his students' learning:

> We have a famous mathematician in our department, Dr. B [a pseudonym for the sake of anonymity]. Who once was talking about learning mathematics and he was thinking, he was telling about back when he was just starting off in college and learning things like calculus, things like that. He said he'd worked ten problems and then suddenly a light would shine up in his head. It would take him ten problems before he understood what was going on. Now probably it was more like two, but he said ten. The point there is repetition. Over and over again, repeating the same kind of structure. And then sometimes, suddenly "Oh!" the intuition is lit up and from that point on you have a totally improved way of looking at what you're doing. And that's what I hope to achieve to the students.

He elaborated on the role of repetition in particular with respect to relating pictures to definitions in real analysis:

> Over and over again drawing pictures, finally at one point "Ah!" they see how the definition of a limit of a sequence, or a limit of a function, is representable in terms of the epsilon error is a picture, is the half-length of a neighborhood.

At another time in the interview, Dr. A highlighted the use of asking questions as a pedagogical strategy to keep students alert:

> The idea is, by asking questions, and asking people by names, they will have their minds alert, saying "he might ask me, I'd better think about what's going on." Now we all fall asleep in classes at times, so it's not clear you're always going to be alert. But hopefully, that *if the lecture is going to be of use to people, that during the lecture at times their minds are picking up something useful. Otherwise they're just copying off the board* [emphasis added], which is what we always do sometimes too. But, it makes it a little more exciting for me to be able to ask questions and talk to the class rather than stand up there and write stuff on the board.

From this excerpt, we are inferring that Dr. A does not think it is conducive to learning for students to simply copy the blackboard proof. If students' minds are to pick up something useful, they need to be alert and, presumably, listening to what Dr. A is saying in the class. One pedagogical technique that he uses to increase the chance of this occurring is asking questions to the class before verbalizing an idea that he believes is important.

S2.2. The Main Ideas the Instructor Aimed to Encode in His Lecture

Method. When Dr. A was observing the video, he stopped the video at every excerpt described in Table S1. His description of the idea that he wished to convey was consistent with the account we provided in section S1.1.2. One particular idea concerns students' mathematical toolbox. The excerpt below is used to corroborate our interpretation that to Dr. A, a mathematical toolbox in the context of real analysis includes techniques used to show a quantity is small because other quantities are assumed to be small. When asked to describe what he meant by *toolbox*, Dr. A gave the following reply:

> Once you get into the area where you're doing approximations, you can't do equal, equal, equal. You have to have bounds, bounds, bounds. […] *The objective is to show how bounds, using the triangle inequality, can be used to show that something is small using information that they're given is small* [emphasis added]. And this instance turns out that the information which is small is given in a form that allows us to use the geometric series as a bound.

Our interpretation of this excerpt is that the algebra involved in a real analysis proof differs from the algebra in students' previous experience. Until this point, students were usually solving equations or proving identities, which often involved substituting one equation with an equivalent equation. In real analysis, one is not solving equations but rather working with bounds. Students need to develop techniques using bounds that they can use to show desired quantities are small from the premise that other quantities are small. We do not discuss the other methods that Dr. A described as they were largely consistent with our interpretation of his lecture proof discussed in the previous section.

Thinking about Cauchy sequences in terms of pictures. When asked about the main ideas he intended to convey to students with this proof, Dr. A emphasized thinking of Cauchy sequences in terms of pictures, using the word *picture* 32 times in a 6-minute span. His initial response to this question was, "What has to be emphasized over and over again is that these definitions, which you might write down in symbols, are not going to make sense to you unless you have a picture associated with it. So I emphasize geometry. Pictures, pictures, pictures." He also indicates this is a regular practice throughout the course, saying, "Every time we have an analytic expression, I try to draw a picture. Back and forth, correspondence between pictures and analytic representations. . . . Pictures, symbolic representations. Pictures, symbolic representations. Back and forth." However, when viewing this particular proof, Dr. A noted that he actually did not include any pictures, saying, "This is a poor example. There are no pictures here! [*Laughs*.]" As Dr. A's stated reason for presenting pictures was to help students gain a conceptual understanding of Cauchy sequences, we coded the use of pictures as conceptual explanation.

Cauchy sequences can be thought of as a sequence whose terms are "bunching up." Dr. A claimed that he wanted students to view these sequences as "bunching up," which from our perspective implied that the terms of the tail of the sequence would become arbitrarily close together and "bunch up" around a particular point. Dr. A explained this as follows:

> So the idea is that first of all, you're given an error bound, an epsilon. And then you have to show that if you go far enough out in the sequence, the difference between any two terms whose index, the *m*, the *n*, are large enough. Will always be less than epsilon. What that says is that they bunch up. [*Dr. A places his hands vertically and parallel to one another and slowly moves his two hands toward each other*.] So the Cauchy property for a sequence is, the property says they bunch up [*Dr. A repeats the gesture described above*] in some place.

When viewing the video, he stopped the video at Line 6 and indicated that he was trying to convey the geometric interpretation of bunching up. In this excerpt, he made the gestures described above, saying, "Now what you expect as the terms [of the sequence] get closer and close, the sequence will converge." As this main idea was conveyed to help students understand the type of sequences in the theorem from a geometric perspective and use this perspective to see why it made intuitive sense for these sequences to converge, we coded this as conceptual explanation.

The terms of the sequence can be viewed as approximations of the limit of the sequence, and epsilons can be viewed as the error term of the approximation. Dr. A stressed that a continual goal in presenting these types of proofs throughout the course is for students to view the terms of the sequence as an approximation for its limit and the epsilon term as the maximum error from this approximation. The following excerpt is representative of several comments that Dr. A made during the interview:

> And the whole objective is to get them to have in their mind a certain way of approaching problems. That's learning how to do mathematics, is learning structures to carrying out certain types of proofs. Having pictures and a structure for how they develop their understanding of the pictures. Because the proofs relate to the pictures. Let epsilon be greater than zero be given tells us that there's a neighborhood. The epsilon is the error, which is the neighborhood in which the approximation will be. It can't be more than epsilon away from where the answer lies. So those are the things they've been seeing over and over again. Repetition, epsilon means error. And if you can make the error smaller than any epsilon, then you know that you have a sequence that's approximating something.

In this excerpt, Dr. A is reiterating the theme of repetition that we discussed earlier. He hopes that by repeating certain ideas in a proof, students will come to learn these ideas and naturally apply them when they write proofs on their own. One theme is "learning structures," which we discussed in section S1.1.2, as a common structure for proving that a sequence is Cauchy. The other is a picture where students should come to view the goal of convergence as approximating the limit of a sequence and the epsilon term specifying the distance between the actual limit and a specific term in the sequence. As this involves providing students with a metaphor to understand the concept of convergence and the definition of limit, we coded this main idea as a conceptual explanation. In viewing the lecture proof we did not observe any explicit mention of these ideas.[1] Indeed, the words *approximation and error* were not uttered by Dr. A during the lecture proof.

A relationship with computer science. Dr. A believed the idea of real analysis proofs were related to computer science, as we illustrate in the excerpt below:

> The idea of approximation is you want to get something that is close enough for you to live with. There's what's called acceptable error. In real life, if you're trying to calculate something that you can only do approximately, then you have to decide beforehand what kind of error can you live with? 10^{-5}? 10^{-6}? That sort of thing. Then what you want to do is to carry the calculation out to the point where the error is below that and then stop it. Because it can't go on forever. See the concept of convergent sequence is one that is an abstract concept. It doesn't relate to real life, because you don't get there forever. So you have to have an acceptable error and you stop it when the program gets within that acceptable error. And a lot of what we do in advanced calculus is talk about acceptable errors, that's the

[1] Upon reading the draft of this article, the lecturer described in section 4.2 disagreed, claiming that Lines 34–35 implicitly refer to this idea. We did not perceive this. Also, Dr. A did not highlight this excerpt as representing that type of conceptual idea. But we invite the reader to inform his or her own judgment.

> epsilon. And that, over and over again, what we want to do is be able to represent the difference between an approximate value and the abstract limit by an error bound. And in some of these problems, we actually determine a representation of error bound. [. . .] And once that error bound is less than the acceptable error which you have programmed into the computer, poof it stops. And that's the way I look at this, because that's what epsilon is all about. Error, acceptable error, things of this kind.

Our interpretation of this excerpt is that many approximation algorithms in computer science will continue indefinitely, over time yielding a better approximation. However, in practice, a computer cannot run for an infinite amount of time and the program must be terminated at some point. This point can be when the approximation is within an acceptable error. Students can view the epsilon in the proof as representing the acceptable error and the N term as knowing when the computer program should terminate. This idea was not coded in the proof itself, but was discussed elsewhere in the lecture. As this metaphor provides a useful way to interpret the epsilon-N definition of convergence, we coded this as conceptual explanation.[2]

S2.3. Summary

Our summary of the intended main ideas of Dr. A's lecture is presented in Table S2. In this expanded list of the ideas that Dr. A wished to convey, it is noteworthy that the ideas that Dr. A wished to convey were predominantly conceptual explanation and method, the goals valued by mathematics educators. Further, with the exception of how to structure a proof, none of these ideas appeared in the blackboard proof; rather they were stated orally. Of the three instances of conceptual explanation, only one was contained in this specific lecture proof. In this case, the mention of Cauchy sequences bunching up was only mentioned with a brief single gesture. It is possible that this idea was conveyed in other lectures.

Table S2
Summary of Ideas Dr. A's Claimed He Conveyed in the Lecture Proof

Idea	Type of idea	Coding
One can relate the concept of limit and Cauchy sequences to a picture.	Conceptual explanation	No pictures were presented in this proof.
Cauchy sequences can be understood as sequences that "bunch up."	Conceptual explanation	Lines 3–7. Dr. A stated orally, with gestures, how these gestures "bunch up."
The terms of a convergent sequence can be thought of as an approximation for its limit and the epsilon term can be thought of as the error of the approximation.	Conceptual explanation	No references to approximation or error were present in the proof presentation.
One can prove a sequence with an unknown limit is convergent by showing it is Cauchy.	Discovery[a]	Lines 9–17. Lines 25–28. Dr. A stated orally why Cauchy's theorem is both useful and necessary to prove this theorem.
How one sets up a proof that shows a sequence is Cauchy.	Discovery, Verification	Lines 21–28. Dr. A wrote out the structure of the proof, explaining what needs to be shown to prove a sequence is Cauchy.
The triangle inequality is useful in proving series in absolute value formulae are small.	Discovery	Lines 39–46. Dr. A stated orally that the triangle inequality is used "over and over again" in these proofs.
The geometric series formula is part of the mathematical toolbox to keep some desired quantities small.	Discovery	Line 59. Dr. A stated orally that the geometric series formula needs to be in the students' mathematical toolbox.
Proofs in real analysis are related to determining when algorithms in computer science should terminate.	Conceptual explanation	Stated elsewhere in the lecture, but not in this proof

[a] By discovery, we are referring to de Villiers' (1990) category that one purpose of a proof is to foster discovery by showing new ideas or methods to prove other theorems.

[2] There was some dispute in our research group on this coding, with some arguing that this would be better placed in a separate category called "application." However, when reading the transcript above, other lecturers of real analysis whom we spoke with all felt strongly that this metaphor was used by Dr. A to understand convergent sequences, not to apply them in other domains. This choice of code does not influence the subsequent analysis.

S3. Students' Perceptions of the Lecture

In each subsection that follows, we report on the ideas that students were able to grasp through each step of the student interview phase, as discussed in the methodology described in the main manuscript.

S3.1. Step 1: Students' Recall of the Main Ideas of the Lecture From Their Notes

S3.1.1. Students' notes. In the first stage of the interview, students revisited their own notes from lecture. We noticed that students' notes usually contained little beyond what Dr. A had written on the blackboard. S1 was an exception; she recorded nearly everything Dr. A said aloud as well as what he wrote. S2 claimed not to take notes that day. The notes of S3, S4, S5, and S6 were a near verbatim transcription of what Dr. A wrote on the blackboard.

S3.1.2. Students' recall of the lecture. Each pair of students was asked to describe what they learned from the proof and what they thought their professor was trying to convey. Their comments were sparse (perhaps predictably). S1 noted, "Like for Cauchy sequences, I remember that there's always like not two variables, but that it has the *m* and the *n*. And that's how I remember how to prove a Cauchy sequence." It appears that S1 is relating that she learned a proving heuristic from observing this proof. One interpretation of this excerpt was that if the sequence was defined in terms of its subscripts (the role that the variables *m* and *n* usually play in this context), one might try to prove it is Cauchy, perhaps because the definition of Cauchy sequences also contains subscripts. Her comments mainly focused on how particular statements in the proof logically followed from other statements. S2 added that "it showcases how you can use the Cauchy sequence proof to show that it's convergent." He concluded his description of the proof by noting, "Yeah, I don't think if this question was given on a test, I'm not sure I would be able to reproduce it."

In the second group, S3, the student Dr. A labeled as "below average," exhibited some confusion about what was proven, appearing to understand the original setup (described in Lines 21–28 of the proof) and the subsequent algebraic manipulations as two separate and independent proofs. She noted, "He used the two different methods to drive home the same thought process, I guess," and she guessed that he did so because students learn in different ways. She said, "I guess people learn differently, and some people may be able to understand . . . where he used the sum. But then some other people may understand it with a strict rigorous proof in the same set up that he always sets up the proofs." S4 observed the proof showed that one can show a sequence is convergent by showing it is Cauchy.

The third group was excited to discuss this proof, instantly recalling that the proof employed geometric series. S5 claimed that from viewing the proof, he learned that "just like simplifying series or sequences into the series so I can convert it with the definition of the limit," with a reference to geometric series in particular. This was consistent with Dr. A's method goal of including geometric series in students' toolbox to keep quantities small. S6 added that the proof illustrated "different approaches to different problems" and that one can rely on background knowledge from previous courses, such as the formula for geometric series, for constructing proofs in real analysis.

S3.1.3. Summary of Step 1. In general, the students did not mention the main ideas that Dr. A aimed to convey in this proof. With the exception of S3, students mentioned material that was accurate and potentially useful, but their interpretations of the proof did not highlight what Dr. A perceived to be most important. For instance, three students—S1, S2, and S4—remarked that the proof illustrated that one could prove a sequence was convergent by showing it was Cauchy. However, no student mentioned the critical point, emphasized thrice in the lecture, that this technique was specifically useful if one did not know what value the sequence converged to. The importance of the triangle inequality and geometric interpretations of Cauchy sequences, or convergent sequences, were not mentioned by any students. The third pair of students mentioned the use of geometric series as a problem-solving tool for working with inequalities, but they did not mention the important idea that this tool can help ensure that a sum of small terms will remain small.

One reason that students might not have recalled what Dr. A and we considered to be the main ideas conveyed in the proof is that five of the six students did not record in their notes what Dr. A said orally. As Dr. A conveyed the conceptual and method ideas of the proof orally but did not do so in the blackboard proof, students did not record these ideas in their notes. Perhaps consequently, they were not able to recall these ideas at a later time. This step is similar to how students would study the material after a lecture, suggesting that many of the main ideas that Dr. A intended to include in the proof would not be retained by the students.

S3.2. Step 2: Students' Perceptions of the Main Ideas After Viewing the Proof

Each pair of students was shown a video of the proof in its entirety and given a sheet of paper containing Dr. A's blackboard proof. The students were again asked what they learned from the proof and what they thought Dr. A was trying to convey. Students' comments in this step were more detailed than in Step 1.

Pair 1 highlighted three things: (a) Cauchy sequences will be on the midterm, (b) the use of geometric series brought in prior knowledge, and (c) the importance of using the triangle inequality. Regarding (c), S2 stated, "When he talked again about the triangle inequality, I think this was right after one of the homeworks where people did badly because of, not using triangle

inequality. So I think it wasn't an accident that he chose one that used it repetitive." Interestingly, S1 followed up on this comment by noting that Dr. A used repetition as a pedagogical device to have students in the right mindset for proof writing. S1 said, "Kind of going off of that, like when you said repetition of the triangle inequality, I feel like he's doing a lot of repetition to get us into the mode of thinking how to prove," which is consistent with Dr. A's description of and motivation for his pedagogical practice.

Pair 2 highlighted three things that they learned from reading this proof: (a) one can show a sequence is convergent by showing it is Cauchy, (b) there is a consistent structure to writing proofs about convergence, and (c) one can use ideas from calculus (specifically geometric series and the triangle inequality) to write proofs in analysis. With regard to (a), immediately after viewing the proof, S3 stated, "We're showing that a sequence converges by showing that it's a Cauchy sequence." an idea that S4 highlighted in Step 1. Regarding (b), S4 observed the repetition of this proof structure by Dr. A:

> Other than showing that a contractive sequence is a Cauchy sequence, I think it's more. He's showing more of the structure of the proof […] A lot of the proofs that we did over the last nine or so weeks basically have the same structure […] So basically like taking the definitions and just kind of making a variation on the structure that we already used for something like six weeks.

With regard to (c), S3 noted proofs like this illustrated the motivation of presenting these proofs in calculus. S3 said what this proof "shows is really the [geometric] sequence and he introduced like, the triangle inequality again. And where that came from and why we were given that formula when we were back in calc."

Pair 3 claimed the proof motivated the following ideas: (a) the proof expanded the students' toolbox of how to simplify expressions and (b) the proof illustrated how students can use prior knowledge from calculus to write proofs in analysis. With regard to (a), S5's first comment about the proof was, "Well he [Dr. A] said in the beginning that he was, he tried to get us to think back in our toolbox that, some of the formulas that we could implement to simplify any given case that seems difficult, at the moment." Although S5 and Dr. A both mentioned the word *toolbox*, it is not clear that they are using this term in the same way. Our interpretation of Dr. A's use of the term (supported by Dr. A's interview comments) was that a toolbox consists of techniques to keep expressions in an inequality small. This is because when one writes proofs in real analysis, one is typically permitted to assume certain quantities are small and are then required to prove that another specific quantity is small. Knowing how to manipulate such algebraic expressions without having them become too large is important for proficiency in writing these types of proofs. S5's comment discussed simplifying equations, which involves algebraic manipulations but not the crucial idea of keeping quantities small.

S3.2.1. Summary of Step 2. There are two points we wish to emphasize from the second step. First, although all pairs of students highlighted important ideas in the proof, none of the students mentioned what we thought was the most important idea in the proof—that showing a sequence is Cauchy is an important method for proving the sequence is convergent *particularly* when one does not know the limit of the sequence. Two pairs of students mentioned that showing the sequence was Cauchy was a way of establishing convergence, but no student mentioned the conditions under which this was likely to be useful. Dr. A emphasized these conditions at three separate times in his lecture.

The second point is that each pair of students said the proof taught them to consider prior knowledge to help them progress in the proof. Neither Dr. A nor our research team noted this idea when viewing the proof ourselves, yet all of the students found this valuable. Nonetheless, this certainly is a valid and useful interpretation of the proof and illustrates how students' interpretation of the proof may differ from the instructor's intentions, but still be useful.

S3.3. Step 3: Students' Interpretations of Specific Video Clips

We showed the student pairs individual clips that Dr. A highlighted as conveying main ideas (summarized in Table S2) and asked the student pairs what ideas they believed Dr. A was attempting to convey.

S3.3.1. Lines 4–8. In Lines 4–8, Dr. A claimed to be trying to give students some geometric intuition for what was being asserted in the theorem and why the theorem was true:

> So since, when n [*points to r^n written on the board*] gets large this [*points to $|x_{n+1} - x_n|$*] gets very small. It will be that these two consecutive terms will get closer and closer together as n gets larger. Now what you expect as the terms get closer and closer [*places hands vertically parallel to one another and brings them closer together*], the sequence will converge. What will the limit be? [*Shrugs shoulders to indicate that he doesn't know*.]

For Pair 1, S2 believed this clip was trying to establish geometric intuition for why the sequence converged, saying, "I just got the idea of starting with intuition. Just, I mean, this is fairly intuitive. You look at it and the r to the ns are going to keep going up and so this interval is going to keep shrinking, so of course it would be natural to suggest Cauchy sequence."

Pair 3 also grasped that Dr. A was trying to convey an intuitive argument prior to the proof. S6 remarked, "Instead of just going over the theoretical proof part, he was just trying to give us a geometric picture like why this has to be true." S5 agreed and said, "The picture is the main idea." S5's comment is especially interesting as no picture was drawn with this proof. The kinesthetic motions of Dr. A, however, did suggest that the given sequence could be represented visually on the real number line with the terms of the sequence becoming closer together.

Pair 2 did not remark on the geometric nature of Dr. A's argument, with both S3 and S4 saying that Dr. A was trying to convey that one can show a sequence is Cauchy without knowing its limit (perhaps because Dr. A shrugged at the end of the clip when asked what the limit of the sequence was).

S3.3.2. Lines 9–17. In this excerpt, Dr. A introduces the idea of Cauchy sequences as a way to show that a sequence is convergent.

Dr. A: How can we proceed to show that this is a convergent sequence? Anybody have a guess?

Student: [*Incomprehensible utterance*]

Dr. A: Well that's not quite the right term. What kind of sequences do we know converge even if we don't know what their limits are? [*pause*] It begins in "c."

Students: Cauchy.

Dr. A: Cauchy! We'll show it's a Cauchy sequence.

Both Pair 1 and Pair 2 believed Dr. A was trying to convey that one can show a sequence is convergent by showing it is Cauchy, which is useful if you do not know the limit of the sequence. For instance, S1 said, "We should recognize it, like to figure out it's a Cauchy, we should know that it's converging, but its limit is not necessarily given. So that we recognize it instantly," and S2 said, "Because we don't have the limit here, or we have no way of figuring out what the limit is. All we have is them in relation to each other. Cauchy makes sense." Pair 3 focused on how Dr. A interacted with the class, believing his intent was to keep students active and gauge what they knew.

S3.3.3. Lines 18–19. In this excerpt, Dr. A explicitly highlights that one can show a sequence is Cauchy without knowing what the limit is.

> We will show that this sequence converges by showing that it is a Cauchy sequence. [*Writes this sentence on the board as he says it aloud, then turns around to face class*.] A Cauchy sequence is defined without any mention of limit.

Pair 1 and Pair 2 repeated that the intent here was to remind students that one can show a sequence is Cauchy without knowing its limit. Pair 3 viewed the intention differently: that Dr. A wanted students to get in the habit of stating what they were trying to prove. For instance, S5 said, "So basically he was trying to set up a routine I guess, so for beginning a proof we should always write up what we are aiming to prove or what we are aiming to show."

S3.3.4. Lines 23–29. Here Dr. A reiterates that one cannot find a limit for the sequence in the theorem; therefore showing the sequence is convergent involves showing that it is Cauchy.

> And now we'll state what it is we have to show. We will show that there is an $N(\varepsilon)$ for which $x_m - x_n$ would be less than epsilon when m and n are greater than this number $N(\varepsilon)$. [*Dr. A writes this sentence on the board as he says it aloud*.] This is how we prove it is a Cauchy sequence. [*Turns around and faces class*.] See there is no mention of how the terms of the sequence are defined. There is no way in which we would be able to propose a limit L. So we have no way of proceeding except for showing that it is a Cauchy sequence or a contractive sequence. So let's look and see how we proceed.

Only Pair 2 remarked that Dr. A was trying to convey that one needed to use Cauchy sequences to establish convergence because one could not propose a limit of the sequence under investigation. Both students in Pair 1 said they were not sure what the intention of this clip was, although they had mentioned what Pair 2 described when watching the previous two clips.

In Pair 3, S6 mentioned that the limit of the sequence could not be determined. He said, "He wanted to emphasize that there is no mention of limit whatsoever, so we won't like confuse it with the concept of limit because we were using that method before this class." S5 followed by saying he had nothing to add. Our interpretation of this excerpt is that they perceived Dr. A as noting that their previous approaches to showing convergence, which relied on knowing the limit of the convergent sequence, would be inadequate for this problem. Pair 3 did not mention, however, that showing a sequence was Cauchy would be useful, since the definition of Cauchy sequences did not involve the definition of limit.

S3.3.5. Lines 33–37. Dr. A indicated that the excerpt below was designed to highlight that many proofs in real analysis involve starting with terms that are known to be small and using them to show that a desired quantity is small.

> Now once again we ask the question. If we were to show this is small, we must represent it in terms of what we know is small. Well what do you know is small? For n large enough [*gestures toward the statement of the theorem*], the difference between two consecutive terms is small. [*Turns and faces the blackboard*.] So what we must do is represent that as a sum of consecutive terms.

When asked to describe what Dr. A was trying to convey in this excerpt, none of the pairs of students mentioned the idea of

showing a quantity was small. For Pair 1, S1 focused on the heuristic of "adding and subtracting terms that weren't there initially" because this has been done before "and also helps with other proofs," with S2 saying he had nothing to add. Pairs 2 and 3 also focused on algebraic manipulations. In Pair 2, S4 said, "Basically manipulating the information that we're given so that we can show that a sequence fits the definition." In Pair 3, S6 said, "Given on the problem to see like what we could, how we can manipulate the problem statement. Just how we can start the proof in general." None of the six students used the wordsmall or any synonym for small in their responses.

S3.3.6. Lines 39–48. Dr. A described the following clip as highlighting the importance of using the triangle inequality, a formula that is often useful in these types of proofs.

Dr. A: Now what would you do next?

Student: The triangle inequality.

Dr. A: The triangle inequality; over and over we have to use the triangle inequality. I should point out that in the homework I just passed out. Again a number of you still are not comfortable using the triangle inequality. You have inside an absolute value sign terms with minus signs. And you drop the absolute value signs and still have terms with minus signs. And somehow, you have to be able to get rid of those using the absolute value given by the triangle inequality. [*Turns to face the blackboard*.] All right so use the triangle inequality. So we're going to have less than or equal to $x_m - x_{m-1} + x_{m-1} - x_{m-2}$ all the way down to $x_{n+1} - x_n$. [*Writes this equation on the blackboard as he says it aloud*.]

All three pairs of students said that Dr. A was stressing the importance of the triangle inequality. For instance, when asked what Dr. A was trying to convey, S4 replied, "The importance of the triangle inequality. Because it really is a godsend when it comes to proving inequalities."

S3.3.7. Lines 53–61. In describing this clip, Dr. A pointed to the importance of having geometric series in one's toolbox for establishing bounds with inequalities.

Dr. A: So let's factor out the smallest term, r^n. What's left is $1 + r + r^2 +$ up to r^{m-n}. [*Writes this equation on the blackboard as he speaks*.] Now we know this is small [*circles* r^n], now what can we say about this expression right here? [*Underlines the geometric series* $1 + r + r^2 + \ldots + r^{m-n}$*, then turns around and faces the class*.] Anybody have a vague idea? I'll give you a hint: Calculus II. [. . .] Thirty or forty years ago.

Student: Geometric series.

Dr. A: Geometric series! [*Turns and faces the blackboard*.] You have to always keep a geometric series in your toolbox. So it's going to be less than r^n, this [*points quickly to the geometric series* $1 + r + r^2 + \ldots + r^{m-n}$ *written above*] then is less than sum from $k = 0$ to infinity of r^k. And now we need to know the formula of a sum of a geometric series.

When asked what Dr. A was trying to convey, no student mentioned the notion of toolbox. Rather, all three pairs of students highlighted the importance of referring back to previous knowledge to complete this proof. For instance, S6 stated Dr. A was "trying to like convert this expression somehow into an expression that we are familiar with or we know about from like our previous courses—in this case it would be geometric series."

S3.3.8. Summary of Step 3. When shown specific clips that Dr. A highlighted, the students were collectively much better at identifying what Dr. A aimed to convey. For instance, two pairs of students were able to articulate the geometric intuition that Dr. A was attempting to convey in the first clip they were shown, two pairs of students identified what the research team felt was the main idea of the proof (i.e., that to prove a sequence is convergent without knowing its limit, one should show that it is Cauchy), and all students recognized the importance of the triangle inequality. As these ideas concerned conceptual explanation and method, the findings above indicate that these students could decode some of the main ideas that Dr. A expressed orally, if asked to do so immediately after viewing his comments.

On the other hand, we note that students' interpretations did not always match Dr. A's intentions. No student explained the sixth clip as indicating that these proofs involved starting with small quantities and using this information to show another quantity was small, and Pair 3 never articulated what we thought was the key reason for Dr. A presenting this proof.

S3.4. Step 4: Did Students Recognize the Main Ideas That Dr. A Included in the Proof?

In this final step of the student interview, the pairs were specifically asked if the main ideas that Dr. A described could have been gleaned from his proof presentation. For two of these main ideas (showing that proofs about convergence have a common structure and Cauchy sequences can be thought of as "bunching up," not discussed in this section for the sake of brevity), students agreed that they could and were able to articulate how the proof demonstrated these ideas. For the other main ideas Dr. A discussed, a general finding was that the students often agreed that the ideas were present in the proof. However, from their comments, we found their interpretations of these ideas differed from Dr. A's.

S3.4.1. Epsilon as error. As noted earlier, Dr. A conceptualized the terms of the sequence as approximations for its limit and epsilons as the errors of these approximations. Each pair of students was asked, "Another thing that you might get from this proof is that the epsilon used is the error in approximations. Is that something that you got from this presentation?" All six students replied that Dr. A repeatedly emphasized that the epsilon was an error term. However, none of the participants described what was meant by this point; none mentioned the word *approximation* in their replies or even that epsilon represented distance from the actual limit. To illustrate, we present Pair 2's response to this question.

[*Both students immediately nod yes to the interviewer's question*.]

S3: Yeah. I think the first day or the second day, he explained what epsilon was so I think every time I see epsilon, I think about the error.

S4: Yeah.

I: So in this, so this proof is included in every time you see the epsilon?

S3: Yeah.

S4: [*Nods*.] Yeah, every time, every time I begin a proof I see let epsilon greater than zero be given. And we've had the same opening for that proof for, ever since we started dealing with like lower bounds and upper bounds.

Note in this excerpt, the students do not describe what epsilon being error would mean. In fact, S4's response appears to have little to do with Dr. A's intentions. These students appear to have appropriated some of the language that Dr. A used in his lectures, but we did not have evidence that the terms they used had a rich conceptual meaning.

S3.4.2. Toolbox for making quantities small. Dr. A stressed the need for students to have in their toolbox techniques for working with inequalities that would keep desired quantities small. The interviewer asked each pair of students, "One last thing you might get from this proof is that mathematics students need to have a toolbox of ideas that help them to prove things are small. Is this something that you got from this presentation?" All six students answered yes. However, in their responses, none mentioned inequalities or making things small. Indeed, from their responses, it appeared that students viewed the components of the toolbox as general techniques for writing proofs in mathematics. For instance, S2 described Cauchy sequences as being part of his toolbox, indicating the toolbox was for proving convergence, not keeping quantities small. S3 described, "I think if he structures the way that he does, and you keep seeing it, it stays in your toolbox memory area [. . .] not just in this specific proof itself, but it carries over to any other areas of math when you want to start to prove something." Again, here it appears that how one structures real analysis proofs is part of one's toolbox. Only one student, S5, mentioned the word *small* in his or response. In the following excerpt, we can see that S5 was not using *small* in terms of a magnitude of a quantity, as Dr. A intended.

S5: We can use Mathematica, or like a tool to convert to make something small.

I: So right so mathematics students need to have a toolbox of ideas to help them prove things are small.

S5: Things are small. Oh you mean that they're not so complicated. When you say that things are small?

I: No I mean like in terms of convergent sequences. Is that something that you think you got from this presentation?

S5: I mean, in terms of simplifying them and deriving for approximating the answer, I think it's on the path, it's like it's working.

By listing Mathematica (a computer algebra system commonly used in college calculus classes but not in real analysis), S5 is referring to general mathematical tools, rather than tools for working with inequalities or keeping quantities small. His response to the next question revealed that S5 did not know what was meant by "small" in this context, guessing that it means a not complicated, or simplified, equation, rather than a quantity with a small magnitude.

S3.4.4. Summary of Step 4. In this stage of the interview, the students were able to articulate what it meant for Cauchy sequences to be "bunching up" and how convergence proofs often had a common structure. For topics such as epsilon representing the error term of an approximation and having a toolbox to keep quantities small, the students generally recalled that Dr. A conveyed these ideas and were able to use these words when discussing these ideas. However, in analyzing what students said, it is apparent that what students meant by these words differs significantly from what Dr. A meant. (The same is true for comments about applications to computer science, but these were not reported for the sake of brevity). To Dr. A, these terms signified rich conceptual structures, while for the students, these terms seemed to be thought of merely as labels.

S3.5. Summary of Students' Perceptions of the Lecture

In Table S3 below, we list each of the main ideas that Dr. A believed he encoded in his lecture and what step in the student interview, if any, that students first articulated these ideas.

Table 3
Steps During Which Student Pairs First Identified Dr. A's Main Ideas

Dr. A's Main Ideas	Step 1	Step 2	Step 3	Step 4
1. Cauchy sequences can be understood as sequences that "bunch up."	—	—	Pairs 1 and 3	Pair 2
2. One can prove a sequence with an unknown limit is convergent by showing it is Cauchy.	—	—	Pairs 1 and 2	—
3. There is a common structure for proofs that show a sequence is Cauchy.	—	Pair 2	—	Pairs 1 and 3
4. The triangle inequality is useful in proving summations in absolute value formulae are small.	—	Pair 1	Pairs 2 and 3	—
5. The geometric series formula is part of the mathematical toolbox to keep some desired quantities small.	—	—	—	—

There were two instances where students described ideas related to those in Table S3, but at a more shallow level than Dr. A intended. First, all three pairs of students observed that the proof illustrated a new way to show a sequence was convergent—namely, by showing that it was Cauchy—and Pair 1 and Pair 2 remarked on this idea in the first step in the student interview. However, the conditions under which this was useful, when a limit for the sequence could not be proposed, were less prevalent in students' responses. Pair 1 and Pair 2 did not mention this until the third step in the interviews, and Pair 3 did not discuss this idea in any step of the interview. Second, all pairs of students remarked on geometric series as being part of a students' mathematical toolbox, yet no student discussed geometric series (or anything else in the toolbox) as being a useful tool for keeping specific quantities small, which was Dr. A's intention. Finally, in the second step of the student interviews, all pairs of students noted that the proof illustrated that one's prior knowledge could be useful in writing these proofs. This was not one of the main ideas that we highlighted, but we feel this was a valuable lesson nonetheless.