SOLUTIONS to calendar

1. 1007, 1008. We observe the sequence presented in the problem: $2^2 - 1^2 = 3$; $3^2 - 2^2 = 5$; and $4^2 - 3^2 = 7$. In each case shown, $a^2 - b^2 = a + b$, where $b = a - 1$. If the observed pattern is generally true, then we need the two consecutive integers whose sum is 2015: $a + (a + 1) = 2015 \to a = 1007$ and $(a + 1) = 1008$. Now confirm that the pattern is true in general. We have $a^2 - (a - 1)^2 = a^2 - (a^2 - 2a + 1) = 2a - 1 = a + (a - 1)$.

2. $1 + 3\sqrt{2}$; $2 + 1\sqrt{3}$; $3 + 2\sqrt{1} = 2 + 3\sqrt{1}$. Approximations for $\sqrt{2}$ and $\sqrt{3}$ to the nearest tenth will be helpful. Students may know that $1.4 < \sqrt{2} < 1.5$ and that $1.7 < \sqrt{3} < 1.8$. One method to show these inequalities depends on recognizing that $10\sqrt{2} = \sqrt{200}$ and finding perfect squares greater and less than 200:

$$\begin{gathered}\sqrt{196} < \sqrt{200} < \sqrt{225}\\ 14 < 10\sqrt{2} < 15\\ 1.4 < \sqrt{2} < 1.5\end{gathered}$$

Likewise, since $10\sqrt{3} = \sqrt{300}$, we see that

$$\begin{gathered}\sqrt{289} < \sqrt{300} < \sqrt{324}\\ 17 < 10\sqrt{3} < 18\\ 1.7 < \sqrt{3} < 1.8.\end{gathered}$$

Use these results to obtain approximations for all six expressions:

$$\begin{aligned}4.4 &< 1+2\sqrt{3} < 4.6\\ 3.7 &< 2+1\sqrt{3} < 3.8\\ 4.4 &< 3+1\sqrt{2} < 4.5\\ 5.2 &< 1+3\sqrt{2} < 5.5\\ 2+3\sqrt{1} &= 5\\ 1+2\sqrt{1} &= 5\end{aligned}$$

We see that $1 + 3\sqrt{2}$ has the only value greater than 5. Both expressions containing the square root of 1 equal 5, and the value of $2 + 1\sqrt{3}$ is less than 4.

Extension: Two of the expressions have overlapping approximations. Determine which is larger, still refraining from using a calculator.

Guess the relationship between the expressions and write a comparative expression—try $1 + 2\sqrt{3} > 3 + 1\sqrt{2}$. Then apply basic operations to both sides of the inequality:

$$\begin{aligned}1+2\sqrt{3} &> 3+1\sqrt{2}\\ 2\sqrt{3} &> 2+1\sqrt{2}\\ 12 &> 4+4\sqrt{2}+2\\ 6 &> 4\sqrt{2}\\ 36 &> 32\end{aligned}$$

The final inequality is true, so the guess was correct, and the inequality can be properly proved by reversing the order of the steps. This method of comparison provides an alternative approach to the entire problem, one that does not require initial approximations of square roots.

3. 88°. The opposite angles of a *cyclic quadrilateral* (another term for a quadrilateral inscribed in a circle) are supplementary. In this case, we have a pair of opposite angles that have the same measure, $2a$, so each has measure 90°. Setting the sum of the remaining pair equal to 180 gives us $7b + 12 = 180 \to b = 24$. One angle has measure $4(24) - 8 = 88$, and the other has measure $3(24) + 20 = 92$.

4. 610. Write the expressions in terms of a and b:

$a + b + c = a + b + (a + b) = 2a + 2b$
$b + c + d = b + (a + b) + (a + 2b) = 2a + 4b$

Solve the system:

$$\begin{aligned}2a + 2b &= 3194\\ 2a + 4b &= 5168\end{aligned}$$

Subtract the first equation from the second to obtain $2b = 1974 \to b = 987$. Substitute this value into the first of the

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two equations in a and b: $2a + 2(987) = 3194 \rightarrow a = 610$.

Alternate solution: Since $a + b = c$, we have $a + b + c = (a + b) + c = 2c = 3194 \rightarrow c = 3194/2 = 1597$. Use the same reasoning to find $d = 2584$. Work backward: $d - c = b = 987$, and $c - b = a = 610$.

5. They are all powers of 2. All odd natural numbers greater than 1 can be written as the sum of exactly two consecutive integers—for example, 3 = 1 + 2 and 17 = 8 + 9. In general, $2n + 1 = n + (n + 1)$. The multiples of 6 (even multiples of 3) can be written as the sum of three consecutive integers—for example, 6 = 1 + 2 + 3. In general, $6n = (2n - 1) + 2n + (2n + 1)$. Thus, we have accounted for twelve of the nineteen integers from 2 to 20. Sums can be found for 10, 14, and 20: 10 = 1 + 2 + 3 + 4; 14 = 2 + 3 + 4 + 5; and 20 = 2 + 3 + 4 + 5 + 6. Sums cannot be found for 2, 4, 8, or 16. These “holdouts” are all powers of 2. (A positive integer is the sum of two or more consecutive integers if and only if it is not a power of 2.)

6. 5/11. The sample space consists of all ordered number pairs such that one of the numbers is 4 and the other is between 1 and 6: {(1, 4), (2, 4), (3, 4), (4, 4), (5, 4), (6, 4), (4, 1), (4, 2), (4, 3), (4, 5), (4, 6)}. Of these eleven outcomes, five have a sum of 8, 9, or 10.

7. \$11.50. Let B be Bert’s hourly rate, and let E be Ernie’s hourly rate. Then $3B - 2E = \$4.60$ and $15B = 12E$. Divide both sides of the second equation by 5 to obtain $3B = (12/5)E$. Substitute $(12/5)E$ into the first equation to obtain $(12/5)E - (10/5)E = \$4.60 \rightarrow 2E/5 = \$4.60 \rightarrow E = \$11.50$.

8. 10. Let F_n and F_{n+1} be two consecutive numbers in the sequence. Then the next term in the sequence must be $F_n + F_{n+1}$, and the sum of these three consecutive terms is $2F_n + 2F_{n+1}$. Create a table to organize successive terms and successive sums.

Solution 8a General Expression for Sum of Terms

Term Number	Value of Term (in terms of the two initial values)	Cumulative Sum of Terms
1	F_n	F_n
2	F_{n+1}	$F_n + F_{n+1}$
3	$F_n + F_{n+1}$	$2F_n + 2F_{n+1}$
4	$F_n + 2F_{n+1}$	$3F_n + 4F_{n+1}$
5	$2F_n + 3F_{n+1}$	$5F_n + 7F_{n+1}$
6	$3F_n + 5F_{n+1}$	$8F_n + 12F_{n+1}$
7	$5F_n + 8F_{n+1}$	$13F_n + 20F_{n+1}$
8	$8F_n + 13F_{n+1}$	$21F_n + 33F_{n+1}$
9	$13F_n + 21F_{n+1}$	$34F_n + 54F_{n+1}$
10	$21F_n + 34F_{n+1}$	$55F_n + 88F_{n+1}$

Solution 8b Counterexamples for Sums as Multiples of 11

Value of k	Summed Terms	Sum
4	13 + 21 + 34 + 55	123
5	13 + 21 + 34 + 55 + 89	212
6	13 + 21 + 34 + 55 + 89 + 144	356
7	13 + 21 + 34 + 55 + 89 + 144 + 233	589
8	13 + 21 + 34 + 55 + 89 + 144 + 233 + 377	966
9	13 + 21 + 34 + 55 + 89 + 144 + 233 + 377 + 610	1576

We see the sum of ten consecutive terms: $F_n + F_{n+1} + F_{n+2} + \cdots + F_{n+9} = 55F_n + 88F_{n+1}$. Since both coefficients are divisible by 11, the sum will be divisible by 11, regardless of the value of the first term.

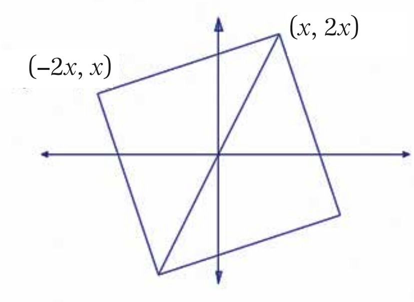
We need to show that for all $k < 10$, the sum of the k terms is not necessarily divisible by 11. That is, we need to find nine counterexamples. Begin with the first Fibonacci number greater than 11—namely, 13—as the starting point for the sequence. For $k = 1$, the sum, 13, is not divisible by 11. For $k = 2$, the sum 13 + 21 = 34 is not divisible by 11. For $k = 3$, the sum 13 + 21 + 34 = 68 is not divisible by 11. The counterexamples continue in the table for solution 8b.

Readers can confirm that none of the sums shown is divisible by 11.

9. 1/3, –3. Assume that the center of the square is located at the origin. Then an endpoint of the given diagonal has coordinates $(x, 2x)$. Rotate this point through 90° counterclockwise to find the coordinates of an endpoint of the other diagonal: $(-2x, x)$. These two plotted points are the endpoints of a side of the square. The slope is

$$m = \frac{x - 2x}{-2x - x} = \frac{-x}{-3x} = \frac{1}{3}.$$

The sides of the square are perpendicular to one another, so the other slope is –3.



10. Examples will vary. Select 21. Its square is 441, which can be written as 220 + 221. So 21-220-221 should be a Pythagorean triple: $21^2 + 220^2 = 441 + 48{,}400 = 48{,}841 = 221^2$. To prove this method in general, we show that the square of an odd integer can be written

as the sum of two consecutive integers: $(2n+1)^2 = 4n^2 + 4n + 1 = (2n^2 + 2n) + (2n^2 + 2n + 1)$. Next we show that the three integers satisfy the Pythagorean relationship, making use of the fact that the difference of squares $a^2 - b^2$ can be written as $(a - b)(a + b)$. We have

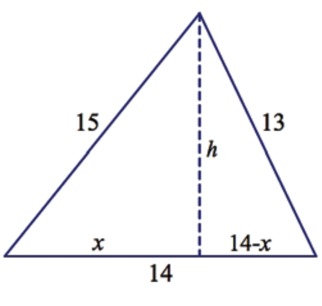
$$\begin{aligned}(2n^2 + 2n + 1)^2 - (2n^2 + 2n)^2 &= [(2n^2 + 2n + 1) - (2n^2 + 2n)] \\ &\quad \cdot [(2n^2 + 2n + 1) + (2n^2 + 2n)] \\ &= 4n^2 + 4n + 1 = (2n + 1)^2.\end{aligned}$$

11. 6, 84. The familiar 3-4-5 triangle is a right triangle; its legs serve as a base and corresponding height, so area $A = (1/2) \cdot 3 \cdot 4 = 6$. The triangle with sides 13, 14, and 15 is clearly not right, so we consider other methods.

Let the side of length 14 serve as the base. The corresponding altitude h divides the base into lengths of x and $14 - x$, as shown. Apply the Pythagorean theorem twice to obtain $h^2 = 15^2 - x^2$ and $h^2 = 13^2 - (14 - x)^2$. Therefore,

$$\begin{aligned}15^2 - x^2 &= 13^2 - (14 - x)^2 \\ 15^2 - 13^2 &= x^2 - (14 - x)^2 \\ (15 - 13)(15 + 13) &= [x + (14 - x)][x - (14 - x)] \\ 2(28) &= 14(2x - 14) \\ 4 &= 2x - 14 \\ x &= 9.\end{aligned}$$

This result gives us $h = 12$ (by another application of the Pythagorean theorem or by recognizing two familiar Pythagorean triples: 9-12-15 and 5-12-13). Finally, the area $A = (1/2)14 \cdot 12 = 84$.



Alternate solution: We know that the area is an integer, so it makes sense to use Heron's formula for the area of a triangle:

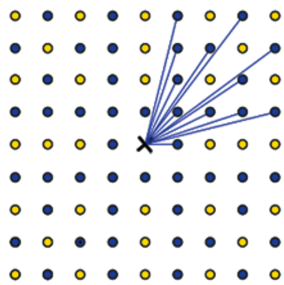
$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where s is the semiperimeter, and a, b, and c are the side lengths. We obtain

$$A = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = \sqrt{3^2 \cdot 7^2 \cdot 4^2} = 84,$$

as before.

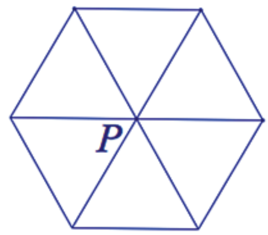
12. 32. Place the array on the coordinate plane so that the X is located at the origin. Examine the "trees" in the first quadrant and those on the positive x-axis. Of the 20 "trees" under examination, a direct line of sight exists for 12; 8 are hidden from view. Then, by symmetry, there must be a total of 32 trees not visible from the center of the orchard. (We could have reduced the region to consider by examining only those "trees" that satisfy $y < x$ in the first quadrant and then reflecting the region across $y = x$.)



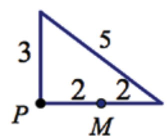
Alternate solution: The figure suggests an approach using slope. For trees represented in the first quadrant, below the diagonal $y = x$ but not on the x-axis, there are five possible slopes of lines to the origin: 1/2, 1/3, 1/4, 2/3, and 3/4. We must also count reciprocals of those as well as slopes of 1 and 0. Therefore, there are twelve unique slopes to consider. Since there are 20 trees in this quadrant, 8 trees must be invisible. By symmetry, we count $4 \cdot 8 = 32$ invisible trees in the orchard. (Problem 12 was inspired by problem 351 at Project Euler; see https://projecteuler.net.)

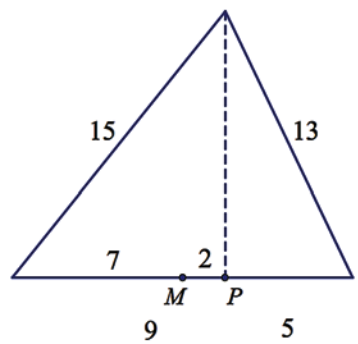
13. If a point lies on the x- or y-axis, it will be hidden from view unless it lies exactly 1 unit from the center, C. For all other coordinate pairs, if x_1 and y_1 have no common factors other than ±1, then the point will be visible from the origin.

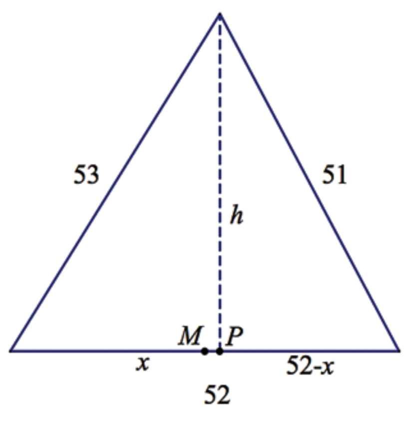
14. We "see" a plane figure, a regular hexagon divided into equilateral triangles. (See also problem 21 from the *MT* February 2014 Calendar.)



15. 2. In the 3-4-5 right triangle, the leg of length 3 is the altitude to the base of 4. The midpoint of the base is therefore 2 units from the foot of the perpendicular. The solution to the problem of September 11 showed that the base of 14 was divided by the altitude into segments of lengths 9 and 5. Half the base length is 7, so the distance from P, the foot of the perpendicular, to M, the midpoint of the base, is $9 - 7 = 2$.







For the 51-52-53 triangle, let the altitude be h, and let the base segments be x and

$52 - x$. By the Pythagorean theorem, $h^2 = 53^2 - x^2$ and $h^2 = 51^2 - (52 - x)^2$. Therefore,

$$\begin{aligned} 53^2 - x^2 &= 51^2 - (52 - x)^2 \\ 53^2 - 51^2 &= x^2 - (52 - x)^2 \\ (53-51)(53+51) &= [x+(52-x)][x-(52-x)] \\ 2(104) &= 52(2x - 52) \\ 4 &= 2x - 52 \\ x &= 28 \text{ and } 52 - x = 24. \end{aligned}$$

Half the base is 26, so the distance PM is $28 - 26 = 2$.

***Alternate solution*:** Use either method shown in the solution for the September 11 problem to find the area. The semiperimeter is $(51 + 52 + 53)/2 = 78$, and Heron's formula gives us the area

$$\begin{aligned} A &= \sqrt{78(78-51)(78-52)(78-53)} \\ &= \sqrt{78 \cdot 27 \cdot 26 \cdot 25} \\ &= 1170. \end{aligned}$$

Now $A = (1/2)bh$, or $h = 2A/b$, so $h = 2(1170)/52 = 45$. Apply the Pythagorean theorem to find that the altitude divides the base into segments of lengths 24 and 28. Half the base is 26, so the distance PM is $28 - 26 = 2$.

16. 210. The coefficients give important information about the roots of a polynomial equation. Let r_1, r_2, and r_3 represent the three roots. Since the leading coefficient is 1, we can write the polynomial in factored form as $(x - r_1)(x - r_2)(x - r_3) = 0$. If we multiply these three linear terms carefully, we see where the coefficients come from:

$$\begin{aligned} &(x^2 - (r_1 + r_2)x + r_1r_2)(x - r_3) \\ &\quad = x^3 - (r_1 + r_2 + r_3)x^2 \\ &\qquad + (r_1r_2 + r_2r_3 + r_1r_3)x - r_1r_2r_3 \end{aligned}$$

The opposite of the sum of the roots, or $-(r_1 + r_2 + r_3)$, is the coefficient of the x^2 term. In this case, we are told that the three roots are consecutive integers, so we let r_1 be the smallest root. Then the sum of the roots is $r_1 + (r_1 + 1) + (r_1 + 2) = 18$ so that $3r_1 + 3 = 18$, or $r_1 = 5$. The constant term of the cubic polynomial is $-r_1r_2r_3$, or the opposite of the product of the roots. In this case, $d = r_1r_2r_3 = (5 \cdot 6 \cdot 7) = 210$.

17. Let k represent the smallest of four consecutive integers. The product of the four is then

$$\begin{aligned} &k(k + 1)(k + 2)(k + 3) \\ &\quad = (k^2 + k)(k^2 + 5k + 6) \\ &\quad = k^4 + 5k^3 + 6k^2 + k^3 + 5k^2 + 6k \\ &\quad = k^4 + 6k^3 + 11k^2 + 6k. \end{aligned}$$

Adding 1 gives $k^4 + 6k^3 + 11k^2 + 6k + 1$, which can be written as a perfect square: $(k^2 + 3k + 1)^2$.

***Alternate solution*:** Reorder the factors as

$$\begin{aligned} &k(k + 3)(k + 1)(k + 2) \\ &\quad = (k^2 + 3k)(k^2 + 3k + 2). \end{aligned}$$

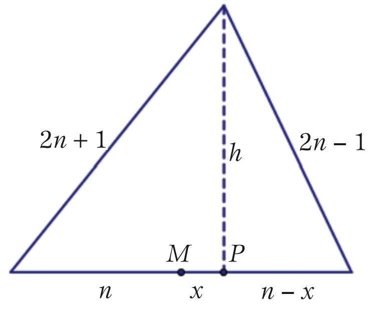
Identify a convenient substitution, $x = (k^2 + 3k)$, to write $x(x + 2) = x^2 + 2x$. Add 1 and factor as $(x + 1)^2 = (k^2 + 3k + 1)^2$.

18. 31. The five pairs with sums from 100 + 101 to 104 + 105 require no carrying. We can replace the 0 in the tens place with 1 or 2 or 3 or 4; thus, we obtain twenty additional noncarry pairs. All pairs with sums from 150 + 151 to 198 + 199 require carrying. That leaves six pairs, in which a tens digit is 0, not yet counted: 109 + 110, 119 + 120, 129 + 130, 139 + 140, 149 + 150, and 199 + 200. The total number of pairs is 31.

19. The solution to problem 15 showed that $PM = 2$ in three particular cases. The method shown there can be used for the general case as well. Label the side lengths—$2n - 1$, $2n$, $2n + 1$—as shown in the figure. Let $PM = x$. The altitude, of measure h, is a leg of two different right triangles. Thus,

$$\begin{aligned} (2n + 1)^2 - (n + x)^2 = h^2 = (2n - 1)^2 - (n - x)^2 \\ (2n + 1)^2 - (2n - 1)^2 = (n + x)^2 - (n - x)^2. \end{aligned}$$

It follows that $8n = 4nx$, and we have $x = 2$.



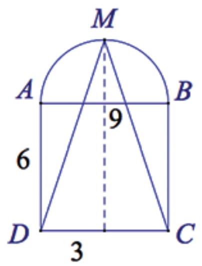
Problems 11, 15, and 19 are based on the work of Raymond Beauregard and E. R. Suryanarayan. Readers can learn more about these triangles in Beauregard and Suryanarayan, "The Brahmagupta Triangles," *The College Mathematics Journal* 29, no. 1 (1998): 13–17.

20. A two-digit number ending in 1 can be squared as $(a + 1)(a + 1) = a^2 + 2a + 1$, where a is a multiple of 10 from 10 to 90. The term a^2 will be a multiple of 100, not affecting the digit in the tens place; nor will the single digit 1 affect the digit in the tens place. The term $2a$ will be an even multiple of 10 from 20 to 180; hence, the digit in the tens place will be even.

The square of a two-digit number ending in 6 can be written as $(a + 6) \cdot (a + 6) = a^2 + 2a + 36 = a^2 + (2a + 30) + 6$, where a is a multiple of 10 from 10 to 90. The product $2a$ will be an even multiple of 10 from 20 to 180; then $2a + 30$ will be an odd multiple of 10. This value contributes an odd digit to the tens place.

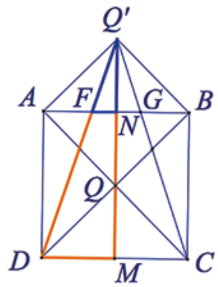
21. 25, 76. The only numbers to consider would end in 0, 1, 5, or 6. We rule out two-digit numbers ending in 0 because their squares will all end in two zeros. The squares of two-digit numbers ending in 1 have an even digit in the tens place, so we look only at the squares of 21, 41, 61, and 81. None of these is automorphic. All two-digit numbers ending in 5 have squares ending in 25, so 25 must be automorphic: $25^2 = 625$. The squares of all two-digit numbers ending in 6 have an odd digit in the tens place, so we look only at the squares of 16, 36, 56, 76, and 96. We find that $76^2 = 5776$, so 25 and 76 are the two-digit automorphic numbers.

22. $6 + 6\sqrt{10}$. The base of $\triangle DMC$ is 6; its height equals the side of the square plus the radius of the semicircle, or 6 + 3 = 9. Apply the Pythagorean theorem to find the length of the hypotenuse: $DM^2 = 3^2 + 9^2 = 90 \rightarrow DM = 3\sqrt{10}$. The perimeter of $\triangle DMC$ is $6 + 6\sqrt{10}$.



23. Note that the line through Q and Q' is a line of symmetry for the figure; this line will intersect $\overline{AB}$ and $\overline{CD}$ at their midpoints, N and M, respectively. Note also that quadrilateral $AQ'BQ$ is a square with its diagonal equal to the length of a side of $ABCD$. Let $AB = s$. Then $AN = DM = Q'N = s/2$. Consider right triangle DMQ' with short leg $s/2$ and long leg $Q'N + NM = 3s/2$. This triangle and right triangle FNQ' have a common angle, $\angle FQ'N$, so the triangles are similar. Thus, $FN : Q'N = 1 : 3$, implying that $FN = (1/3)(s/2) = s/6$. Since $FG = 2FN$ by symmetry, $FG = s/3$. We know that $AF = GB$ (also by symmetry); each must be

$$\frac{s-(s/3)}{2} = \frac{s}{3}.$$



Alternate solution: Use a different pair of similar triangles—namely, $\triangle Q'FG$ and $\triangle Q'DC$. They are similar because $\overline{FG} \parallel \overline{DC}$.

24. 19, 37, 73. Observe that $c > b > a$. The largest two-digit prime is 97. Suppose that $c = 97$. Then

$$\frac{97+1}{2} = 49 \text{ and } \frac{49+1}{2} = 25.$$

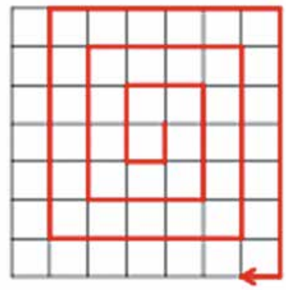
Clearly, 49 and 25 are not prime, but we have learned that a must be a two-digit prime less than 25. There are exactly five such primes: 11, 13, 17, 19, and 23. Rearrange the second formula to write $b = 2a - 1$. Test the five candidates for a: $2(11) - 1 = 21$; $2(13) - 1 = 25$; $2(17) - 1 = 33$; $2(19) - 1 = 37$; $2(23) - 1 = 45$. Only 37 is prime. Confirm that $2(37) - 1 = 73$ also yields a prime.

25. 6. Write the condition as a quadratic inequality:

$$2x^2 - 2x \le 12$$
$$x^2 - x - 6 \le 0$$
$$(x - 3)(x + 2) \le 0$$

The product equals 0 when $x = 3$ or -2. The product is less than 0 when the two factors have opposite signs—that is, when $x - 3 < 0$ and $x + 2 > 0$. Thus, $x < 3$ and $x > -2$. The integers $\{-1, 0, 1, 2\}$ satisfy this conjunction. (Note that $x - 3$ is always less than $x + 2$; no numbers satisfy the conjunction $x - 3 > 0$ and $x + 2 < 0$.) The solution set contains the six integers between -2 and 3, inclusive.

Alternate solution: Graph the corresponding quadratic function, $f(x) = 2x^2 - 2x - 12$. When the graph of the parabola intersects the x-axis or lies below the x-axis, the values of the function are less than or equal to zero.



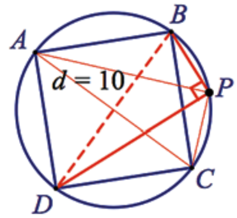
26. 40 turns. Because the diagram is drawn to scale, the grid is uniform, with the distance between consecutive vertical lines the same as the distance between consecutive horizontal lines. Four miles is equivalent to 5,280(4) = 21,120 ft. Examining David's walk, we observe that as he makes his 2nd turn, he has walked $(1 + 1)50 = (1)2 \cdot 50 = 100$ ft. As he makes his 4th turn, he has walked $(1 + 2)2 \cdot 50 = 300$ ft., and as he makes his 6th turn, he has walked $(1 + 2 + 3)2 \cdot 50 = 600$ ft. In general, as he makes his $2n$th turn, he has walked $(1 + 2 + 3 + \cdots + n)2 \cdot 50$ ft. The sum of the integers from 1 to n is $(n^2 + n)/2$, so we wish to have $((n^2 + n)/2) \cdot 2 \cdot 50 \le 21{,}120$. Simplify the inequality to $5n^2 + 5n \le 2{,}112$. Solve the quadratic for positive n to obtain $n \le 20.1$. To confirm the value $n = 20$, calculate that when David makes his 40th turn, he has walked $(1 + 2 + \cdots + 20)100 = 21{,}000$ ft. The distance he would walk between the 40th and 41st turns is $21 \cdot 50 = 1{,}050$ ft., so David completes 4 miles before the 41st turn.

27. Integers 1, 4, and 16 with divisors $\{1\}$, $\{1, 2, 4\}$, $\{1, 2, 4, 8, 16\}$, respectively. We could begin by searching for all geometric sequences made up of the integers from 1 to 20. There are at least 10 such sequences, so we turn instead to Divya's clue about the number of

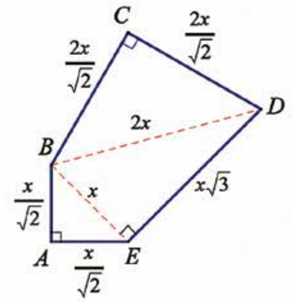
divisors. Since each number has an odd number of divisors, the clue implies that the three numbers must be perfect squares. So a, b, and c are elements of the set $\{1, 4, 9, 16\}$. The three integers must be 1, 4, and 16. The integer 1 has exactly one divisor—itself; $\{1, 2, 4\}$ is the set of the divisors of 4; and $\{1, 2, 4, 8, 16\}$ is the set of the divisors of 16. So $d(a) = 1$, $d(b) = 3$, and $d(c) = 5$, forming an arithmetic sequence.

We did not need the last clue to solve Divya's puzzle, but we use the opportunity to make three observations. The number 1 is the only number to have a single divisor; only squares of primes have exactly three divisors; and only perfect squares have an odd number of divisors.

28. 200. Label the vertices A, B, C, and D. Note that the segment joining opposite vertices B and D forms a diameter of the circle. Therefore, since $\angle BPD$ is inscribed in a semicircle, it must be a right angle, and $\overline{PB}$ and $\overline{PD}$ are two legs of a right triangle with hypotenuse 10. The sum $PB^2 + PD^2 = 100$. Use the same reasoning to conclude that $PA^2 + PC^2 = 100$.



29. 6. Draw the two diagonals from B and note that they are a leg and the hypotenuse of a right triangle with right angle $\angle BED$. Let $BE = x$. Then $BD = 2x$, and $DE = x\sqrt{3}$. Now $\triangle ABE$ and $\triangle CBD$ are isosceles right triangles. In terms of x, $AB = AE = x/\sqrt{2} = x\sqrt{2}/2$, and $CB = CD = 2x/\sqrt{2} = x\sqrt{2}$. So the perimeter is $x\sqrt{3} + 3x\sqrt{2}$, which contains two unlike radicals. To match the form $a + a\sqrt{a}$, we try scaling by a factor of $\sqrt{2}$ or $\sqrt{3}$ to obtain $x\sqrt{6} + 6x$ or $3x + 3x\sqrt{6}$, respectively. Use the latter expression with $x = 2$ to obtain $6 + 6\sqrt{6}$, which matches the given form.



30. 1414. The number pattern indicates that the sum of the cubes from 1^3 to n^3 equals the square of the sum of the integers from 1 to n. So we need n such that

$$\begin{aligned}(1 + 2 + 3 + \cdots + n)^2 &\geq 10^{12}\\ 1 + 2 + 3 + \cdots + n &\geq 10^6\\ (n^2 + n)/2 &\geq 10^6\\ n^2 + n &\geq 2(10^6).\end{aligned}$$

Completing the square results in

$$\begin{aligned}(n + 0.5)^2 &\geq 2(10^6) + 0.25\\ n + 0.5 &\geq \sqrt{2 \cdot 10^6 + 0.25}\\ n &\geq 1000\sqrt{2} - 0.5 \approx 1413.7.\end{aligned}$$

Round up to 1414.