Transforming Perceptions of Proof: A Four-Part Instructional Sequence*

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Mathematics teachers are expected to engage their students in critiquing and constructing viable arguments. These classroom expectations are necessary, given that proof is a central mathematical activity. However, mathematics teachers have been provided limited opportunities as *learners* to construct arguments and critique the reasoning of others, and hence have developed perceptions of proof as an object that must follow a strict format. In this article, we describe a four-part instructional sequence designed to broaden and deepen teachers' perception of the nature of proof. We analyzed participants' reflections on the instructional sequence in order to gain insight into (a) the differences between this instructional sequence and participants' previous proof learning opportunities and (b) the ways this activity was influential in transforming participants' perceptions of proof. Participants' previous learning experiences were focused on memorizing and reproducing textbook or instructor proofs, and our sequence was different because it actively and collaboratively engaged participants in constructing their own arguments, critiquing others' reasoning, and creating criteria for what counts as proof. Participants found these activities transformative as they became more clear about what counts as proof, began to view proof as socially negotiated, and expanded their conception of proof beyond a rigid structure or format.

Key words: Reasoning and proof; Learning to prove; Constructing arguments; Critiquing arguments; Criteria for proof

Secondary mathematics teachers are expected to provide their students with opportunities to construct viable arguments and critique each other's reasoning across all mathematics courses and topics (CCSSM, 2010; NCTM, 2000, 2009). These are noteworthy recommendations given the central role of proof in mathematics, and how engagement in reasoning and proving has the potential to deepen students' understanding of mathematical concepts (Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002; Hanna, 1995; Hersh, 1993). Furthermore, the treatment of proof in classrooms should be a negotiated communal activity that promotes understanding (Ball et al., 2002; Hanna, 1995; Hersh, 1993). To reach this end, G. Stylianides (2008) explains how classrooms can access deductive reasoning through a set of activities such as generating examples, identifying patterns, and making a generalization before constructing an argument. He explains these activities as the reasoning-and-proving framework that aligns with stages mathematicians pass through to produce a proof. While progress is underway (e.g. G. Stylianides & A. Stylianides, 2009; Karunakaran, Freeburn, Konuk & Arbaugh, 2014), more work is needed before a communally negotiated view of proof is materialized across secondary classrooms (Bieda, 2010; Furinghetti & Morselli, 2011; Steele & Rogers, 2012).

A current challenge with integrating proof as a central curricular activity is that based on prior experiences, undergraduate mathematics and mathematics education majors and practicing secondary mathematics teachers have developed narrow views and abilities with proof (Bleiler, Thompson, & Krajčevski, 2014; Furinghetti & Morselli, 2011; Knuth, 2002a, 2002b; Kotelawala, 2009; Tabach et al., 2011). Mathematics teachers have been given limited opportunities as *learners* to construct arguments and critique the reasoning of others, and hence have developed narrow perceptions of proof. For example, some undergraduate students and practicing secondary mathematics teachers believe that proof must follow a strict format

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(Bleiler et al., 2014; Tabach et al., 2011). Some practicing secondary mathematics teachers view proof as a topic of study in geometry (Knuth, 2002a; Kotelawala, 2009) or think that it is only appropriate for the most advanced high school students (Knuth, 2002a; Furinghetti & Morselli, 2011). Thus, teachers either present proofs to students as a final product or avoid proofs altogether (Furinghetti & Morselli, 2011), often believing that their students are unable to prove (Knuth, 2002a).

Instruction that focuses on presenting proof as a completed product limits learners' access with how to construct and/or assess arguments, as students in these learning environments are passive participants (Hanna, 1995; Harel & Sowder, 1998; Solomon, 2006). Traditionally, the instructor or textbook serves as the sole producer and arbiter of proof in a classroom, which Harel and Rabin (2010) identify as the authoritative approach. Learning to construct proofs in an authoritative learning environment contributes to narrow views and limited understanding (Hanna, 1995, 2000; Harel & Sowder, 2007). We agreed with Harel and colleagues (Harel & Sowder, 1998; Harel & Rabin, 2010) that when students (K–16) experience proof through the authoritative perspective, they develop limiting views of proof and these views stay with them even after they become teachers. As we summarized below in Figure 1 (authoritative perspective), the beliefs that teachers harbor, as shown in the first trapezoid, toward the nature of proof in schools translates into classroom practices and assessment routines that lead to unproductive results. The student learning outcomes (i.e., students view proof as a rigid, formal object; students learn that they are unable to prove) are direct outputs of the classroom engagement and assessment routines where students view their role as passive consumers of presented proofs (Solomon, 2006). These student outcomes lead some educators to advocate for proof to become more reclusive, and others have called for it to be eliminated from curricula altogether (Hanna, 1995). To address this ongoing cycle grounded within the authoritative perspective, we find it essential to engage teachers in productive proof learning opportunities. The goal is to transform teachers' perspectives away from the authoritative approach, so that proof has the potential to become a central part of the curricula.

We contend that if prospective and practicing teachers have opportunities to learn proof as a communal, negotiated, and sense-making process as recommended (Ball et al., 2002; Hanna, 2000), they will be better equipped to foster their students' development of proof. In this article, we offer a general four-part instructional design that has the potential to transform learners' perceptions of proof. We explain in detail our specific enactment of an instructional sequence and then provide a discussion of the key generalized features of the sequence that can be implemented across instructional contexts with different student populations. We seek to answer the following two research questions:

- In what ways do participants perceive this instructional sequence as different in relation to their prior experiences with mathematical proof?
- How (if at all) do participants perceive that the activities in this instructional sequence changed their perceptions of proof?

Background

The pervasive authoritative perspective directly conflicts with research recommendations regarding how to support students' access and ability to construct proofs (Harel & Sowder, 2007; Hanna, 1995; Lannin, 2005; A. Stylianides, 2007a). The research indicates that students at all levels (K–16) should be provided opportunities to engage in proof as a process (e.g., generating examples, looking for patterns, and making a generalization) before developing a valid argument, as mathematicians do (Stylianides, 2008).

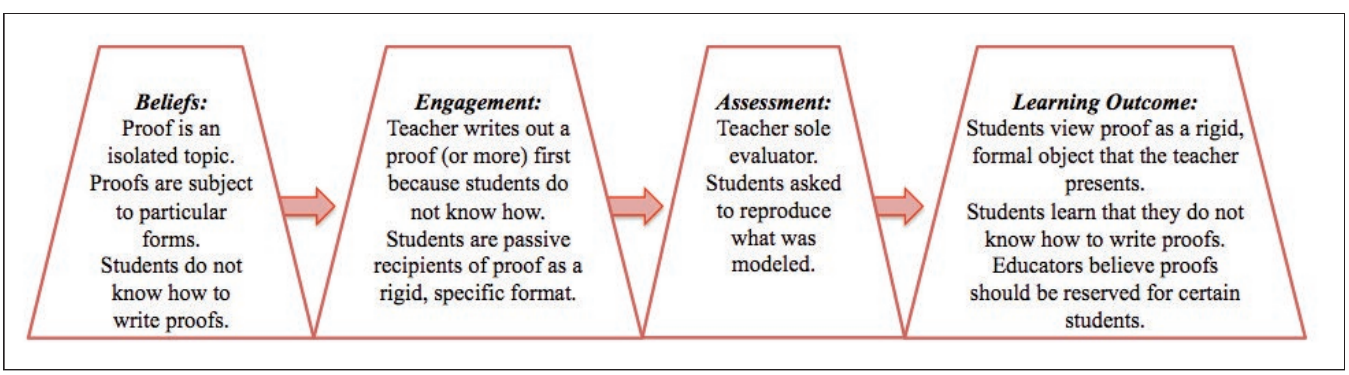

Figure 1. **Authoritative perspective of proof.**

If students have a general idea about what is needed for an argument to count as proof, they can understand what they are working toward (Stylianides, 2007a, 2007b). The disconnect between research and classroom practice requires attention, and a productive way to address the disconnect is through changing students' and teachers' learning opportunities. We designed Figure 2 below to summarize our conceptualization of the research recommendations for proof, positioning proof as a communal activity. By contrasting Figure 1 and Figure 2, we can see how research recommendations (Figure 2) conflict with the authoritative perspective (Figure 1).

Starting with beliefs (as shown in Figure 2), students should be afforded opportunities to engage in proof as a process, even if their initial attempts to construct arguments are not proofs (G. Stylianides & A. Stylianides, 2009; NCTM, 2009, 2014). Teachers often believe they should focus on what students produce as correct (proof) or incorrect (nonproof) (Furinghetti & Morselli, 2011; Knuth, 2002a), as opposed to working from what students produce as part of a learning process. Lannin (2005) conducted a teaching experiment to develop the ability of 25 sixth-grade students to construct proofs. Lannin found that students mostly depended upon empirical arguments when they constructed arguments on their own. However, during the whole-class discussion, students were able to verbalize a proof. The students in the teaching experiment were provided access through the use of pattern tasks where students could first examine examples before constructing a generalized argument. Some secondary mathematics teachers view proof as a topic of study in geometry or that it should be reserved for the most advanced students (Furinghetti & Morselli, 2011; Knuth, 2002a, 2002b; Kotelawala, 2009), but Lannin shared how proof is possible outside geometry as he worked from what sixth-grade students produced toward communally constructing a valid argument.

Moving beyond beliefs, teachers also want to know how to productively engage students in reasoning and proving. An important part of engagement is selecting a task (NCTM, 2014; Smith & Stein, 2011). Following the reasoning-and-proving framework (i.e., Stylianides, 2008), students are afforded access to proof when they are familiar with reasoning activities similar to those that mathematicians follow. Since students may not know to generate examples on their own in order to identify patterns, the written task can promote reasoning activities explicitly before prompting students for a proof. Karunakaran and colleagues (**Karunakaran, Freeburn, Konuk, & Arbaugh, 2014**) taught a course to preservice secondary teachers (PSTs) while implementing the Cases of Reasoning and Proof in Secondary Mathematics[1] (CORP) materials (Smith, Boyle, Arbaugh, Steele, & Stylianides, 2014) to develop teachers' (practicing and/or preservice) knowledge and ability to implement reasoning-and-proving tasks. In their study, Karunakaran et al. determined that they were able to improve PSTs' ability to assess and construct proofs and noted, similar to Lannin (2005), that the tasks used supported PSTs' access to developing arguments.

As students work toward developing a proof, the classroom community should collaborate to assess proposed arguments to determine what is valid. A. Stylianides

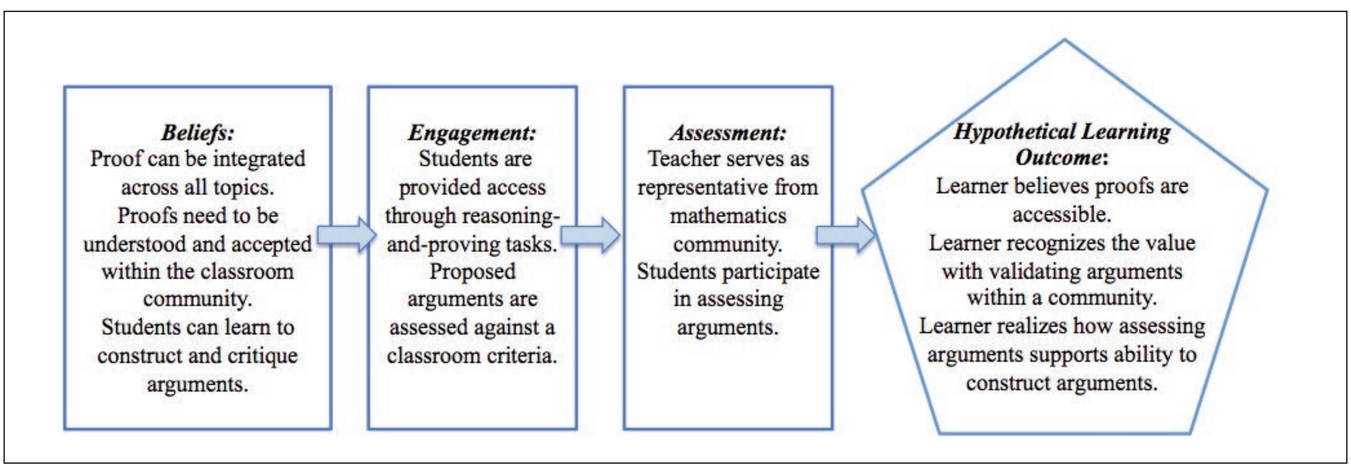

Figure 2. **Communal perspective of proof.**

1 A set of curricular materials developed under the auspices of the National Science Foundation-funded CORP (Cases of Reasoning and Proof in Secondary Mathematics) Project, grant DRK–12 #0732798, directed by Margaret (Peg) Smith and Fran Arbaugh.

(2007a) developed general criteria of proof in school mathematics:

> **Proof is a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:**
>
> **1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification;**
>
> **2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and**
>
> **3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community. (p. 291)**

The idea is that teachers would use A. Stylianides's characteristics to communally develop classroom criteria of proof as a way to critique arguments. Moreover, the aim is "to achieve a defensible balance between two (often competing) considerations: *mathematics as a discipline and students as mathematical learners"* (Stylianides, 2007b, p. 294). Teachers assume the role as a representative of the mathematics community to support students with developing criteria and apply the accepted criteria to critique presented arguments.

While a goal would be to have students construct proofs, a learning outcome is for teachers to understand that proofs are accessible for all students across all content and to recognize how to support students' engagement in constructing and critiquing arguments.

Healy and Hoyles (2000) studied high school students' ability to construct and assess arguments with respect to proof. The findings include accepting and constructing empirical arguments as proof, but the authors also found that successful provers were those who relied on everyday language as opposed to trying to compose a formal solution consisting of algebraic symbols. An interesting take-away is that the students accepted the algebraic nonproof arguments as proof since they thought those were the ones their teachers would give the highest grade. Therefore, students intuitively understand the value of general arguments that promote understanding, which is what should be the role of proof in classrooms (Hanna, 1995; Hersh, 1993); thus, teachers need to learn to promote a sense-making process of proof where validation goes beyond examples.

Designing an Instructional Sequence to Reach Learning Outcomes

G. Stylianides and A. Stylianides (2009) designed and implemented an instructional sequence to foster teachers' understanding that while examples are helpful with finding patterns, arguments based solely on examples fall short of proof. Their instructional sequence was revised over a 4-year period and implemented in an undergraduate mathematics course for elementary teachers. Practically, the instructional sequence included a four-part design across three different mathematics tasks and a reflection after the first three. The fourth part involved the participants in revisiting the initial task to develop a viable argument. Similar to Lannin (2005), G. Stylianides and A. Stylianides found that with scaffolding, the class community was able to construct a proof.

A main theoretical component of the G. Stylianides and A. Stylianides (2009) design was based on cognitive conflict to have preservice teachers experience the limitation of empirical arguments toward a more secure form of validation. Their assumption was that while solving the first task, the preservice elementary teachers would become satisfied with a generalization based on a few examples. The next two tasks support students with realizing the limitations of empirical arguments. The learning outcome is that empirical arguments are insufficient, which is the cognitive conflict, since the assumption is that learners start with assuming empirical arguments are proof.

Similar to G. Stylianides and A. Stylianides (2009), we designed an instructional sequence, but focused on transforming perceptions of proof away from an authoritative toward a communal perspective. Theoretically, our design draws on the situated learning perspective, where knowledge is shared and negotiated communally (Lave & Wenger, 1991). The design assumes that mathematics and mathematics education majors would produce a range of empirical to deductive nonproof and proof arguments depending on their prior experiences. We surmised that students would collectively accept general arguments that are understandable and convincing, and as a class community would dispute those that were based on examples and nonproof deductive arguments along with the instructors' guidance. The participation in discussions would act on and alter participants' current thinking about what is an acceptable proof. Given this theoretical approach, our practical design included four learning activities (the four diamonds), as shown in Figure 3.

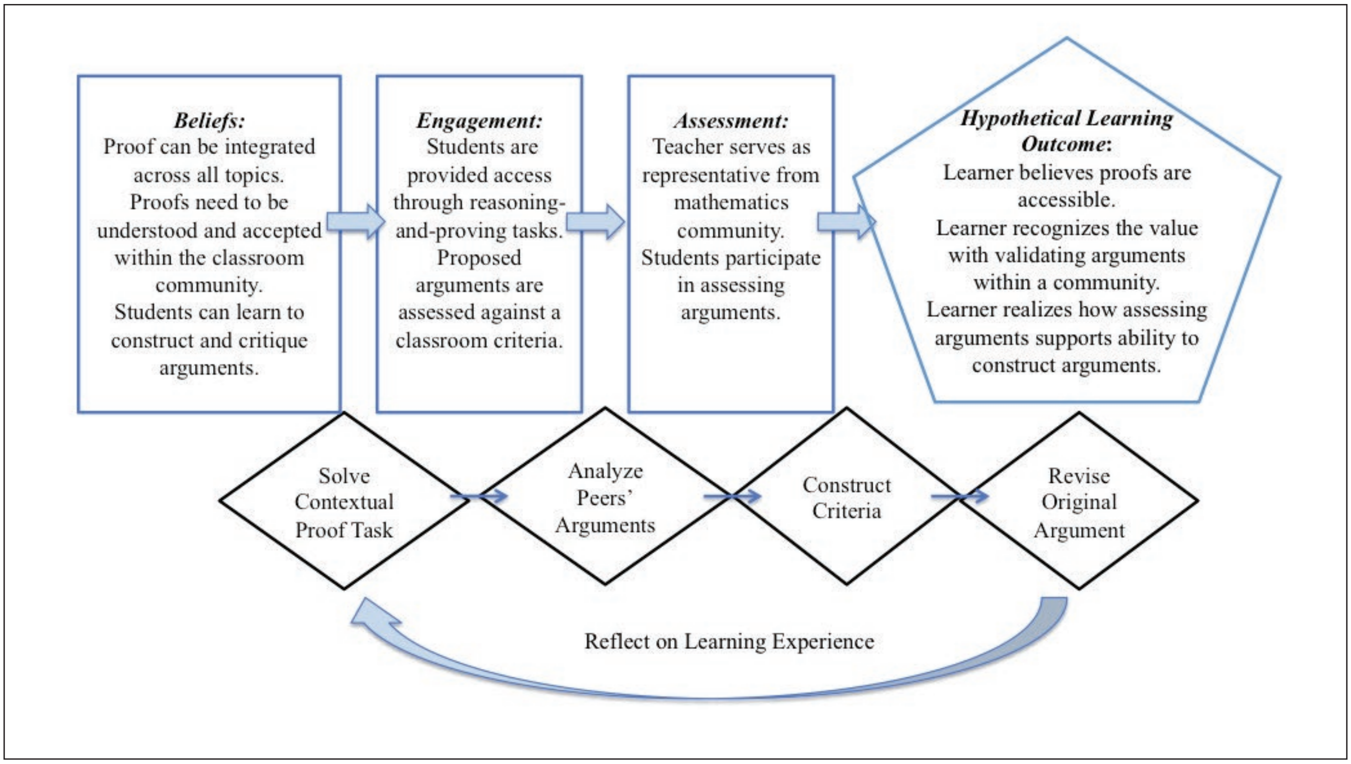

Figure 3. **Conceptual plan to transform teachers' perceptions of proof.**

Each of our four activities was designed to transform perspectives away from authoritative toward a communal perspective of learning. First, we asked participants to prove a nongeometric contextual situation. We wanted teachers to learn that for students to learn to construct proofs then they need to be *doing the proving* (Karunakaran et al., 2014; Lannin, 2005; Smith et al., 2014; G. Stylianides and A. Stylianides, 2009). The first diamond in Figure 3 is positioned between both the practices and the beliefs rectangles, as it aligns with and addresses both beliefs and engagement. Second, for participants to learn what arguments are considered valid, then they need to *do the assessing* (e.g., Healy & Hoyles, 2000; Karunakaran et al., 2014; Smith et al., 2014). Third, as the participants evaluated each presented argument, they included their rationale for why each argument was or was not valid. The rationales from analyzing each specific argument are generalized within small groups to form criteria for proof. The small group criteria are synthesized into one list of communal criteria. The idea here is that the participants will begin to reflect on their originally produced argument in light of the criteria. Finally, they use the criteria to explicitly assess their original argument and revise it to construct a proof. It is important to note that these activities relate to those designed in the CORP[2] project (Smith et al., 2014). Based on this conceptual description, we share our specific implementation in the next section.

Methods

The participants (N = 58) in this study spanned four separate courses across four universities and included both undergraduate and graduate students. The authors designed the instructional sequence and were the instructors of the four courses. Two of the courses (taught by Authors Boyle and Ko) were secondary mathematics methods classes, and the other two (taught by Authors Bleiler and Yee) were mathematics content courses. One of the mathematics content courses was specifically for students planning on becoming teachers (taught by Author Yee), and the other mathematics content course was an introduction to proof course for mathematics majors (taught by Author Bleiler). Therefore, the participants in this study were either pursuing undergraduate degrees in mathematics (N = 24) or working toward becoming secondary mathematics teachers at the time of this study

2 The first author was supported by the CORP grant and participated in the development of the CORP materials, so his contributions to the design of this instructional sequence were strongly influenced by his work on the project.

(*N* = 34). However, since the students across the four courses were not all in a teacher preparation program at the time of this study, we will refer to them more generally as *participants*.

The four-part instructional sequence designed for this study was implemented across three separate time segments: *Before Class*, *During Class*, and *After Class*, as depicted in Figure 4. It was implemented early in the semester across the four classes. Prior mathematical experiences varied among the participants, but they had all completed at least the first two calculus courses and had completed an introduction to proof course except those in an introductory proof course (Author Bleiler). Most of the participants had not experienced an inquiry-style mathematics class (i.e., Smith, 2006), and some participants in two of the courses (Author Yee [content] and Author Ko [methods]) were initially opposed to working in a format of communal engagement.

Before Class

The *Before Class* component was a take-home assignment distributed 3 to 7 days before the *During Class* activity. Participants were instructed to solve the Sticky Gum problem (Fendel, Resek, Alper, & Fraser, 1996) (see Figure 5). In addition to solving the task, they answered the following questions that aimed to promote reflection of their prior experiences about what counts as proof and how they had previously learned about proof:

1. What do you believe are some characteristics of arguments that count as proof?
2. What do you believe are some characteristics of arguments that do NOT count as proof?
3. Based on your past learning experience with mathematical proof (either high school or college), how did you learn about what makes a good mathematical proof? Be specific in your response.

Participants were instructed to submit their responses electronically at least 2 days prior to the *During Class* activity. Each instructor then purposefully selected five solutions to the Sticky Gum problem from the submitted work. These five solutions would be presented and analyzed in the *During Class* activity. More specifically, we wanted the participants to evaluate the five arguments and then, based on those evaluations, develop a communally accepted set of characteristics to serve as the class criteria of proof.

Each instructor selected the five arguments that would highlight the diversity of the participants' responses in their class (see appendixes A–D for all five selected responses from each class). Moreover, instructors selected arguments that they believed had the potential to provoke debate during the class discussion, such as about accepting empirical arguments as proof and the tendency to evaluate arguments based on form. For example, Argument 3 from Author Yee (see **Appendix C**) is an example of an argument that "looks" like a proof in form, but it lacks mathematical sense and fails to address the problem statement. Argument 2 in Author Bleiler's class (see **Appendix B**) is an empirical argument that combines algebraic symbols and seems to be attempting to follow a structure of proof by mathematical induction. Additionally, each instructor included arguments that spanned a

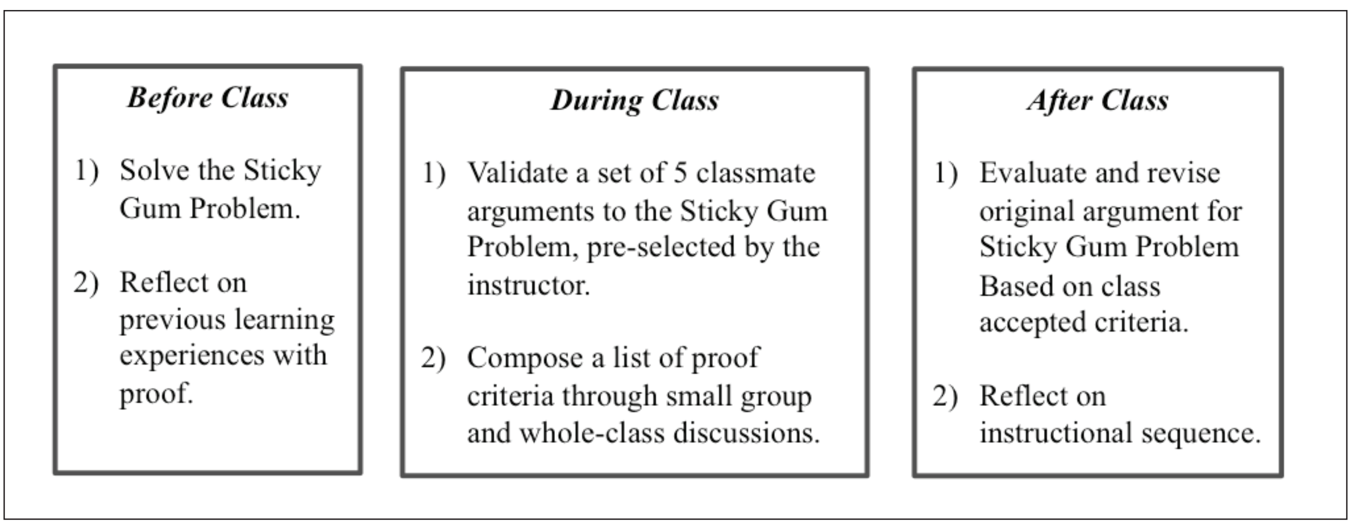

Figure 4. **Implemented instructional sequence to change perspective of proof.**

The Sticky Gum Problem

Ms. Hernandez came across a gumball machine one day when she was out with her twins. Of course, the twins each wanted a gumball. What's more, they insisted on being given gumballs of the same color. The gumballs were a penny each, and there would be no way to tell which color would come out next. Ms. Hernandez decides that she will keep putting in pennies until she gets two gumballs that are the same color. She can see that there are only red and white gumballs in the machine.

1. Why is three cents the most she will have to spend to satisfy her twins?
2. The next day, Ms. Hernandez passes a gumball machine with red, white, and blue gumballs. How could Ms. Hernandez satisfy her twins with their need for the same color this time? That is, what is the most Ms. Hernandez might have to spend that day?
3. Here comes Mr. Hodges with his triplets past the gumball machine in question 2. Of course, all three of his children want to have the same color gumball. What is the most he might have to spend?
4. Generalize this problem as much as you can. Vary the number of colors. What about different size families? Prove your generalization to show that it always works for any number of children and number of gumball colors.

Figure 5. **The Sticky Gum problem.**

range of sophistication from nonproof to proof arguments. In the case that few of the submitted participant arguments approximated a mathematical proof, the instructors included a researcher-prepared generic example proof as one of the five. This occurred in the classrooms of Author Boyle and Author Ko (see **Appendix A** and **Appendix D**).

During Class

At the beginning of the *During Class* activity each instructor asked the entire class, "Based on your past learning experience with mathematical proof (either high school or college), how did you learn about what makes a good mathematical proof?" Participants in each class were then asked to share some of their responses from this question. After this discussion, they worked individually to decide whether each of the five arguments (Appendixes A–D) was or was not a proof and to provide a rationale for their decision. Then participants were placed in groups of three or four to discuss their assessments and rationales. They were instructed to go beyond labeling "proof" or "not proof" and develop a rationale for each decision. Finally, they drew on the insights from their discussions to create three to seven characteristics that they believed were important for constructing a proof. During these small-group discussions, the instructor asked questions within the small groups about how they assessed each argument and/or listened while the individuals in the small groups shared and discussed their rationales. These interactions among the groups allowed for the participants to verbally articulate their positions and supported the instructor with learning how the participants within each group and across the groups were thinking about the validity of each argument.

A table is included in each appendix (A–D) to show how all of the participants in each course individually evaluated the five arguments before their small-group discussions. A "1" in a cell indicates that the participant identified the argument as proof, and a "0" indicates that the participant labeled the argument as nonproof. The top row of each table in all four appendixes includes the participant that created the argument. For example, in Appendix A, three of the participants initially labeled their own argument as proof.

To launch the whole-class discussion, each group posted its list of characteristics of proof, and the instructor asked the participants to compare and contrast their list with the others. Common themes were shared across the groups while the instructor facilitated the conversation to create a "common class list of characteristics for good proof writing." We then used the "common list" to evaluate two arguments as a whole class where most students thought the argument counted as proof and some assessed it as a nonproof argument. Using the criteria to evaluate an argument served two purposes: (1) to determine if the characteristic is sufficient to determine the truth of an

argument and (2) to support the entire community with understanding about what the listed characteristics mean with respect to assessing an argument. Therefore, the goal was not to reach 100% agreement, but we believe that having participants engage in the assessment process develops their sense of having a voice within the community and develops their ownership of the criteria.

As instructors, we agreed to limit the number of criteria to five to fit the time limitations of the classrooms and to establish an initial set of agreed upon characteristics. While the criteria that emerged in each class were different (see Figure 6), we aimed to engage the participants in active reflection on what they believe should count as proof based on their classroom community, instead of reaching consensus on the criteria across the four classes (cf. Stylianides, 2007a). The characteristics are arranged vertically to align with each class and horizontally to highlight commonalities across the classes (see Figure 6).

We were careful not to create an environment where any characteristic was acceptable. Our aim was to balance generating communally accepted characteristics while also attending to what is accepted within the larger mathematics community (A. Stylianides, 2007a, 2007b). As instructors, we drew on our expertise along with the participants' experiences to negotiate individual characteristics to build a shared understanding (i.e., A.Stylianides, 2007a, 2007b). For instance, when Author Boyle's students offered a characteristic, he wrote it and then asked other participants to share their thinking. Since he questioned their thinking during small-group discussions, he could strategically select particular perspectives to be shared across the class in addition to allowing volunteers to share. In some cases, the original wording was modified before adding a new characteristic to the list. For example, one participant suggested the characteristic of a "clearly defined domain," as shown in column 1 of Figure 6. Later in the discussion, another participant added the need to "state definitions."

Author Boyle's Methods Class	Author Bleiler's Mathematics Class	Author Yee's Mathematics Class	Author Ko's Methods Class
Logical Progression. No mathematical errors.	Chain of evidence (clear, step-by-step logical flow).	Organization – Structure of argument.	Proof needs to follow a logical order.
True for all cases.	Generalization (Does proof hold true for all cases and answer "why?").	Convincing-Exhaustive, not ambiguous, audience appropriate.	Proof should be always true for any case and not be verified by specific examples.
Clearly define domain, definitions, and assumptions.	Clearly identifies parameters/ constraints/ variables.	Definitions.	
	Simplified/concise.	Clarity-Concise, "math language" and "grammar."	Proof needs to include clear explanations that are concise.
Clearly stated conjecture / hypothesis statement.			Proof includes clear statement of what you are trying to prove.
	Valid/true/correct.	Correctness – Supporting evidence, overall structure. (proof built from definitions, theorems, etc.)	
State a conclusion that follows from the argument.			
			Argument clear for audiences to follow based based on their community

Figure 6. **Communally developed criteria of proof across the four classrooms.**

The instructor returned to the former participant's characteristic and added "define definitions" to merge the two ideas into one category. That is, some of the characteristics were pieced together from multiple participants, and the instructor organized the criteria that was discussed and accepted within the community.

After Class

After developing the classroom-based criteria of proof, all participants were asked to complete the *After Class* activity using their class-constructed criteria of proof (see Figure 6). The *After Class* activity was another out-of-class activity where the participants would evaluate their original arguments to the Sticky Gum problem based on their class criteria of proof, revise their arguments, and reflect on their experiences throughout this instructional sequence.

The focus of this paper is on the participant responses to questions 7 and 8 in the *After Class* activity (see Figure 7). A total of 52 of the 58 participants responded to the two reflection questions, and the written responses were analyzed following the principles and techniques of grounded theory (Glaser & Strauss, 1967; Strauss & Corbin; 1990) to develop themes. We chose to ask open-ended reflections questions (questions 7 and 8 in Figure 7), since we did not want to assume past experiences, and we wanted to learn if new themes (related to our hypothetical learning outcomes in Figure 2) might arise that we did not

Postactivity Assignment: Applying class criteria of proof to your argument

Name: ___________________________________

Please complete this assignment by answering the four questions below with a full description. All responses must be submitted digitally. You may scan and email your response.

Question 5. Copy our class-developed criteria for a compelling/convincing argument in the first column, and then complete the remainder of the table based on your original written argument for the Sticky Gum problem.

Class-developed criteria for proof writing.	On a scale of 0–5 (0 = not at all, 5 = completely), how well does your argument meet each criterion?	Explain your rationale for your numeric rating by making explicit reference to your original argument.

Question 6. Revise your original argument based on your response to Question 4 so that your new argument would count as a proof based on our class-developed rubric.

Question 7. How has this engagement with the Sticky Gum problem and the classroom activity impacted your understanding of proof?

Question 8. Describe the similarities and differences between your past learning experiences with mathematical proof and the series of activities we have done with the Sticky Gum problem.

Figure 7. **After class activity.**

anticipate. The first author listed the themes and revised them while coding the pairs of responses for each of the 52 participants. After narrowing the number of themes to nine across the two research questions, the second author and first author discussed four participant responses after coding them individually. A "1" was placed in a cell that matched a theme with a response and "0" was listed in a cell where there was no match. We discussed our codes until we reached agreement; then Author Boyle and Author Bleiler independently coded 21 participant responses. During our discussion of these 21 pairs of responses, we decided to eliminate a code and change the wording of two others. This resulted in a difference of two or fewer participant-comments for each of the remaining eight themes. The first author revisited each discrepancy and made a final decision, then coded the remaining 27 pairs of responses. In the next section, we share the most common themes along with samples of the participants' written responses to answer the two research questions.

Results

The results provide insight into how the participants viewed their experience throughout our instructional sequence compared to their previous experiences with proof, and the ways participants perceived that the activities in the instructional sequence changed their perceptions of proof. Representative responses from the reflection questions (questions 7 and 8 in Figure 6) across the four courses are included to highlight how the participants explained their thinking. It is important to state that participants' responses were coded under a particular theme only when they explicitly wrote about that theme. Therefore, it is possible that more of the participants could find some of the same themes to have an impact on their learning, but they were not specifically asked to comment on them, given the general nature of the questions. Additionally, a pseudonym is listed after each reflection and the instructor of the participant is in parentheses. Portions of some responses are italicized. This is done to highlight the connection with the theme. We include the full response or a majority of each response to share how most of the responses span several themes. We conclude this section by sharing four cases to compare a participant's written argument against his or her reflection responses.

RQ1: In what ways do participants perceive this instructional sequence as different in relation to their prior experiences with mathematical proof?

Table 1 includes the five most common themes for the 52 participants related to our first research question. We explain each of the five themes and provide sample responses. The first three themes align with the tasks in our instructional sequence. The fourth theme relates to how participants engaged in the tasks. The fifth theme aligns with how some participants have previously engaged in proof. The total number of responses exceeds 52 since many of the participants referenced more than one perceived difference.

Discussing what is needed for an argument to count as proof. Most participants (73%) had never worked from implicit or explicit criteria of proof, and they explained it was helpful for them to generate characteristics of what counted as proof. They identified the development of common criteria as a supportive activity that helped them to construct a holistic view of proof. Beyond just identifying the usefulness of criteria in general, the three participants below spoke about specific characteristics such as clarity and generalizing beyond a set of examples. Gina also adds that she now has a greater understanding of what the characteristics mean. The following responses are representative of how most participants reflected on the development of criteria as being different and important to them:

> **I have never seen a rubric for a proof. Creating and thinking about a rubric helped me make sure that my proof was of quality for someone to read and understand. This rubric helped guide my thinking in creating and evaluating proofs, but**

Table 1

Participant Identified Differences Between the Instructional Sequence and Previous Experiences

1) Discuss what is needed for an argument to count as proof.	2) Constructing and/or revising an argument.	3) Analyzing and discussing the validity of arguments.	4) Working collaboratively in class.	5) Previously asked to memorize and reproduce the textbook or instructor proof.
38/52 (73%)	18/52 (35%)	15/52 (28%)	18/52 (35%)	20/52 (38%)

it did not set complete restrictions on the exact format of the proof. Brandon (Author Ko)

Now I have a better understanding of what the proof, regardless of community, needs to entail. I need to be sure that everyone will be able to follow the explanations clearly, thus they need to be clear and concise. They also need to be logical. Even though I may provide all the explanations and evidence in the proof, they need to be in a logical order so that the audience can follow systematically. Additionally, the proof needs to be generalized. My past experience and this experience are similar in that I realized they must have a conjecture, explanations, and evidence that prove the conjecture. However, I now have a better understanding of what exactly this means. Gina (Author Ko)

I feel that this activity gave me the opportunity to learn the basics of proof. I feel that in the past it wasn't explained to me that proof writing is not specific. In a sense it gave me an abstract way to write a proof in a way that made it concrete. I really liked the idea of comparing others work so that we could determine the criteria for a proof. I liked that we came up with a list of the criteria for future references. Carmelo (Author Boyle)

Constructing and/or revising an argument. About one third of the participants (35%) indicated that constructing an argument on their own prior to being provided sample arguments and/or being asked to revise their original argument based on the class criteria was different. Many participants mentioned that proof now seems much more accessible to them as learners than it has been in the past. Most important, they appreciated the opportunity to learn from their mistakes as they revised their original argument against the classroom criteria. The following are representative of the participants' responses:

First, I understand what proof means more clearly now. *When rewriting a proof, I tried to incorporate or use what I learned during the class discussion. I tried to generalize it more because I learned that generalizing was my weakest part in my previous proof.* This engagement is helping me to understand the problem. Most importantly, I understand that trying and struggling are very important processes in order to fully understand the task. *Without trying to solve the task alone, before interruption of teacher, is important for me to find out what my weaknesses and strengths are.* Mona (Author Boyle)

In the past, we only worked with examples and then the teacher telling us to do problems when we have been lectured on the right way to do them. *In this case, we found our errors basically by ourselves with the teacher prompting and developed our own understanding of what proofs are and now have a better understanding on how to write proofs.* We know why we do the things that we do so that it will be a clear concise proof. We saw that if you don't write proofs well, they either don't make sense or are very hard to read. Carla (Author Ko)

I thought it was very helpful to go back and revise my proof. Something different that I experienced this time was being able to clearly map out the structure of proof before going back to revise it. This improves the proof tremendously. Stacey (Author Yee)

Analyzing and discussing the validity of arguments. Twenty-eight percent of the participants identified reviewing their classmates' arguments as new and very supportive to their learning. Through evaluating and discussing the validity of their peers' work, participants realized that they tended to think differently from one another. The variety of the instructor-selected arguments helped them to identify characteristics that were similar to and different from their solution method and supported them with improving their ability to distinguish between those that qualify and do not qualify as proof. Below is a sample of what was written related to this theme:

It really made me realize how differently this problem can be viewed based on the 5 examples given in class. The same problem can be proved in numerous ways, depending on the reader's interpretation of the questions. Also, there is no right and wrong way to do it, but rather many ways where some may be more sufficient and effective than others. Jacqui (Author Yee)

The Sticky Gum Problem and the activities that went along with it really taught me the importance of clarity and generalization in a proof. *Looking at proofs written by other students through a critical lens really has helped me understand what works and what does not work in a proof.* One of the most common mistakes in the Sticky Gum proofs was the use of examples to verify that a general rule was correct. Brandon (Author Ko)

In the past I would just jump to an applicable solution. This often led to an incomplete

> **knowledge of the proof. *Diving into writing and dissecting these proofs has given me a deeper understanding and a tool set to go back to older proofs.* Gabriel (Author Bleiler)**

Working collaboratively in class. Thirty-five percent of the participants explained that being afforded the opportunity to discuss their thinking with their peers was unusual from prior proof experiences. Furthermore, it helped them discuss the shortcomings of their solution path and negotiate their thinking about what counts as proof. Specifically, they found it empowering to be part of the process with both assessing arguments and generating characteristics for the class criteria. Sharing their thinking about proof with their peers was new and powerful, and this is what a sample of them wrote:

> **[The activity] has given me time to struggle with what makes a good proof or not by myself and with others. Listening to others has really helped me to see where my thinking and generalizations tend to come from. Danielle (Author Bleiler)**

> **Some differences were the way we went about discussing what makes a proof. Normally, as a student, I am lectured as to what makes a good proof from a teacher/professor; in our class we thought individually, then as a small group, and concluded as a whole class. I like the way we discussed proofs, because then I thought that I was actually making up the rules, rather than being told what to write or expect. Naomi (Author Ko)**

> **I have never had the opportunity to compare other students' proofs of a problem with my own as a class. I have never discussed what others believe constitutes a valid proof as a class and heard the explanations for why these specific criteria are what others consider important for proof writing. This experience let me see some of my proof writing weakness by allowing others to express their doubts about certain aspects of the proofs our classmates wrote. Antonella (Author Yee)**

Previously asked to memorize and reproduce the textbook or instructor proof. More than one third (38%) of the participants shared that their previous experiences focused on reproducing textbook or instructor produced proofs. In other words, proof seemed difficult or confusing since they were never provided an opportunity to engage in the proving process. With the Sticky Gum sequence, the participants explained that developing an argument themselves and learning the limitations of their arguments enhanced their understandings of proof in general. Here is a sample of participants' thoughts:

> **The activities we did helped me to feel more comfortable with writing proofs. . . . *When proofs were first introduced to me, it was in geometry, and my teacher did most of the writing and we watched. Proofs didn't come up again until last fall in Calculus III, and again, my teacher did most of the writing.* . . . I simply just familiarized myself with different common manipulations for writing a convincing argument. Having to come up with/critique proofs gave me a way of trouble shooting what they are and how to write them. Through learning from mistakes and trial and error I was able to get a better understanding. Calvin (Author Bleiler)**

> **Before class, I had viewed proof writing as a purely academic endeavor that had little or no real world application. Not to sound bitter, but *in previous classes, the process of proof seemed like a large amount of academic back-patting and teachers saying, "Hey, look at me, I can do this really complicated piece of math on the board."* Peter (Author Bleiler)**

> ***In my past, learning proofs has been very difficult. We were basically shown examples with no guidance as to what should be included in a proof.* These activities have helped to demonstrate the key concepts that should be included in proofs. As I look back, all of the key things we described to make up a good proof were included in the proofs my teachers did but weren't explained. *Teachers simply showed us different proofs and then expected us to write some on our own.* I think I would have had a much easier time with proofs had I been taught what should be included to make a proof valid. Jessica (Author Ko)**

RQ2: How (if at all) do participants perceive that the activities in this instructional sequence changed their perceptions of proof?

Forty-seven of the 52 participants across all four classes identified at least one activity from the instructional sequence as supportive. In this section, we share three themes that address new insights and transformed perspectives of proof (see Table 2). The first theme relates to the participants gaining a clearer understanding about what is needed for an argument to count as proof or how to construct a proof themselves. The second theme

Table 2

Themes with How the Participants Perceived the Instructional Sequence Changed Their Perceptions of Proof

1) More clear about what counts as proof.	2) Provided with an opportunity to see proof as socially negotiated.	3) Expanded thinking beyond a particular format.
44/52 (85%)	28/52 (53%)	27/52 (52%)

addresses how proofs should be assessed so that the entire community is a participant in the sense making and assessment process. The third theme highlights a previous misconception, namely, an overreliance on format. Many of the participant responses were coded into more than one theme. After sharing a few quotes for each of these three themes, we select one participant from each that had his or her argument assessed to learn how they might be thinking about their initial argument in relation to their reflection responses.

More clear about what counts as proof. Most participants (85%) explained that they never really understood where proofs came from or what was required for an argument to count as proof. They shared that the instructional sequence helped them to better understand what is needed for an argument to count as proof. Some explained that some of the ideas they thought were important were made more explicit and secure for them. Others wrote about how the instructional experience supported them with beginning to learn what counts. While many participants still believe they need more experience, they feel they have a better understanding with how to start and what is expected. Sample responses are given below:

> ***Before this assignment, I had no clue what made a good proof nor how to write one by myself.*** **Previous math classes only stated the proof and said how it was true. This activity stated out point by point how to make a proof valid and understood. Natasha (Author Bleiler)**

> ***I feel like as a class now we have a better grasp on what we all believe is a good proof.*** **The activity has broken down proof writing so that it is not as formal. We state what we want to show and then show it. We have guides also to help examine how we can strengthen our proofs off of other criteria and not just what we were used too. Michael (Author Ko)**

> ***This engagement along with the discussion of proofs has helped me understand proofs a little more.*** **I know how different people will have a different take on what makes a good proof and how to prove it.** ***I also know that most of us only have a basic understanding of proof, and will continue to learn what makes a good proof as well as the process.*** **Naomi (Author Yee)**

Provided with an opportunity to see proof as socially negotiated. Just over half of the participants (53%) identified the social interactions with their peers as supportive to reshaping their thinking. Analyzing their peers' solutions supported them in realizing that not all viable arguments to this problem need to be done the same way. The social interactions among the participants helped them to begin to gain ownership of proof as something that must adhere to agreed-upon criteria as opposed to just satisfying an instructor. Overall, they seem to now view proofs as arguments that should make sense to them and their community while also attending to the agreed-upon criteria. Below are representative responses:

> **This engagement with the Sticky Gum Problem and classroom activity has impacted my understanding of proof by** ***giving me a more solid foundation of what not only a teacher sees as a valid proof, but what my peers see as a valid proof. I was able to compare my work that I created on my own with work from my peers for the very same problem. It helped me realize that initially not everyone goes about proving a problem the same way.*** **I think after having the classroom discussion I understand different aspects of a proof, such as organization and clarity a bit more than I did before. Antonella (Author Yee)**

> **This engagement has given me a sort of "foundation." Knowing that these are the criteria that every proof should meet,** ***I can now check my proofs to make sure they are meeting all the qualifications for a good proof. With the classroom activity I can see how other people are thinking when doing proofs. It's not just the teacher's opinion and mine but now it is a community of provers.*** **Hilary (Author Bleiler)**

> **In Calculus III we were trying to manipulate an equation to somehow find "delta" to make**

it look like the equation we started with. *Just as that probably doesn't make sense to you, it didn't to me either!* It was more of memorizing different methods and comparing problems that look just alike. Never was "criteria for proofs" really stressed! *We broke down the Sticky Gum Problem, and we were able to take each piece and work with it to see the bigger picture. As we kept breaking down pieces the bigger picture gets larger and finally you have something that makes sense! Analyzing together in groups creates a much bigger brain flow than any you could have created yourself, and it was very helpful.* Calvin (Author Bleiler)

Expanded thinking beyond a particular format. About half of the participants (52%) shared that they previously believed that proof needed to follow a particular format, such as using a certain proof method, or that it needed to include certain symbols or notation. Through participating in this sequence, they learned that the content of what is written matters more than the format. In other words, they realized that learning to prove goes beyond simply employing specific proof techniques and including mathematical symbols. A few specific thoughts are shared below:

I was under the impression, previously that proofs were subject to "standard" types of proofs. Implying that a type of proof was to be chosen and a strict format to be followed. In this activity there is a format but it is much more flexible than I initially thought, requiring key elements instead of outlines. Henry (Author Ko)

I feel that this activity gave me the opportunity to learn the basics of proof. *I feel that in the past it wasn't explained to me that proof writing is not specific.* In a sense it gave me an abstract way to write a proof in a way that made it concrete. I really liked the idea of comparing others works so that we could determine the criteria for a proof. Carmelo (Author Boyle)

The requirements to a proof are much simpler than I have always thought. I also associated the fancy language with proofs when it's more about explaining a concept/theory. *I've always been taught that the language and formatting of the proof was more important than the basics [criteria] we discussed.* Rachel (Author Bleiler)

Comparing original arguments with reflection responses: Four case examples. In this section, we provide four case examples, one from each of the four courses. With these cases we intend to illustrate how participants' reflective comments relate to their initial arguments. By looking more closely at a participant's initial argument, we can better understand the context of their reflective comments. We selected these cases examples from participants who initially identified their invalid argument as proof (at the beginning of the During Class activity). In addition, we selected the case examples from participants whose responses provided us with insights into how they were making connections to their original arguments.

In Appendix A, Ji-min (Author Boyle, argument 1) individually labeled her argument as proof, although none of her peers believed the argument was valid. After the class discussion she shared thoughts that caused her to change her thinking about the validity. She wrote, "A proof has to hold true for every possible situation. . . . Re-writing my proof I was reminded of how much thought and effort goes into a proof." Prior to this instructional sequence, she accepted generalizations based on a set of examples as proof. Her original argument was empirical, and after the instructional sequence, she seemed to be clear that her original argument was not true for all cases, which was a listed characteristic in this class.

In Appendix B, the table shows that Keyshawn (Author Bleiler, argument 2) labeled his argument as proof. About 65% of Keyshawn's peers in his class also labeled his solution valid, even though it included misunderstandings about both proof by mathematical induction and the use of variables. For example, he wrote that the $x + 1$ case holds true, but he only checked a specific case (3). After the instructional sequence, he wrote, "Proofs can mean several things; there isn't one standard for proof writing." This comment suggests that he changed his perspective away from believing a proof needs to follow a specific format. In the past he only followed what his instructors produced without trying to make sense of proofs. He added, "Past proofs have been much more teacher-led rather than this one which was almost all class-led in group discussions." Thus, Keyshawn now understands that he needs to write arguments that are acceptable to the class community. While it is not completely evident from what he wrote, it seems as though he realized from the class discussion that his original argument was not a proof.

In Appendix C Andy (Author Yee, argument 3) and about 56% of his classmates labeled his argument as proof. While the argument does not make sense mathematically, he included mathematical symbols and a format that includes multiple cases. After the instructional sequence, Andy wrote, "This activity has given me more of an

understanding on what is an actual proof. As a proof writer I have to be careful to write proofs in a way that is understandable not only to me, but also to whoever is going to read it." He also added, "Some differences about the activity would be to have more of an open mind about how to do a proof." Andy does seem to realize that his original argument required revisions, as he wrote that it needs to be understandable to others. Additionally, the comment about being more open-minded seems to suggest that he, like Keyshawn, realizes that proofs are not confined to a specific format. Therefore, Andy may have realized the limitations of trying to make an argument proof like (i.e., using symbols and following a particular structure), and through engagement in the instructional sequence, he has a new view about what is acceptable and should be understandable to others.

Finally, in Appendix D, Brandon (Author Ko, argument 1) and about 30% of his peers labeled his argument as proof. Brandon made a generalization and attempted to explain why his formula works in every possible case, but as written it is invalid. After the instructional sequence, Brandon wrote, "The Sticky Gum Problem and the activities that went along with it really taught me the importance of clarity and generalization in a proof." Here, like Ji-min, he identifies generalization as a limitation, and this may mean that he realized that his argument fell, short since it did not include a generalized argument. He also added, "One of the most common mistakes in the Sticky Gum proofs was the use of examples to verify that a general rule was correct. Although it is good to check your generalization with examples, a proof should be able to stand alone without them." These comments suggest that Brandon realized that his original argument was not a proof.

These four cases, one from each course, provide additional support that participants shared not only that their perspective of proof changed, but that their new perspective was explicitly aligned with how they originally constructed their argument. For example, Brandon and Ji-min both constructed empirical arguments and shared after the instructional sequence that they now understand proofs need to be true for all cases. In addition, Andy and Keyshawn tried to fit their arguments into some type of deductive form without attending to mathematical meaning. After engaging in the instructional sequence, they both wrote about how a strict format is less important than they had originally thought. All four participants from each of the four courses shared specific comments related to the arguments they originally produced, which suggests that they each have changed their perspectives about how to construct a proof that was different prior to the instructional sequence.

Discussion

Reasoning and proving should be central activities throughout the K–12 curricula, since it has the potential to develop a deeper understanding of mathematics. However, in order for this to become reality, teachers need opportunities to learn reasoning and proving to support their change away from the authoritative perspective, as it contributes to limiting perceptions and misunderstandings about the nature of proof. We shared an instructional sequence that we implemented with mathematics and mathematics education students. The mathematics task promoted access to proof since it explicitly calls students to examine cases leading to the development of a wide variety of arguments based on their prior experience. As expected, none of the participants explained that they did not know how to start the task, but many realized what they originally produced required revision to count as proof. Through active engagement with constructing an argument, analyzing a set of five arguments, and developing a communal proof criteria, many participants were able to identify their own prior misunderstandings and share how the learning opportunity supported them in gaining a better understanding about what is needed for an argument to count as proof. The 24 responses in the results section across the four different classes provide examples of the impact of the instructional sequence.

The reflections suggest that the participants experienced movement away from the authoritative perspective of proof and that they are embracing a communal perspective as learners, as shown in Table 3. It is encouraging to read how some participants came to dismiss previous beliefs, such as proofs needing to follow a particular format. Some also shared that assessing their peers' arguments helped them to rethink their approach to the problem and specifically realize why empirical arguments are not proof. Previous research reported that teachers hold narrow and limiting views of proof (e.g., Karunakaran et al., 2014; Knuth, 2002b; Steele & Rogers, 2012), and these perceptions have an impact on their instructional practices (Bieda, 2010; Furinghetti & Morselli, 2011). We agree with Harel and colleagues (Harel & Sowder, 1998; Harel & Rabin, 2010) that when secondary or undergraduate students experience proof in authoritative settings, they develop limiting views of proof and these views stay with them even after they become teachers. We believe the research-based design of our instructional sequence jumpstarted a transformation from the authoritative toward a communal perspective of proof. The participant reflections highlight some of the key distinctions between authoritative and communal engagement with proof, which we sought to capture in Table 3.

Table 3
Comparing the Authoritative and Communal Engagement Method

Learning activity	Authoritative engagement	Communal engagement	Shifts in perspectives
Producing a proof	Instructor or textbook proof serves as the one correct and/or model proof. Learner expects to be provided a completed proof.	Learners produce an argument on their own that may or may not be valid.	***Beliefs:*** More confident about what is required of a proof. Not as formal as previously thought.
Analyzing proof and nonproof arguments	Learners compare the argument they produce against the textbook or instructor proof (model) to align their argument.	Learners analyze a set of peer-produced arguments to consider what counts as proof.	***Engagement:*** Proofs need to be understood by others. Constructing an argument without being shown examples was different, but it helped to identify shortcomings after analyzing other arguments.
Explicitly discuss criteria for constructing and evaluating arguments	Learners are expected to individually reproduce the correct proof and or use the model proof for similar statements. Instructor responds to questions to direct learners to model proof.	Learners negotiate criteria of proof while the instructor holds the criteria accountable to the mathematics community.	***Assessment:*** Assessing peer arguments helped to learn that others think differently and how we all struggle with proof. Developed ownership of criteria since we participated in creating it.
Revising argument	Learner practices the provided proof (and or similar types) until it is memorized.	Learners apply their understanding of the negotiated criteria to revise their original argument.	

We view authoritative engagement and communal engagement as existing along a spectrum, rather than being two isolated options. Believing that students should be afforded proving opportunities is a productive first step, but productive engagement and assessment routines are required to promote student learning. Productive engagement requires an accessible task that allows for multiple solution paths (NCTM, 2014; Smith & Stein, 2011). Time is also required for students to analyze strategically selected arguments. Students should be encouraged to share their current thinking, as opposed to the "correct solution," for the community to assess and provide feedback. We believe that these negotiated, communal, sense-making engagement experiences afford opportunities for our participants to change their thinking about how they view proof (shown in column 4 of Table 3). Most, if not all, of the participants experienced changes in their perspective of proof from an individual and or memorization activity that must follow a strict (possibly single) format to viewing proof as collaborative process where proposed arguments are critiqued and revised based upon negotiated criteria. For example, participants stated that they previously experienced proof as passive recipients in settings where their instructor provided the correct proof and that a proof needed to follow a particular structure or format. These prior experiences could be why the participants also thought that proof had to follow a specific format or structure. Our results suggest that transforming engagement with proof can lead to important changes in learners' perceptions of proof. Just as authoritative learning experiences explain why teachers develop narrow beliefs and in turn repeat a similar authoritative enactment, we hope that this learning experience can cause teachers to change their instructional practices toward a communal perspective of proof.

We are excited by the results of this study, and although the data we have analyzed here suggest participants' movement in a positive direction, we realize that more work is needed. These participants displayed similar mathematical misunderstandings found in studies with secondary students, such as accepting empirical arguments as proof and believing that the form (e.g., including specific symbols) is more important than making sure others can follow and make sense of your argument. The four cases shared in the results section highlight how the instructional sequence seems to have affected their thinking about the limitations of their original arguments. Even though the instructional sequence supported them with expanding their understanding about what counts as proof, many participants realized that more experience is needed to develop their ability to construct proofs. For instance,

Linda in Author Boyle's class wrote, "I believe; however, I need much more practice to feel confident in doing other proofs." Our aim was for this learning experience to serve as a starting point that could be used in a variety of settings to provide more students with access to opportunities to construct and critique arguments and to deepen their understanding of mathematics, and we believe this instructional sequence serves this purpose.

Recommended Next Steps for Teacher Educators

A strength of this instructional sequence is that it was implemented at four different institutions with participants possessing varying degrees of mathematical experience. Regardless of the class context across the four instructors, the participants shared how the instructional sequence affected their perspective of proof. One of the primary challenges students across varying grade levels face in developing an understanding of proof is the *authoritative proof scheme* (Harel & Sowder, 1998); instructors often perceive that their students are unable to produce a proof on their own (Harel & Sowder, 2009) causing them to decide to present their own proofs for students to reproduce (Stylianou, Blanton, & Knuth, 2009). This instructional sequence acknowledges this instructor-student tension but embraces the perspective that learning meaningfully requires active student engagement. Although providing learners with opportunities to collaborate with peers and make sense of mathematics takes time, the result shows that participants gain a deeper understanding of the nature of proof and, therefore, likely of the nature of mathematics.

While participants pointed to specific activities such as developing a criterion or working collaboratively, we believe the collection of activities sequenced in the way described is critical to gaining similar results. In other words, we find it advantageous to start with having learners solve a task before developing criteria for proof or evaluating the validity of arguments. While the Sticky Gum problem does not need to be the task used in the sequence, we recommend choosing a task that provides opportunities to look for patterns, generate examples, make a generalization, and construct an argument. These reasoning-and-proving activities (Stylianides, 2008) provide access into the problem, promote different types of solution paths, and allow for misconceptions to surface, such as proofs based on empirical examples. As Lannin (2005) found with middle school students and G. Stylianides and A. Stylianides (2009) with undergraduate elementary majors, others will most likely need support with developing a general argument that connects their formula to the general context of the problem. Having learners evaluate their peers' argument may seem contentious, but explicitly explaining that their work has the potential to be shared and that their names would be removed can help them feel comfortable. Also, this practice seemed to directly support Ji-min, Keyshawn, Andy, and Brandon (four cases previously shared) as we compared their original solution against their reflection after engaging in the instructional sequence. One possible extension to the sequence might be to spend time reviewing and discussing students' revised arguments with respect to the criteria.

In moving this work forward, we believe it is important to continue to think critically about how this new knowledge about proof can be parlayed to secondary mathematics teachers supporting their students to construct and critique arguments (CCSSM, 2010). While some participants realized this next step, it was not an explicit part of this instructional sequence. For instance, Francine in Author Ko's class wrote:

> **This particular problem has really opened my eyes to how broad and important proofs can be. Proofs are not only used to prove theorems or equations to be true. I also understand now that proofs can be used from very early on in education rather than only in geometry at the high school level.**

Therefore, some participants did realize our ultimate implicit goal, but more work is needed to support participants after changing their perceptions of proofs to also transfer this instructional perspective into secondary classrooms to develop students' capabilities in constructing and critiquing arguments. Nonetheless, independent of the generalizability of our qualitative results, the versatility of this activity sequence can allow teacher educators to genuinely connect with teachers and help them view teaching and learning proof as a communally constructed, sense-making mathematical activity.

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Appendix A: Author Boyle

Argument 1

We may come up with the following chart using the number data(#3) we get from diagrams.

C: # of color
F: # of family size
P: # of penny needed

F \ C	2	3	4
2	3	4	5
3	5	7	9
4	7	10	13

P

F (# of Family) means that we have to get same # of same color gumball. In diagrams, we may notice that we have to repeat (F−1) times of subset and add 1 time at the end to get F th gumball.

In Subset (⌒), there are all different color gumballs (C: # of colors).

Therefore, we can infer this diagram and number chart to one formula (or equation)

$$P = C \cdot (F-1) + 1$$

Argument 2

Please complete this assignment by answering the three questions below with a full description. All responses must be submitted digitally. You may scan and email your response.

Sticky Gum Problem

Ms. Hernandez came across a gumball machine one day when she was out with her twins. Of course, the twins each wanted a gumball. What's more, they insisted on being given gumballs of the same color. The gumballs were a penny each, and there would be no way to tell which color would come out next. Ms. Hernandez decides that she will keep putting in pennies until she gets two gumballs that are the same color. She can see that there are only red and white gumballs in the machine. (*The following questions are to help guide you towards answering question 1. You should complete all of the bulleted questions, but you only need to email complete responses for Questions 1-3 below. Based on your solution to question 1 below, I may ask you to turn in your bulleted responses in class.)*

- Why is three cents the most Ms. Hernandez will have to spend to satisfy her twins?
- The next day, Ms. Hernandez passes a gumball machine with red, white, and blue gumballs. How could Ms. Hernandez satisfy her twins with their need for the same color this time? That is, what is the most Ms. Hernandez might have to spend that day?
- Here comes Mr. Hodges with his triplets past the same gumball machine with red, white, and blue gumballs. Of course, all three of his children want to have the same color gumball. What is the most he might have to spend?

Question 1. Generalize this problem as much as you can. Vary the number of colors. What about different size families? Prove a generalization that always works for any number of children and any number of gumball colors. I encourage you to think about using alternative representations within your proof in order to create your most convincing/compelling argument. R-red, W-white, B-blue, Y-yellow, G-green these are the color of gumballs

k RWB/RWB/RWB/RWB/RWB

3 gum balls, 2 kids, spend $0.04

3 gum balls, 3kids, spend $0.07

3 gum balls, 4 kids, spend $0.10

RWBY/RWBY/BYWR/YBRW/RWYB

4 gum balls, 2 kids, spend $0.05

4 gum balls, 3 kids, spend $0.09

4 gum balls, 4 kids, spend $0.13

4 gum balls, 5 kids, spend $0.17

RWBYG/RGBYW/YBGRW/WBYRG/GBWRY

5 gum balls, 2 kids, Spend $0.06

5 gum balls, 3 kids, Spend $0.11

5 gum balls, 4 kids, Spend $0.16

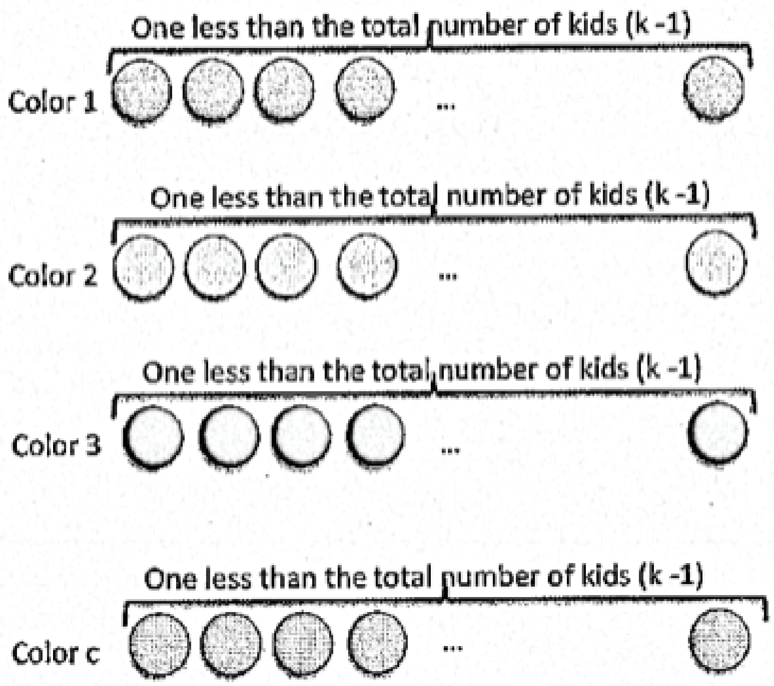
5 gum balls, 5 kids, Spend $0.21

For every child you add, the amount of money you spend increases by the number of colored gum balls present.

Argument 3

Question 1:
Conjecture: For any number of kids, k and any number of gumball colors c, where each kid must have the same gumball color the total worst case scenario cost at 1¢ per gumball is c(k -1) + 1

Assume the same color is drawn for each kid except one kid because if every kid got the same color it would not be the worst possible case. Then a new streak of color (2nd color) is drawn for every kid, but again it stops one short to match every kid. Then a third color begins to be drawn from the machine and again the number of gumballs of this third color is one less than is needed.
Here let me show you a picture of what I mean.



Where c is the total number of colors and is a natural number

So after every gumball color, c is drawn k – 1 times then every kid but one has a matching gumball and since there are no new colors in the machine, the very next gumball (+1) will be a match so that every kid has the same color gumball at the worst possible case.
So the conjecture holds true c(k -1) + 1

Argument 4

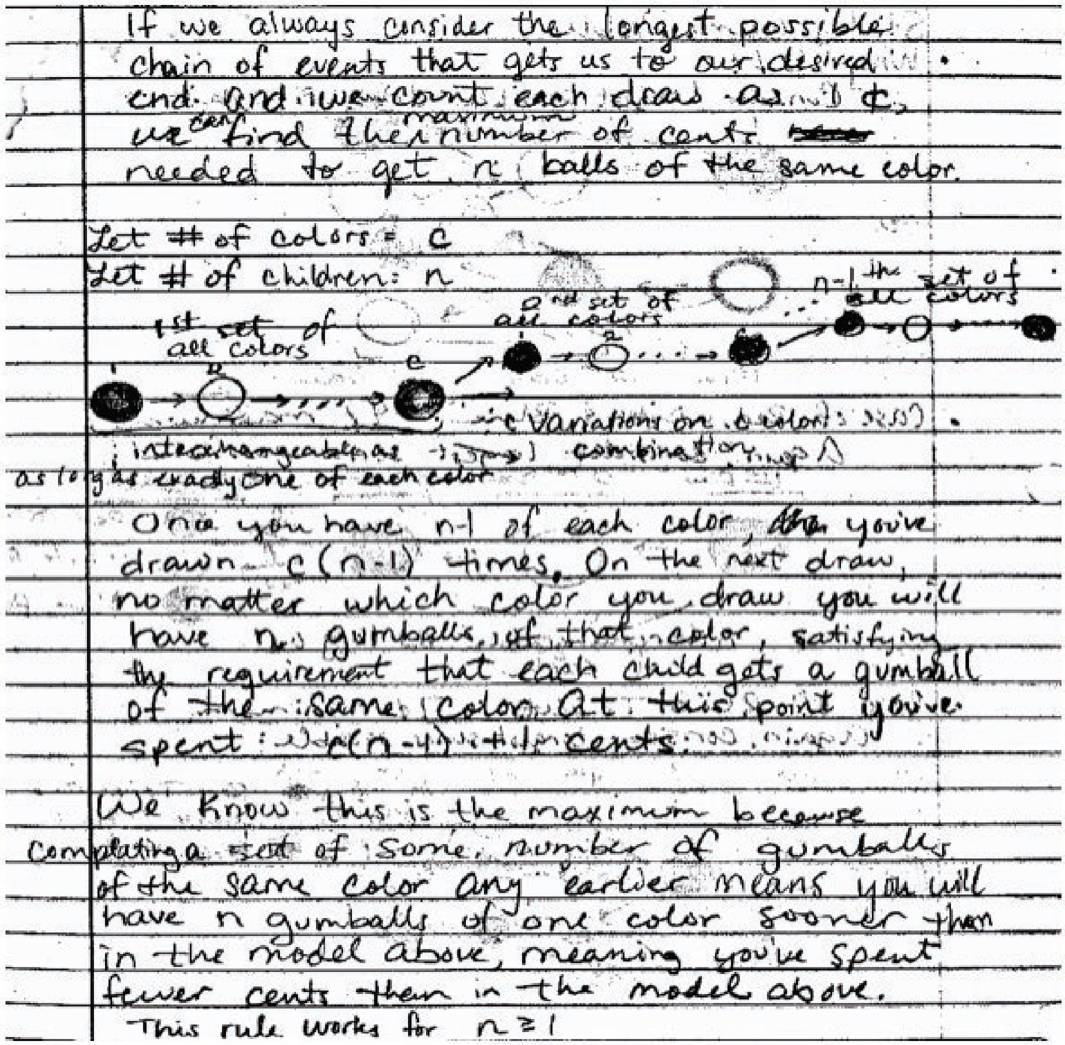
WHEN I INCREASED # COLORS AND KEEP # KIDS CONSTANT, BUT INCREASE # KIDS TO 3 FROM LAST EXAMPLE, MAXIMUM SPENT WENT DOWN. THE FOLLOWING CHART REPRESENTS THIS:

COLORS	KIDS	MAXIMUM SPENT ($)
2	3	5
3		4

A GENERAL EQUATION THAT I FOUND THAT WORKED FOR # COLORS & # KIDS TO = MAXIMUM SPENT IS THE FOLLOWING:

[(# of COLORS) * (# of KIDS)] − [(# of COLORS) − 1]

Argument 5


If we always consider the longest possible chain of events that gets us to our desired end and we count each draw as 1 ¢, we can find the maximum number of cents needed to get n balls of the same color.

Let # of colors = c
Let # of children: n

1st set of all colors ... 2nd set of all colors ... n-1 the set of all colors

c variations on c colors
interchangeable as long as exactly one of each color ... combination

Once you have n-1 of each color, you've drawn c(n-1) times. On the next draw, no matter which color you draw you will have n gumballs of that color, satisfying the requirement that each child gets a gumball of the same color. At this point you've spent c(n-1)+1 cents.

We know this is the maximum because completing a set of some number of gumballs of the same color any earlier means you will have n gumballs of one color sooner than in the model above, meaning you've spent fewer cents than in the model above.

This rule works for n ≥ 1

	A1 - Ji-min	A2 – Mona	A3 - instructor	A4 - Linda	A5 – Natalia
Carmelo	0	0	1	0	1
Darren	0	0	1	0	1
Linda	0	0	1	1	0
Natalia	0	0	1	0	1
Mona	0	0	1	0	1
Ji-min	1	0	1	0	1
Average	0.167	0	1	0.167	0.833

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Appendix B: Author Bleiler

Argument 1

Ms. Hernandez came across a gumball machine one day when she was out with her twins. Of course, the twins each wanted a gumball. What's more, they insisted on being given gumballs of the same color. The gumballs were a penny each, and there would be no way to tell which color would come out next. Ms. Hernandez decides that she will keep putting in pennies until she gets two gumballs that are the same color. She can see that there are only red and white gumballs in the machine. (*The following questions are to help guide you towards answering question 1. You should complete all of the bulleted questions, but you only need to email complete responses for Questions 1-4 below. Based on your solution to Question 1 below, I may ask you to turn in your bulleted responses in class.)*

- Why is three cents the most Ms. Hernandez will have to spend to satisfy her twins?
- The next day, Ms. Hernandez passes a gumball machine with red, white, and blue gumballs. How could Ms. Hernandez satisfy her twins with their need for the same color this time? That is, what is the most Ms. Hernandez might have to spend that day?
- Here comes Mr. Hodges with his triplets past the same gumball machine with red, white, and blue gumballs. Of course, all three of his children want to have the same color gumball. What is the most he might have to spend?

Question 1. Generalize this problem as much as you can. Vary the number of colors. What about different size families? Prove a generalization that always works for any number of children and any number of gumball colors. I encourage you to think about using alternative representations within your proof in order to create your most convincing/compelling argument.

C = children

\$ = cost

X = number of different colored gumballs

$$\$ = (c-1)\cdot x + 1$$

Argument 2

Ms. Hernandez came across a gumball machine one day when she was out with her twins. Of course, the twins each wanted a gumball. What's more, they insisted on being given gumballs of the same color. The gumballs were a penny each, and there would be no way to tell which color would come out next. Ms. Hernandez decides that she will keep putting in pennies until she gets two gumballs that are the same color. She can see that there are only red and white gumballs in the machine. (*The following questions are to help guide you towards answering question 1. You should complete all of the bulleted questions, but you only need to email complete responses for Questions 1-4 below. Based on your solution to Question 1 below, I may ask you to turn in your bulleted responses in class.)*

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Question 1. Generalize this problem as much as you can. Vary the number of colors. What about different size families? Prove a generalization that always works for any number of children and any number of gumball colors. I encourage you to think about using alternative representations within your proof in order to create your most convincing/compelling argument.

# of colors = n , # of children = x

# of times to get x amount of similar colors = T

$n(x-1)+1 = T(n,x)$

- Mrs. Hernandez w/ R, W gumballs: n=2, x=2
  $T = 2(2-1)+1 = 3$
- Mrs. Hernandez w/ R, W, B gumballs: n=3, x=2
  $T = 3(2-1)+1 = 4$
- Mr. Hodges w/ R, W, B gumballs: n=3; x=3
  $T = 3(3-1)+1 = 7$

The expression $T(n,x) = n(x-1)+1$ is valid for the values of (n=2, x=2) and (n+1=3, x=2) as well as (n+1=3, x+1=3), so we can assume the expression is valid for all positive integers of n and x.

Argument 3

Ms. Hernandez came across a gumball machine one day when she was out with her twins. Of course, the twins each wanted a gumball. What's more, they insisted on being given gumballs of the same color. The gumballs were a penny each, and there would be no way to tell which color would come out next. Ms. Hernandez decides that she will keep putting in pennies until she gets two gumballs that are the same color. She can see that there are only red and white gumballs in the machine. (*The following questions are to help guide you towards answering question 1. You should complete all of the bulleted questions, but you only need to email complete responses for Questions 1-4 below. Based on your solution to Question 1 below, I may ask you to turn in your bulleted responses in class.)*

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Question 1. Generalize this problem as much as you can. Vary the number of colors. What about different size families? Prove a generalization that always works for any number of children and any number of gumball colors. I encourage you to think about using alternative representations within your proof in order to create your most convincing/compelling argument.

Let x = # of colors
y = # of children
z = most amount spent

$$[(y-1) * x] + 1 = z$$

In order to find what the most amount needed to be spent to satisfy given conditions, we must look at the worst case scenario. Because the objective is to find y of a given color, the worst possibility is to get one less than y of every color. Then causing an additional gumball to be purchased in order to obtain a y^{th} matching color.

Argument 4

Ms. Hernandez came across a gumball machine one day when she was out with her twins. Of course, the twins each wanted a gumball. What's more, they insisted on being given gumballs of the same color. The gumballs were a penny each, and there would be no way to tell which color would come out next. Ms. Hernandez decides that she will keep putting in pennies until she gets two gumballs that are the same color. She can see that there are only red and white gumballs in the machine. (*The following questions are to help guide you towards answering question 1. You should complete all of the bulleted questions, but you only need to email complete responses for Questions 1-4 below. Based on your solution to Question 1 below, I may ask you to turn in your bulleted responses in class.)*

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Question 1. Generalize this problem as much as you can. Vary the number of colors. What about different size families? Prove a generalization that always works for any number of children and any number of gumball colors. I encourage you to think about using alternative representations within your proof in order to create your most convincing/compelling argument.

X - represents # of colors
Y - represents # of children
z - # of tries

$xy - x + 1$

PPl	colors	tries	
2	2	3	RW R
2	3	4	RWB R
3	3	7	RWB RWB R

Ex: If 4 Kids x 3 colors (R, W, B)

$12 - 3 + 1 = 10$

R W B — equals 10
R W B
R W B
~~R W B~~ — subtract
R
+ 1 color
4 colors of red

Argument 5

Ms. Hernandez came across a gumball machine one day when she was out with her twins. Of course, the twins each wanted a gumball. What's more, they insisted on being given gumballs of the same color. The gumballs were a penny each, and there would be no way to tell which color would come out next. Ms. Hernandez decides that she will keep putting in pennies until she gets two gumballs that are the same color. She can see that there are only red and white gumballs in the machine. (*The following questions are to help guide you towards answering question 1. You should complete all of the bulleted questions, but you only need to email complete responses for Questions 1-4 below. Based on your solution to Question 1 below, I may ask you to turn in your bulleted responses in class.*)

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- Here comes Mr. Hodges with his triplets past the same gumball machine with red, white, and blue gumballs. Of course, all three of his children want to have the same color gumball. What is the most he might have to spend?

Question 1. Generalize this problem as much as you can. Vary the number of colors. What about different size families? Prove a generalization that always works for any number of children and any number of gumball colors. I encourage you to think about using alternative representations within your proof in order to create your most convincing/compelling argument.

This is for positive integers only.
C = # of colors to choose from.
X = # of specific colors.

For any situation where you need X number of a single color of gumball with c colors available, there is an equation that tells you the maximum number of gumballs you'll have to buy to get X of the random single color.

$[(x-1) \cdot c] + 1$ = max number of gumballs needed.

The idea was to find an equation to show the max number of gumballs that wouldn't satisfy the conditions and then add one. Add one because then you'd only need one more of any of the colors to have enough.

	A1 - Phillip	A2 - Keyshawn	A3 - Wendy	A4 -Monique	A5 - Penelope
Andrew	1	1	1	0	(no response)
Brenda	0	1	0	1	0
Calvin	0	1	1	0	1
Danielle	0	0	0	0	0
Devin	0	1	0	1	(no response)
Elizabeth	0	1	1	1	1
Gabriel	0	0	1	0	1
Hilary	0	1	1	0	(no response)
Jeremy	1	1	1	0	1
Joshlyn	0	0	0	0	0
Keyshawn	0	1	1	0	0
Monique	0	0	1	1	0
Martin	0	1	1	0	1
Natasha	0	0	0	0	(no response)
Penelope	0	0	1	0	1
Peter	0	0	0	0	1
Phillip	0	1	1	1	0
Rachel	0	1	0	0	0
Rhonda	0	1	1	0	1
Steven	0	1	1	0	1
Stacey	0	1	1	1	0
Tyrone	0	1	1	0	0
Thomas	0	0	1	0	1
Wendy	(no response)	(no response)	(no response)	(no response)	(no response)
Average	0.087	0.652	0.696	0.261	0.526

(Return to page 37)

Appendix C: Author Yee

Argument 1

First assign the variables.

Let g be the number of colors of the gumballs.
Let k be the number of children.
Let y be number of pennies.

A general formula for this problem would be

$$y=g(k-1)+1.$$

This formula gives us the amount of most pennies needed to buy enough gumballs such that each child has the same color gumball.

To prove that this formula holds we can use an induction proof. This will be a two case proof. The first case is where the gumballs are fixed at a certain amount and the second case is where the children are fixed at a certain amount.

Case 1: Let p(k) represent the equation $y=g(k-1)+1$ and let g be fixed. Then we need to show that p(k) is true for all k. For the basis step we show p(1) is true, so g(1-1)+1=1. This is true, since there is only 1 child the matching criteria of the gumball colors is nullified so it will only take 1 penny to get a that child a gumball. Now we need to assume that p(k) is true for all k and prove p(k+1) is true. If we apply k+1 we obtain g(k-1)+1+g(k+1-1)+1=g(2k-1)+2 , which is what we wanted and will be true. So by induction this equation is true for all sizes of families when the gumball colors are fixed

Case 2 follows almost exactly the same except fixing the children and letting the amount of gumball colors vary.

Argument 2

# of Children	# of Colors	Amount Spent (¢)
2	2	3
2	3	4
2	4	5
2	5	6
3	3	7
3	4	9
3	5	11
3	6	13
4	4	13
4	5	16
4	6	19

From the table above we can see a pattern going on. When we have 2 children the amount spent goes up by one penny. When we have 3 children and increase the amount of colors the amount of pennies spent goes up by two. Example: When we have 4 children and 6 colors the first 6 pennies will give me one of each different color. The next 6 pennies will give me one of each different color again. This process is the same for the third time 6 more pennies will be spent. Until the fourth time we know we will get any color matching 4 color gumballs.

Amount Spent=A
Children=K
Colors =k
A=(K-1)(C))+1
When we have 2 children we see that the amount spent is simply C+1. But then when we have 3 children the amount spent is by two pennies. This is where the K-1 comes in place.

Argument 3

Let K be the number of kids in a family and let C be the color of gumballs in the machine.
We need to find an equation that satisfies any number of kids to any number of gumball colors in the machine.

So if we have K kids minus 1 times the number of c colors plus 1 gives us the desired number of p pennies.

$$k = kids$$
$$c = color$$
$$p = pennies$$
$$(k-1)(c) + 1 = p$$
$$\forall k, c, p \in \mathbb{N}[(k-1)(c)+1 = p]$$

Pf:

Let k, c and p be arbitrary natural numbers.

We need to show that we will always get an even or odd number for the value of p. If we were to choose k to be either even or odd and c to be even or odd p will still be an even or odd number. We have four cases to check.

Case 1: k is even and c is even then p is odd.
Let k be 2n for all n in ℕ and c be 2m for all n inℕ, then

$$(2n-1)(2m) + 1 = p$$
$$4nm - 2m + 1 = p$$
$$2(2nm - m) + 1 = p$$
$$2r + 1 = p \ \ where\, r = 2nm - m\, \forall r \in \mathbb{N}$$

Therefore p is odd.

Case 2: k is odd and c is even then p is odd.
Let k be 2n+1 for all n in ℕ and c be 2m for all m inℕ, then

$$((2n+1)-1)(2m) + 1 = p$$
$$(2n)(2m) + 1 = p$$
$$2(2nm) + 1 = p$$
$$2r + 1 = p \ \ where\, r = 2nm\, \forall r \in \mathbb{N}$$

Therefore p is odd.

Case 3: k is even and c is odd then p is even.
Let k be 2n for all n in ℕ and c be 2m+1 for all m inℕ, then

$$(2n-1)(2m+1) + 1 = p$$
$$(4mn + 2n - 2m - 1) + 1 = p$$
$$4mn + 2n - 2m = p$$
$$2(2mn + n - m) = p$$
$$2r = p \ \ where\, r = 2nm + n - m\, \forall r \in \mathbb{N}$$

Therefore p is even.

Case 4: k is odd and c is odd then p is odd.
Let k be 2n for all n in ℕ and c be 2m for all n inℕ, then

$$((2n+1)-1)(2m+1) + 1 = p$$
$$(2n)(2m+1) + 1 = p$$
$$4mn + 2n + 1 = p$$
$$2(2mn + n) + 1 = p$$
$$2r + 1 = p \ \ where\, r = 2nm + n\, \forall r \in \mathbb{N}$$

Therefore p is odd.

Therefore we showed that p will either be even or odd in the natural numbers, and p will give us the amount of coins need to get the same color.

Argument 4

Let c be the number of colors of gumballs in the gumball machine. Let k be the number of kids that want the same color gumball in a family. Then g(c,k)=[(c)(k-1)]+1 represents the maximum number of pennies a parent would have to spend to satisfy their kids.

Proof:

Let c be the number of colors of gumballs in the gumball machine. Let k be the number of kids that want the same color gumball in a family. We need to show that g(c,k)=[(c)(k-1)]+1 represents the maximum number of pennies a parent would have to spend to satisfy their kids. If we multiply c by (k-1) then either

1. Each color came out (k-1) times and (k-1) kids have all c colors.
2. Or at least one color came out at least k times, because another color came out less than (k-1) times.

If **2.** happens then all of the kids have the same color gumball and they are happy. If **1.** happens then one more kid needs a matching gumball. In this case the next gumball will satisfy the k^{th} kid because the first (k-1) kids already have all c colors. Thus the most a parent would have to spend is g(c,k)=[(c)(k-1)]+1 colors.

Argument 5

If you want to have 2 gumballs that match, then the worst-case scenario to obtain 2 matching gumballs given n different colors possible is n+1 gumballs. To show this, it is clear that with up to n gumballs, it is possible for all n to be different colors, given n different colors. But by the pigeonhole principle, once you get the n+1th gumball, it has tc match one of the other colors, and you have your 2 matching gumballs.

If you need m matching gumballs, it is a similar method. If you want to match m gumballs, given n colors, then clearly you can get a maximum of m-1 gumballs for each of the n colors, giving n(m-1)total gumballs. When you then obtain one more gumball, again, by the pigeonhole principle, you will obtain m many of a certain color. Thus, for m matching gumballs given n total colors, the maximum number of gumballs needed is n(m-1)+1.

	A1 - Sonia	**A2 - Vinny**	**A3 - Andy**	**A4 - Rose**	**A5 - Luke**
Andy	1	0	1	1	0
Abby	0	0	1	1	1
Connie	1	1	1	0	1
Guadalupe	1	1	1	1	0
John	1	1	1	1	1
Jacqui	1	1	1	0	0
Luke	0	0	0	1	1
Michael	0	0	0	1	0
Antonella	1	0	1	1	1
Rose	0	0	0	1	1
Ryan	0	0	0	1	1
Sonia	0	0	0	0	0
Stacey	0	0	0	1	1
Tanya	0	0	0	1	1
Elizabeth	1	0	1	0	1
Vinny	1	0	1	1	1
Average	0.5	0.25	0.5625	0.75	0.6825

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Appendix D: Author Ko

Argument 1

Children = c
Gumballs = g
Pennies = p
P = g (c -1) +1

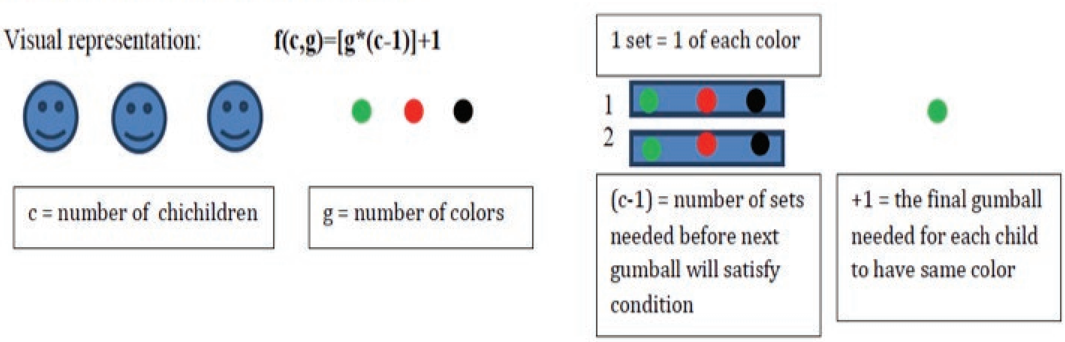
In order to make sure you get enough gumballs for every child to have one of the same color you must buy one of all the colors for every child but one. Then no matter what color you get next every child will have one of the same color. Therefore multiply the number of gumball colors by one less than the number of children, after that add one.

2c and 2g = 3p
2c and 3g = 4p
2c and 4g = 5p
3c and 2g = 5p
3c and 3g = 7p
7c and 2g = 13p

Argument 2

c (number of children)	g (number of colors of gumballs)	f(c,g) (maximum cost)
2	2	3
2	3	4
3	3	7
4	3	10

After reading and thinking through the given scenario, I believe it is best proved by generalizing the examples that were given. We know that the maximum cost of the gumballs will depend on both the number of children AND the number of colors in the machine. The maximum cost will measure the worst case possible: when the family receives all colors of gumballs before receiving repeats of colors. Therefore, for this proof we will refer to a group of gumballs that contains one and only one gumball from each color as a "set". Thus in the worst case possible, a family will accumulate one set of gumballs before receiving repeats of colors. Understanding this, if a family accumulates (c-1) sets, the next gumball purchased from the machine will allow for each child to have the same color because each previous set will include the color that comes out as well. Thinking about this in terms of a functions, f(c,g)=[g*(c-1)]+1. In this function, [g*(c-1)] represents the total number of gumballs that will be in all the (c-1) complete sets when added together. Therefore, the +1 represents the next gumball that will allow all of the children to have the same color. Because each gumball is one penny, the total number of gumballs purchased will equal the amount of money that is spent.



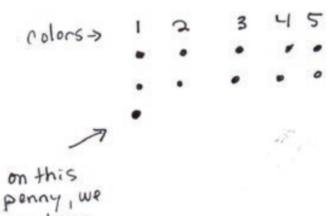
Argument 3

In order to generalize this problem we need to think about it flexably with varying colors and children.

Visualize:

Ex:

3 children 5 colors:



In order to achieve 3 of the same color, 2 rows of colors should be filled up, then + one for the final color to make a set of three for one color.

on this penny, we now have 3 of the same color.

Since the number of rows depends on the number of children, then we can say Rows = n−1 since we do not need the last row filled up completely (that would give us 3 ~~color~~ of the same color for each color) we take away one for the rows. ~~and~~ we need to multiply this ~~[illegible]~~ by the # of colors of gumballs to get the number of gumballs that we will recive and then add one more to represent the final gumball in the last row. So the formula should look something like this:

(# color of gumballs)(rows) +1 = \$ spent or

$c(n-1)+1 = P$ where c = colors, P = pennies. →

In order to prove this we need to look at the information:

For 5 differnt colors and n # of children we have:

Colors	# of sets	# of gumballs
5	1	5
	2	10
	3	15
	4	20
	5	25

Example-not proof

So we can see that if we need 3 sets of the same gumballs colors we would need 15 gumballs.
The relation ship would be sets = (n−1) = K where n is number of chidren so
5K = # of gumballs received.

so: where c = # of colors then:

Colors	# of sets	# of gumballs received
c	K	cK

Since we only need <u>one</u> extra gumball to acheive our goal we add one more gumball:

$cK + 1$ = final number of gumballs

remember, $K = n-1$ so:

examples to show correctness

Colors	rows	+1	# of gumballs received
c	n−1		P
2	2		3
4	2		5
5	3		11
3	3		7
8	5	↓	33

$2(2-1)+1=3$
$4(2-1)+1=5$
$5(3-1)+1=11$
$3(3-1)+1=7$
$8(5-1)+1=33$

End Remark: If you have 5 children, do not buy them gumballs. □

Argument 4

Proof: m=kc-(c-1)

Let m= amount of change it cost
c= colors of gumballs
k= amount of children

By spending 1 cent a R or W ball will be chosen. The next 1 cent will represent the ball in the opposite color. The third 1 cent will be either R or W, but regardless it will be a match to one of the previously chosen colors. Thus, the maximum amount of money being spent by Ms. Hernandez for her twins will be 3 cents.

By spending 3 cents she will receive three gumballs. They must be either red or white, and therefore there must be at least two of the same color. The possible outcomes of receiving three gumballs is: 3 red, 3 white, 2 red 1 white, or 1 red 2 white. Either way there will be two of the same color for her twins.

X Y	R	W
1	1 cent	I cent
2	1 cent	

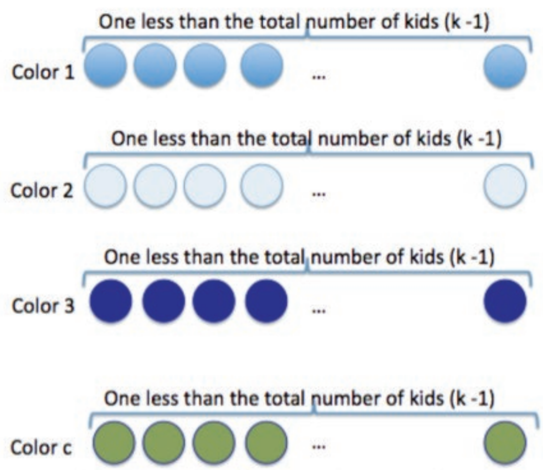
This same idea will work for however many colors and amount of children. For three kids and three different colors, spending 6 cents will give two of every color and the next 1 cent will give a match to any of the three colors. Thus, a correlation can be made to say the maximum amount of money that will be spent will be the amount of kids times the amount of colors. Then subtract the amount of colors-1 from the product. This color minus 1 results from not having to have all the possible options. 3 kids and 3 colors will be 9 for the amount of money being spent, but since just spending 7 will give three of one of the colors, 2 can be subtracted from the 9. This leads to (c-1) being a part of m=kc-(c-1)

Argument 5

Conjecture: For any number of kids, k and any number of gumball colors c, where each kid must have the same gumball color the total worst case scenario cost at 1¢ per gumball is c(k -1) + 1

Assume the same color is drawn for each kid except one kid because if every kid got the same color it would not be the worst possible case. Then a new streak of color (2nd color) is drawn for every kid, but again it stops one short to match every kid. Then a third color begins to be drawn from the machine and again the number of gumballs of this third color is one less than is needed.

Here let me show you a picture of what I mean.



Where c is the total number of colors and is a natural number

So after every gumball color, c is drawn k – 1 times then every kid but one has a matching gumball and since there are no new colors in the machine, the very next gumball (+1) will be a match so that every kid has the same color gumball at the worst possible case.
So the conjecture holds true c(k -1) + 1

	A1 - Brandon	A2 - Dennis	A3 - Gina	A4 - Henry	A5 - Instructor
Brandon	1	1	0	0	1
Carla	1	1	1	0	1
Dennis	1	1	1	0	1
Francine	1	1	1	0	1
Greg	0	1	1	0	1
Gina	0	1	1	0	1
Henry	0	0	0	0	1
Joel	0	0	1	1	1
Jessica	0	1	1	0	1
Katherine	0	0	1	0	1
Lasondra	0	1	1	1	1
Naomi	(no response)	(no response)	(no response)	(no response)	(no response)
Average	0.307692308	0.769230769	0.846153846	0.153846154	1

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