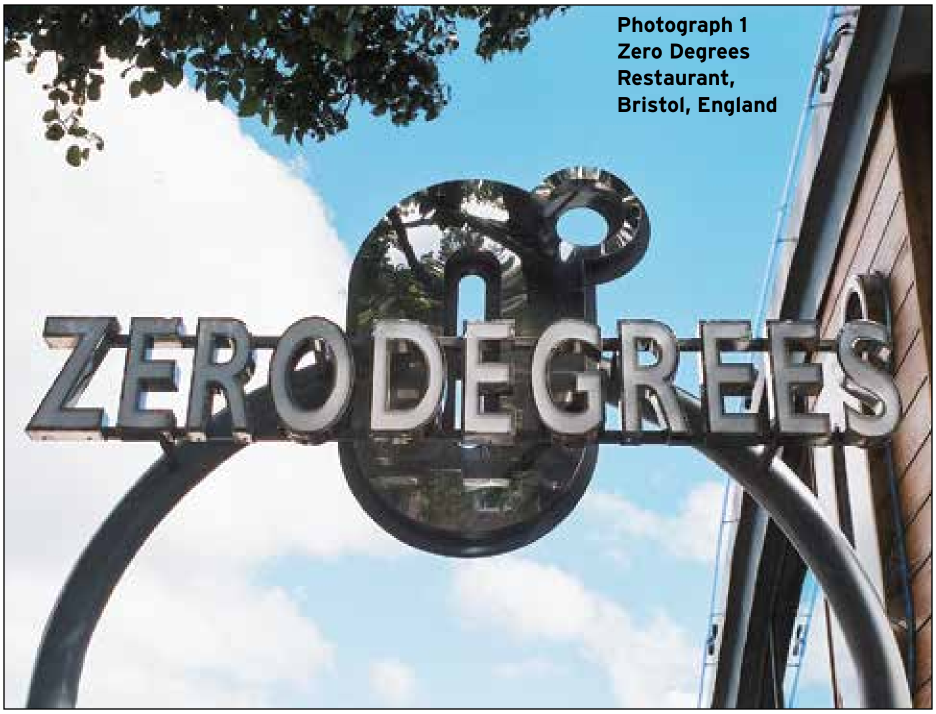
# Zero Degrees

During a trip to Bristol, England, Ron Lancaster came upon a restaurant called Zero Degrees and was intrigued with its logo—0° (see **photographs 1** and **2**). He knew this symbol was meant to be another way of expressing the restaurant's name, but he wondered if he was the only one who saw it as zero to the power zero.




Photograph 1
Zero Degrees Restaurant, Bristol, England

RON LANCASTER

Photograph 2
Zero Degrees Restaurant

RON LANCASTER

1. (*a*) Use a graphing calculator such as the TI-84 to evaluate each of the values in **table 1**. On the basis of your answers, suggest a definition for $0^0$. Use a graphing calculator to evaluate $0^0$ and then compare the result with your suggested value.

   (*b*) Graph $y = x^x$ in a [0, 1] by [0, 1] window. Use the Trace tool to study the graph near and at $x = 0$ and compare what you see with the results from part (*a*).

   (*c*) Use the calculator to find the approximate value of $x$ between 0 and 1 for which the value of $y$ is the smallest. What is the minimum value of $y$? Or, if you have studied calculus, find the exact

"Mathematical Lens" uses photographs as a springboard for mathematical inquiry. The goal of this department is to encourage readers to see patterns and relationships that they can think about and extend in a mathematically playful way.

*Edited by* ***Ron Lancaster***
*ron2718@nas.net*
*University of Toronto*
*Toronto, Ontario M5S 1A1*
*Canada*

***Brigitte Bentele***
*brigitte.bentele@trinityschoolnyc.org*
*Trinity School*
*New York, NY 10024*

| Table 1 Closing in on $0^0$? | |
|---|---|
| $0.1^{0.1}$ | |
| $0.01^{0.01}$ | |
| $0.001^{0.001}$ | |
| $0.0001^{0.0001}$ | |
| $0.00001^{0.00001}$ | |
| $0.000001^{0.000001}$ | |
| $0.0000001^{0.0000001}$ | |

**Table 2 Evaluating $y = (x^r)^{(x^r)}$**

| *r* | *a* (value of *x* for which the function is the smallest) | *b* (smallest value of *y*) |
|---|---|---|
| 1 | | |
| 2 | | |
| 3 | | |
| 4 | | |
| 5 | | |

value of $x$ for which $y$ is as small as possible.

2. ($a$) Let

$$y = \left(x^r\right)^{\left(x^r\right)}.$$

For each of the values of $r$ in **table 2**, find the value of $x$ between 0 and 1 for which $y$ is as small as possible and find the smallest value of $y$.

($b$) Study the results in **table 2** and suggest a rule for $a$ and $b$ in terms of $r$. Use this rule to predict the values of $a$ and $b$ for

$$y = \left(x^{30}\right)^{\left(x^{30}\right)}.$$

Graph this function. Then calculate the values of $a$ and $b$ and compare them with your predictions.

($c$) If you have studied calculus, find the exact values of $a$ and $b$ in terms of $r$ for which $y$ is as small as possible. Are these results consistent with those from part ($a$)?

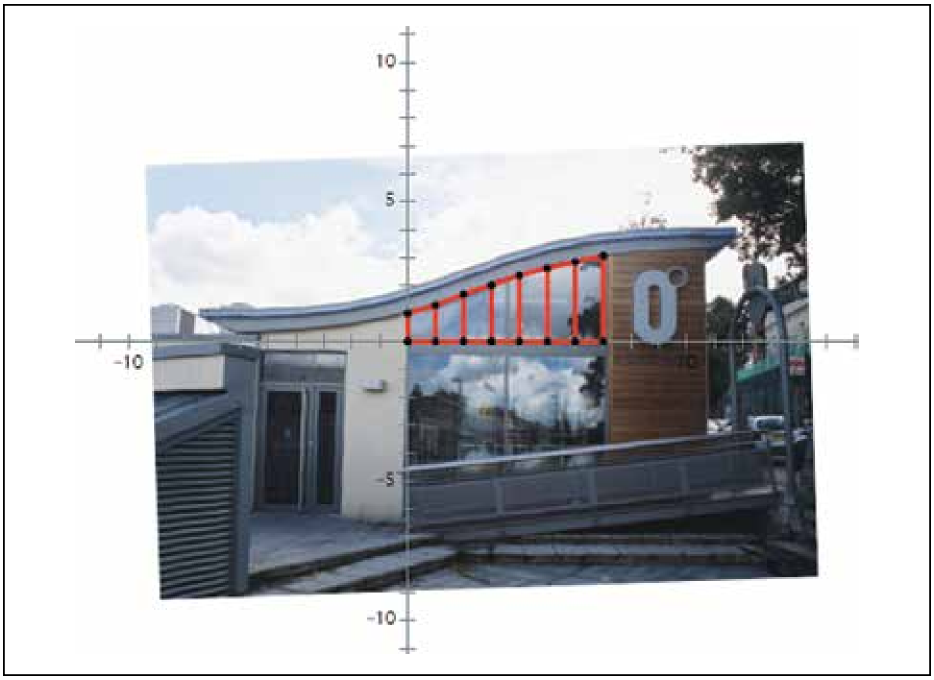



**Fig. 1** Positioning the photograph in GSP so that the origin coincides with the lower-left corner of the upper window makes it easier to find areas.

3. The window on the side of the building (see **photograph 2**) consists of two sections. The bottom section is a rectangle, and its area can easily be calculated. The top section has a curved edge; if you have not studied calculus, it may not be obvious to you how to find its area. In answering this question, you will learn how to find the area of the top section by using an equation for the curved edge.

**Photograph 2** was pasted into a blank Geometer's Sketchpad® file and positioned so that the point (0, 0) coincided with the lower-left corner of the top section of the window (see **fig. 1**). Eight points—$A$, $B$, $C$, $D$, $E$, $F$, $G$, and $H$—were placed along the top curve of the window and then used to form seven trapezoids. The coordinates of these points were measured and recorded in **table 3**.

| Table 3 Coordinates of the Vertices of the Trapezoids | |
|---|---|
| $A$ | (0, 1.00) |
| $B$ | (1, 1.29) |
| $C$ | (2, 1.69) |
| $D$ | (3, 2.01) |
| $E$ | (4, 2.37) |
| $F$ | (5, 2.62) |
| $G$ | (6, 2.85) |
| $H$ | (7, 3.07) |

($a$) Find the combined area of the seven trapezoids. How is this area related to the area of the curved section of the window as it appears in **photograph 2**?

($b$) Find an equation for the curve that passes through the eight points in **table 3**.

($c$) Describe how the equation of the curve could be used to improve the accuracy of the answer obtained in part ($a$).

($d$) If you have studied calculus, find the area of the window by finding the area under the curve.

($e$) So far you have found the area of the window by using the coordinates as established by using GSP, and your answer was in generic square units. What would you need to do to find the actual area of the curved section of the window?

**Table 4** Answers for Table 1

| | |
|---|---|
| $0.1^{0.1}$ | 0.7943282347 |
| $0.01^{0.01}$ | 0.954992586 |
| $0.001^{0.001}$ | 0.9931160484 |
| $0.0001^{0.0001}$ | 0.99907939 |
| $0.00001^{0.00001}$ | 0.9998848774 |
| $0.000001^{0.000001}$ | 0.9999861846 |
| $0.0000001^{0.0000001}$ | 0.9999983882 |

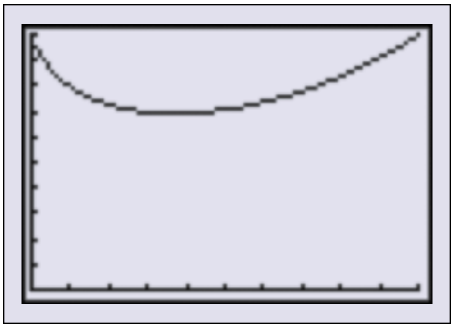


**Fig. 2** The graph of $y = x^x$ appears in a [0, 1] by [0, 1] window.

1. (*a*) On the basis of the results in **table 4**, it appears that $0^0$ should be defined as 1. But the TI-84 returns the message "domain error" when an attempt is made to calculate $0^0$.

   (*b*) The graph of $y = x^x$ is shown in **figure 2**. When $x = 0$, the value of $y$ seems to be 1. But when the Trace tool is activated and the cursor is moved to the point where $x = 0$ (on the TI-84, key in 0 and press the Enter key), no value for $y$ appears. This result is consistent with the error obtained when $0^0$ is evaluated but not with the trend toward 1 seen in **table 4**. The quantity is what mathematicians call an indeterminate form.

   (*c*) The value of $y$ is smallest when $x \approx 0.36787894$. The minimum value of $y$ is approximately 0.69220063. These results were obtained by using the minimum command from the Trace menu on a TI-84 calculator with LeftBound = 0, RightBound = 1, and Guess = 0.5. Slightly different results will be obtained if other choices are made for the LeftBound, RightBound, and Guess or if other technology is used.

   Using calculus, we can see that the value of $y$ is as small as possible when $x = e^{-1}$. Using logarithmic differentiation, we obtain the following:

$$
\begin{aligned}
y &= x^x \\
\ln y &= \ln\left(x^x\right) \\
\ln y &= x \ln x \rightarrow \\
\frac{y'}{y} &= 1 + \ln x \rightarrow \\
y' &= x^x\left(1 + \ln x\right) = 0 \rightarrow \\
x^x\left(1 + \ln x\right) &= 0 \rightarrow \\
\left(1 + \ln x\right) &= 0 \\
x &= e^{-1}
\end{aligned}
$$

   Thus, the minimum value of $y$ is

$$\left(e^{-1}\right)^{\left(e^{-1}\right)}.$$

**Table 5** Answers for Table 2

| $r$ | $a$ (value of $x$ for which the function is the smallest) | $b$ (smallest value of $y$) |
|---|---|---|
| 1 | 0.36787894 | 0.69220063 |
| 2 | 0.60653034 | 0.69220063 |
| 3 | 0.71653287 | 0.69220063 |
| 4 | 0.77880089 | 0.69220063 |
| 5 | 0.81872865 | 0.69220063 |

2. (*a*) The values of $a$ and $b$ for each value of $r$ are shown in **table 5**. The graphs for the last two cases are shown in **figures 3a** and **3b**.

   (*b*) Surprisingly, the minimum value of $y$ for each case is the same, so the value of $b$ appears to be independent of the value of $r$. The connection between $a$ and $r$ is more complicated. One thing that can be said is that the value of $a$ increases when the value of $r$ goes up. The value of $a$ seems to be getting closer and closer to 1 (**fig. 4** shows the graph of $r$ versus $a$). Regarding the graph of

$$y = \left(x^{30}\right)^{\left(x^{30}\right)},$$

   a reasonable prediction for the values of $a$ and $b$ is $a = 1$ and $b = 0.69220063$. The graph of

$$y = \left(x^{30}\right)^{\left(x^{30}\right)}$$

   (see **fig. 5**) shows that this prediction is accurate.

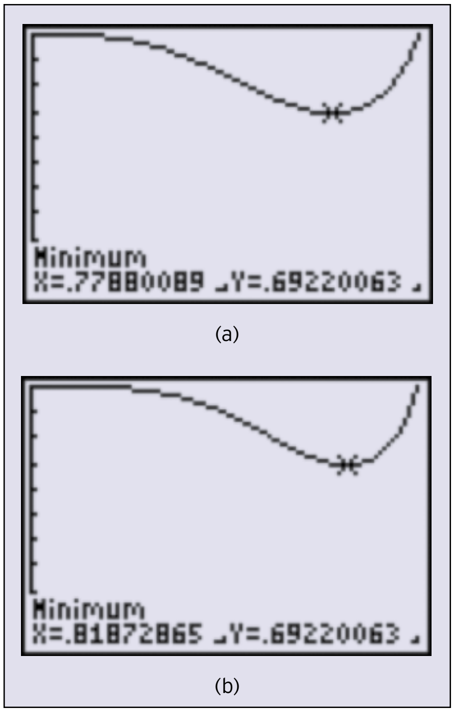



**Fig. 3** The minimum $y$ value is the same no matter what the value of $r$ is; here graphs of the function when $r = 4$ (**a**) and $r = 5$ (**b**) are shown.

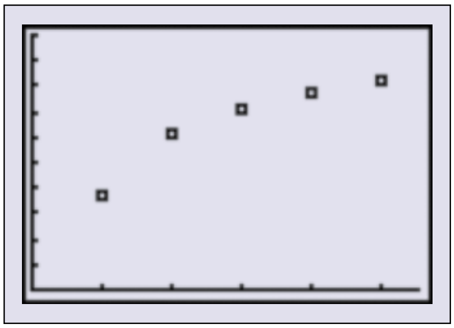


**Fig. 4** This graph shows $r$ versus $a$ in a [0, 6] by [0, 1] window.

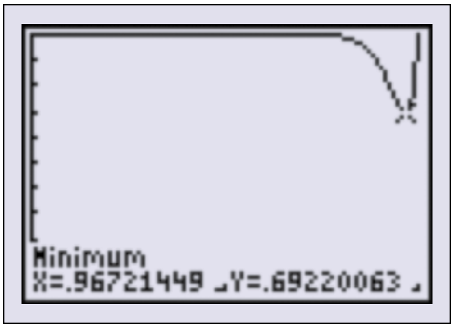



**Fig. 5** The graph confirms that the minimum value occurs when $x$ is close to 1.

($c$) Let

$$f(x) = (x^r)^{(x^r)}$$

for $0 < x \le 1$. Differentiate as follows:

$$\ln f(x) = x^r \ln(x^r)$$
$$\ln f(x) = r x^r \ln x \rightarrow$$
$$\frac{f'(x)}{f(x)} = r x^{r-1} + r^2 x^{r-1} \ln x$$
$$f'(x) = (x^r)^{(x^r)} \left( r x^{r-1} (1 + r \ln x) \right)$$
$$= 0 \rightarrow$$
$$x = e^{\frac{-1}{r}}$$

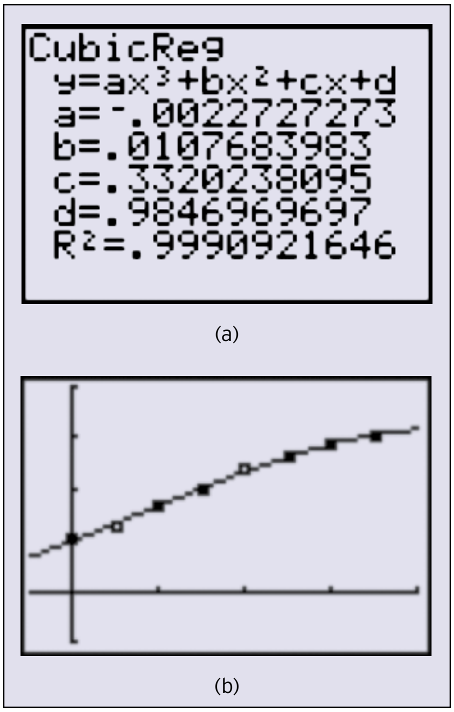
The second derivative test can be used to show that $f$ has a minimum at $x = e^{-1/r}$. Note that as the value of $r$ increases, the value of $e^{-1/r}$ gets closer and closer to $e^0$, or 1. So it is true that the value of $a$ discussed in problem 2($b$) approaches 1 when the value of $r$ goes up. Note that

$$f\left(e^{\frac{-1}{r}}\right) = \left(e^{-1}\right)^{\left(e^{-1}\right)},$$

implying that the minimum value of $f$ does not depend on the value of $r$, results that are consistent with the results from problem 2($b$).

3. As shown below, the combined area of the seven trapezoids is equal to about 14.87 square units. This area is slightly less than the area of the curved section of the window as it appears in **photograph 2**. The area can be computed by repeatedly applying the formula for the area of a trapezoid, $A = 0.5(b_1 + b_2)$:

$$\text{Area} = \frac{1}{2} \cdot 1(1 + 1.29) + \frac{1}{2} \cdot 1 \cdot (1.29 + 1.69) + \frac{1}{2} \cdot 1 \cdot (1.69 + 2.01) + \frac{1}{2} \cdot 1 \cdot (2.01 + 2.37) + \frac{1}{2} \cdot 1 \cdot (2.37 + 2.62) + \frac{1}{2} \cdot 1 \cdot (2.62 + 2.85) + \frac{1}{2} \cdot 1 \cdot (2.85 + 3.07)$$
$$= 1\square.865$$




(a)

(b)

**Fig. 6** When a cubic regression is done on the data, the correlation coefficient is high (**a**) and the curve fits the data closely (**b**).

($b$) Many functions can be used to fit the data in **table 3**. The "Math Lens" editors chose to use a cubic polynomial (see **figs. 6a** and **6b**).

($c$) The accuracy can be improved by using 15 points on the curve with $x$-coordinates that range from 0 to 7 in increments of 0.5. The $y$-coordinates can be calculated for each value of $x$, and then the combined area of the 14 trapezoids can be calculated.

($d$) The area can be determined by finding the area under the curve from $x = 0$ to $x = 7$ either by hand or by using the calculator. The area of the window is about 14.89 square units.

($e$) To find the actual area of the curved section of the window, you need to have a measurement of a certain object in the photograph (the height of the door, for example). Then you can use ratios and proportions to calculate the actual area of the curved section of the window.

## LEARNING MORE ABOUT $0^0$

It would be a huge mistake to say that attempts to define the value of $0^0$ have been much ado about nothing. Many great mathematicians have grappled with what to do with $0^0$, and various proposals for its value have been mired in controversy and confusion. For more information, see the following article and Web sites:

- Lipkin, Leonard J. "On the Indeterminate Form $0^0$." *The College Mathematics Journal* 34, no. 1 (January 2003): 55–56.
- "Exponentiation." en.wikipedia.org/wiki/0%5E0#Zero_to_the_zero_power.
- "Zero to the Power of Zero." hotmath.com/hotmath_help/topics/zero-power-zero.html.
- "Zero to the Zero Power." www.empiwifo.uni-freiburg.de/lehre-teaching-1/winter-term-08-09/materialien-wirtschaftsmathematik/wiki_00problem.pdf.

For a mathematical photograph for which you may create your own questions, go to the NCTM Web site: www.nctm.org/mt. Send your questions to the "Mathematical Lens" editors.