

LEARNING ABOUT FUNCTIONS THROUGH

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LEARNER-GENERATED

In an introductory algebra course, LGEs engage students in writing about

In many mathematics classrooms, the teacher provides “worked examples” to demonstrate how students should perform certain algorithms or processes. But are these examples really helping students learn the concepts? Do students really see and conclude what the teacher is hoping they will see? Some students find it difficult to generalize from the examples that teachers provide and cannot apply what they have learned in new situations (Watson and Mason 2002). Instead, teachers might consider asking students to generate their own examples.

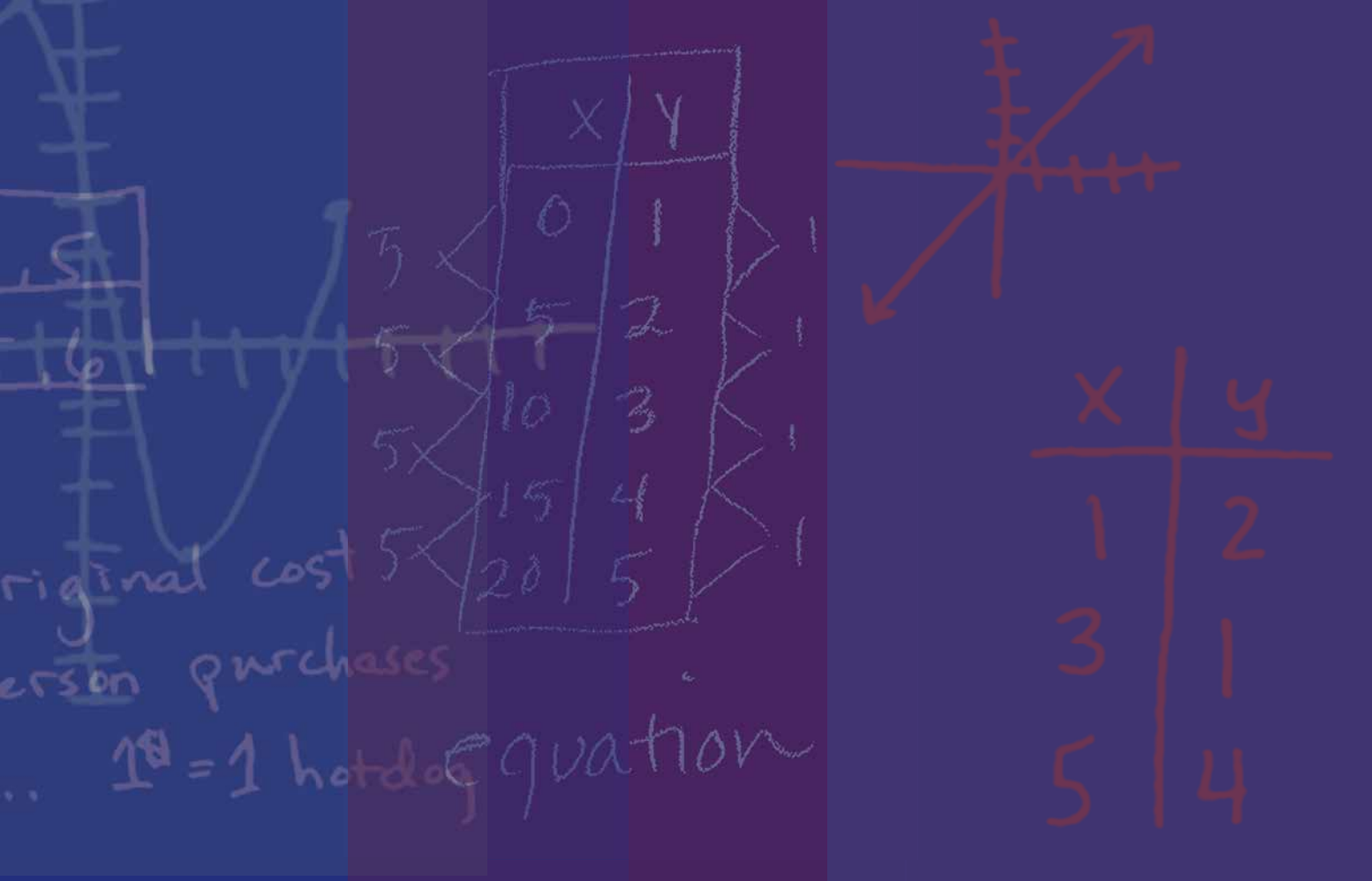
Examples created by students are often called learner-generated examples (LGEs) and have proven to be a powerful teaching tool (Dinkelman 2013; Bills et al. 2006; Meehan 2007; Watson and Mason 2002; Watson and Shipman 2008). When students generate their own examples, they behave more like mathematicians, drawing on connections and taking ownership of the concepts. As a result, generating examples can be motivating for students at all levels (Watson and Mason 2002; Watson and Shipman 2008). Students benefit by becoming better problem

solvers (Watson and Mason 2002) and developing a rich array of example types (Dinkelman 2013). Further, when students generate their own examples, they reveal information about their thinking that is not readily available otherwise (Sinclair et al. 2011). Thus, LGEs are informative tools for assessing student thinking.

Classroom tasks that involve LGEs are easy to design and fold into existing curriculum materials; see Watson and Mason (1998, 2005) for a list of different types of tasks and questions. They also can be tailored to address one or more of the Standards for Mathematical Practice (SMPs) within the Common Core State Standards for Mathematics (CCSSI 2010).

Consider this sample LGE task:

1. Give an example of a linear equation.
2. Change one feature in your example to give a different straight line.
3. Make similar alterations to get new straight lines.



ED EXAMPLES

mathematics and can be used as a formative assessment tool.

By working through the task, you may have noticed that repeatedly changing the same feature encouraged you to pay attention to mathematical structure (SMP 7, CCSSI 2010, p. 8)—in this case, the structure of linear equations. Because the task invites the learner to generate multiple examples of the same type, it creates an opportunity for students to make generalizations about certain features of linear equations (SMP 8, CCSSI 2010, p. 8). If the teacher were then to ask students to share their examples and observations with others, then students would have the opportunity to communicate their mathematical ideas and critique the reasoning of others (SMP 3, CCSSI 2010, pp. 6–7).

In this article, we share some of the LGE tasks we developed for an instructional unit on functions for students in an introductory Algebra 2 course—a slower-paced course that introduces most but not all the concepts in the regular Algebra 2 course. The student examples shared come from two sections of the course in which a total of fifty students were enrolled. The lead author taught

both sections. About 58 percent of the students were seniors; the rest were juniors, about 50 percent of whom were male. We focus our discussion on the how LGE tasks can be used as a formative assessment tool, providing information about how students' ideas develop over time.

LGE TASKS FOR FUNCTIONS

The concept of functions is arguably one of the most important topics in mathematics. It can serve as a connection between different areas of mathematics, such as algebra and geometry, and affords a direct connection to other disciplines, such as environmental sciences, when using functions as mathematical models. Unfortunately, many students have a narrow view, thinking of functions as simply an operation or a procedure (Clement 2001; Oehrtman, Carlson, and Thompson 2008). We created a series of LGE tasks to formatively assess student thinking about functions throughout the unit. The LGE tasks were designed to regularly engage students in communicating about what counts as

TASKS BASED ON LEARNER-GENERATED EXAMPLES FOR A FUNCTIONS UNIT

LGE Task 1 (Context)

Write down two real-life quantities that are related in any way.

Example: Time spent studying and grades on tests

Now describe how they are related.

Example: For the most part, the more time spent studying a particular subject, the better grade you will get. Of course, if you study too much and don't get sleep, that relationship will decline.

LGE Task 2 (Modify)

Given a table of values, change something about this function that would make it a nonfunction.

LGE Task 3 (Everything)

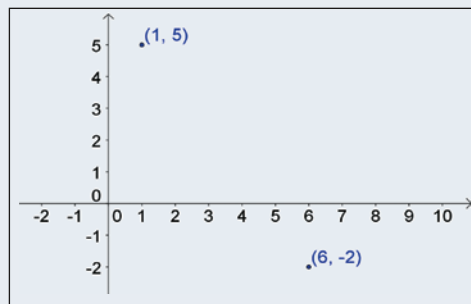
Tell me everything you know about functions.

LGE Task 4 (Is and Is Not)

Write down an example of something that is a function and something that is not a function.

LGE Task 5 (Construct and Specify)

(a) In the given coordinate system, draw a graph of a function that connects the two given points.



(b) What is the domain of your function?

(c) What is the range of your function?

LGE Task 6 (Modify)

What must be removed or changed to make $\{(0, 1), (2, 3), (4, 7), (5, 7), (2, 6)\}$ a function?

LGE Task 7 (Construct to Specifications)

Find an example of a function whose domain is $-2 \leq x \leq 5$ and whose range is $-6 \leq y \leq 7$.

LGE Task 8 (Representation \times 4)

Write down an example of a function.

Represent that same function in a different way.

Represent that function in third way.

Represent that function in a fourth way.

LGE Task 9 (Construct to Specifications)

Draw a relation that is a function for $x > 0$ but not for all real numbers.

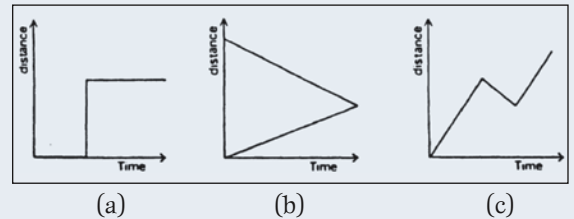
LGE Task 10 (Modify)

Change the following table in some way so that it represents a linear function:

x	y
-1	10
0	13
3	19
4	25

LGE Task 11 (Context)

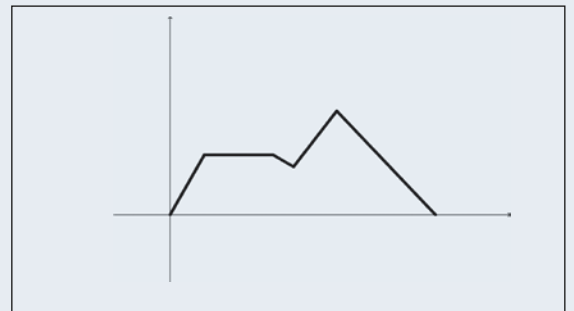
Which of the graphs below represent journeys? Describe what happens in each case.



Source: Leinhardt, Zaslavsky, and Stein (1990), p. 39

LGE Task 12 (Context)

Tell me a story that could be represented by the graph below.



LGE Task 13 (Representation \times 2)

Give an example of a function. Write that function using two different representations, both of which reveal the rate of change of the function.

LGE Task 14 (Construct to Specifications)

Can you find an example of a function for which one representation looks misleading? (For example, can you find an example of a function for which one representation looks linear but another does not?)

Fig. 1 Many of these tasks can be modified to address different mathematics topics.

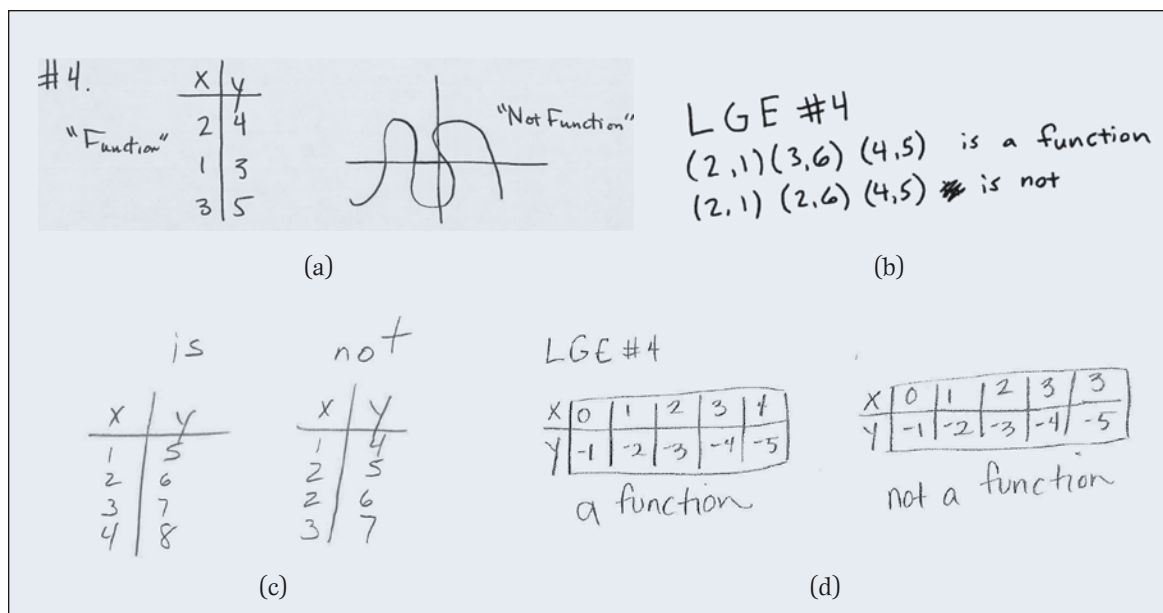


Fig. 2 Sample work by Alex (a), Brandi (b), Cole (c), and Danielle (d) addresses the Is and Is Not task (LGE task 4).

an example of a function as well as the use of different representations.

Students were required to complete fourteen LGE tasks during the unit, including tasks that prompted students to consider nonexamples, determine whether a given relation was a function, or generate a function given particular constraints. A complete list of the LGE tasks used during the unit is included in **figure 1**. Some of the tasks were direct, requiring students to give an example of a function that meets particular restraints (see LGE task 7). Other tasks were more open-ended, asking students to write down everything they know about functions (see LGE task 3). The nature of the tasks depended on the assessment goal. At times, it was important to know whether students could generate something particular; at other times, it was important to know what ideas and examples students associated with functions.

Students kept their responses to all the LGE tasks in a spiral notebook. Each time they completed a task, their responses were assessed using a three-point scale: zero points were awarded for no attempt; 1 point was awarded for an incorrect attempt; and 2 points were awarded for a correct attempt.

Four of the tasks were more generally focused on the function concept and are considered more carefully here. Students addressed LGE tasks 3 and 4 early in the unit, after the class had learned the definition of function. Near the middle of the unit, after reviewing different types of representations—numerical (tabular and other lists of ordered pairs); graphical; symbolic; and narrative (using words to describe a contextual situation)—students encountered LGE task 8. The focus on multiple representations was maintained in LGE task 13, which

occurred toward the end of the unit.

STUDENTS' EXAMPLES DEMONSTRATE IMPROVED UNDERSTANDING

On the second day of the unit, students were asked to respond to the Everything task (LGE task 3). Their responses varied widely and thus did not provide helpful feedback about how the class as a whole was thinking about functions. Some students provided examples, and some attempted to restate the definition. A handful wrote that they did not know anything, yet they proceeded to give some information about functions. That same day the students were asked to complete the Is and Is Not task (LGE task 4); by doing so, they demonstrated that they had some understanding of what counts as a function and what is not a function (68 percent of students earned a 2 on this task).

The evidence from both tasks indicates that students who were successful understood a fundamental aspect of the function concept—that a given domain value must be mapped to one and only one value in the range. They could represent that idea with examples but struggled to articulate that idea when prompted to “tell everything you know” earlier that day. One implication we took from these results was that students needed additional experience and support with writing about mathematics.

Students' responses to the Is and Is Not task presented in **figure 2** show their overwhelming tendency toward numerical representation. More than 80 percent of students gave their example in tabular form, and more than 90 percent used some type of numerical format (tables or ordered pairs) to represent their function. Interestingly, Alex (all names are pseudonyms) provided a graph of a nonfunction

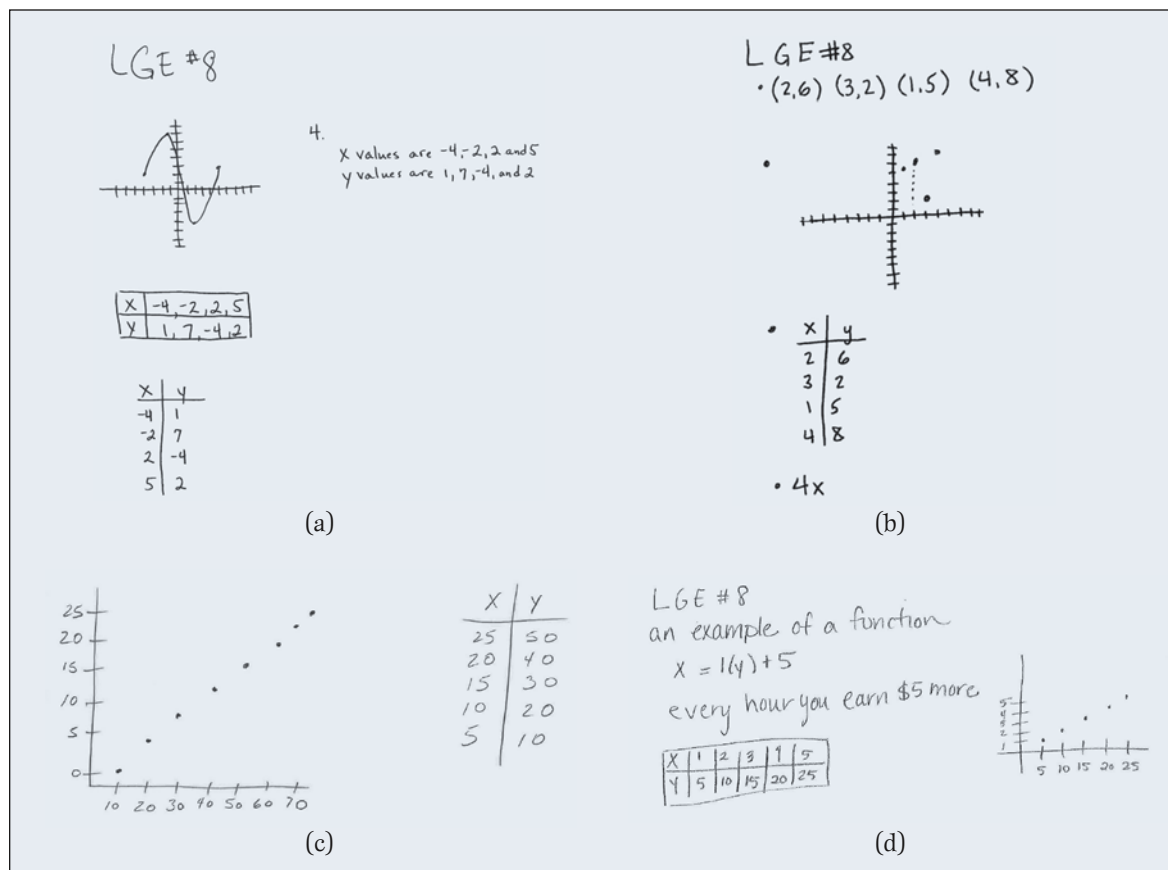


Fig. 3 Sample student work by Alex (a), Brandi (b), Cole (c), and Danielle (d) addresses the Representation \times 4 task (LGE task 8).

that was not an example provided in class. Other students used tables to illustrate how a relation can fail to be a function by assigning two y -values to a given x -value.

By the time the students responded to the Representation \times 4 task (LGE task 8), more than 96 percent were able to generate a correct example of a function but could not quite think of four ways to represent that function. Just as in the Is and Is Not task, many students' first representation was often a table. Approximately 63 percent of students generated a table first, and 75 percent gave some type of numerical representation (either a table or ordered pairs) as their first response. **Figure 3** shows the work of four students and is representative of others who were successful in generating an example of a function to start this task.

When examining the responses to the Representation \times 4 task, we noticed that about 40 percent of students interpreted "a different representation" as simply a different way of listing the ordered pairs. For example, Alex created two tables, one oriented horizontally and the other oriented vertically. His response was certainly creative and came close to the desired outcome in the sense that each representation appeared to match the information provided in the others; notice, however, that the list of x - and

y -values on the upper right of his paper does not explicitly communicate the correspondence between them (see **fig. 3a**).

Every student could generate a function in numerical form, but not all could represent the function using all four types of representations discussed in class (numerical, graphical, symbolic, and narrative). Brandi and Danielle did make attempts to represent their functions symbolically, but those representations do not match their other representations (see **figs. 3b** and **3d**). Interestingly, Danielle did generate four different types of representations and seemed to have a productive idea about a context that could be modeled with a linear function. Notice that her symbolic representation, although not written correctly, appears to be expressing the change in function values—that is, a recursive view of the function (see **fig. 3d**). In this sense, Danielle's work came very close to meeting the desired outcome.

One primary unit goal was for students to be able to use all four representations of functions. Students' responses (toward the end of the unit) to the Representation \times 2 task (LGE task 13) in relation to their responses to the Is and Is Not task and the Representation \times 4 task (LGE tasks 4 and 8), show clear improvement in their ability to generate functions in all forms. Although the Representation \times 2 task did

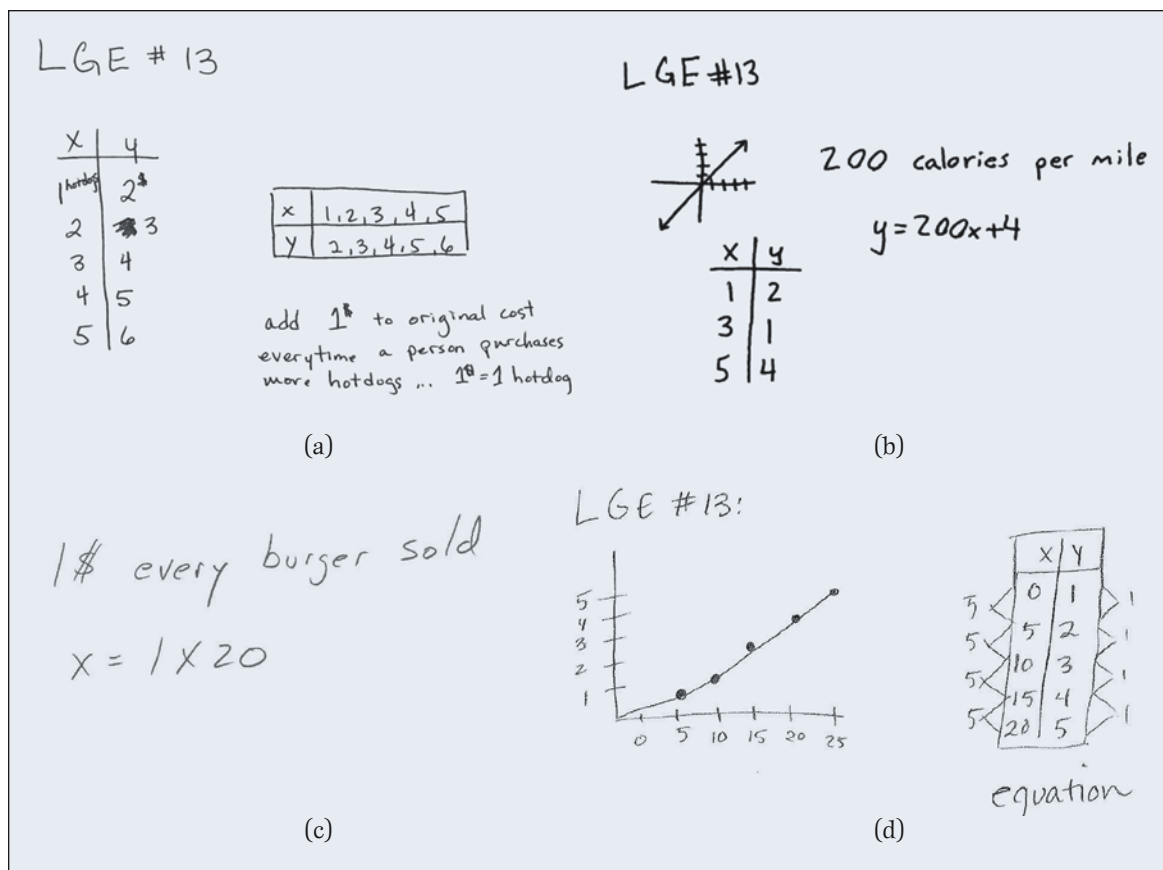


Fig. 4 Sample work by Alex (a), Brandi (b), Cole (c), and Danielle (d) addresses the Representation \times 2 task (LGE task 13).

not require a specific format, it elicited more variety in representations (see **fig. 4**). Students still used tables (80 percent), but graphs, equations, and words appeared in roughly similar proportions (between 40 and 55 percent each).

Notice that Alex, Brandi, and Cole all attempted to generate an example of a function in written form in the Representation \times 2 task (LGE task 13), whereas they had not in the previous tasks. Danielle, however, seemed to be comfortable with all representations from the beginning of the unit. We were encouraged to see students demonstrating their knowledge of function with a range of representation types, especially their examples of contextual situations, given the earlier results from the Everything task (LGE task 3).

LGE TASKS AS A FORMATIVE ASSESSMENT

These student-generated examples illustrate how teachers can use LGE tasks to determine what their students know and whether their knowledge of a certain topic improves throughout a unit. By using these tasks, we could easily determine whether students understood simple definitions or procedures and how their understanding changed as the unit progressed. At the beginning of the unit, students

rarely represented their functions in any form but numerical. Thus, it seemed that students had a lot of experiences with numerical form and needed more exposure to the three other types of representations. This information was extremely valuable for informing instruction. The students needed more time and instruction to master the use of all representations. Danielle and other students like her felt comfortable with functions in all four types of representations from the beginning of the unit but needed more experience generating four representations of the same function.

Engaging in these tasks was, for some students, something very different from what they had been asked to do before—to think for themselves—and they were not always sure what to do. In some cases, especially at the beginning of the unit, it was necessary for the teacher to explain the task prompts and sometimes give hints. The students' struggle in trying to generate their own examples was not bad in an instructional sense. In fact, seeing them grapple with mathematics was satisfying from the teacher's perspective. Students were having mathematical discussions and were verbally wrestling with mathematics. As the year and the unit progressed, students became more comfortable with the LGE tasks and required less coaching. In this way, the use of LGE

tasks enabled the teacher to assess students' ability to adapt to new ways of doing mathematics.

The LGE tasks also afforded opportunities to assess students' skills in writing mathematically. Writing with a mathematical purpose seemed to be a new experience for them, and they needed more practice. Although a struggle at first, writing became easier for the students as the year progressed.

GETTING STUDENTS TO THINK MATHEMATICALLY

Using learner-generated examples to engage students in writing about mathematics not only is effective but also takes very little planning time, once the unit goals are decided. Many tasks used during the functions unit can be adapted according to the topic of instruction. Giving examples and nonexamples was a quick way to see whether students could work with a definition. Asking students to change a given nonexample to an example of a function requires them to identify the faulty characteristic and then figure out what change to make. This type of task is quite complicated and took more attention from the students than originally anticipated, even though it is a quick task to create. Both these tasks can apply to a variety of concepts and levels.

As we think about ways to address the Standards for Mathematical Practice, LGEs offer teachers a strategy that can easily be incorporated to show how students are thinking about a concept and whether they really grasp the important ideas. With a little help, all students can successfully generate examples of their own. And teachers are better able to find misconceptions using these types of tasks than with traditional worksheets or homework assignments.

Using LGE tasks is an effective way to assess students' understanding, their comfort level in grappling with mathematical ideas, and their skills in writing with a mathematical purpose.

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