Exploring New Ge


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Familiar sets of points defined by distances appear as unexpected shapes in larger-distance geometry.

Wayne Nirode

After my honors geometry students have studied Euclidean plane geometry, I like to broaden their conception to include different geometric worlds. One way is to have students explore a new way of measuring distance on the plane (e.g., taxicab geometry). When students work with a non-Euclidean distance formula, geometric objects such as circles and segment bisectors can look very different from their Euclidean counterparts. Students and even teachers can experience the thrill of creative discovery when investigating these differences among geometric worlds.

DISTANCE AND SHAPE IN EUCLIDEAN AND TAXICAB GEOMETRIES

Although Euclid defines the distance between two points with coordinates (x_1, y_1) and (x_2, y_2) as the familiar $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, taxicab geometry uses a different calculation: $d_T = |x_2 - x_1| + |y_2 - y_1|$. For example, if a taxicab were traveling from point A to point B, it travels 5 city blocks (see **fig. 1**). Thus, the taxicab distance between the two points is 5. (For more information on taxicab geometry, see Krause [1986], Dreiling [2012], and Smith [2013].)

After studying Euclidean geometry for most of the school year, my students worked through exercises in Krause (1986) that guided them to create shapes from different sets of points that satisfied constraints based on distance. The sets listed here describe familiar objects in Euclidean geometry, identified in parentheses. For example, the first solution is "the set of all points P such that the distance from P to A is equal to k (a constant)," the

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definition of a circle. Five sets of interest follow:

1. $\{P|d(P, A) = k\}$ (circle)
2. $\{P|d(P, A) + d(P, B) = d(A, B)\}$ (segment, points between two given points)
3. $\{P|d(P, A) = d(P, B)\}$ (perpendicular bisector of a segment, points equidistant from two given points)
4. $\{P|d(P, A) + d(P, B) = k\}$ (ellipse)
5. $\{P|\ |d(P, A) - d(P, B)| = k\}$ (hyperbola)

Students explored these sets in both Euclidean and taxicab geometry.

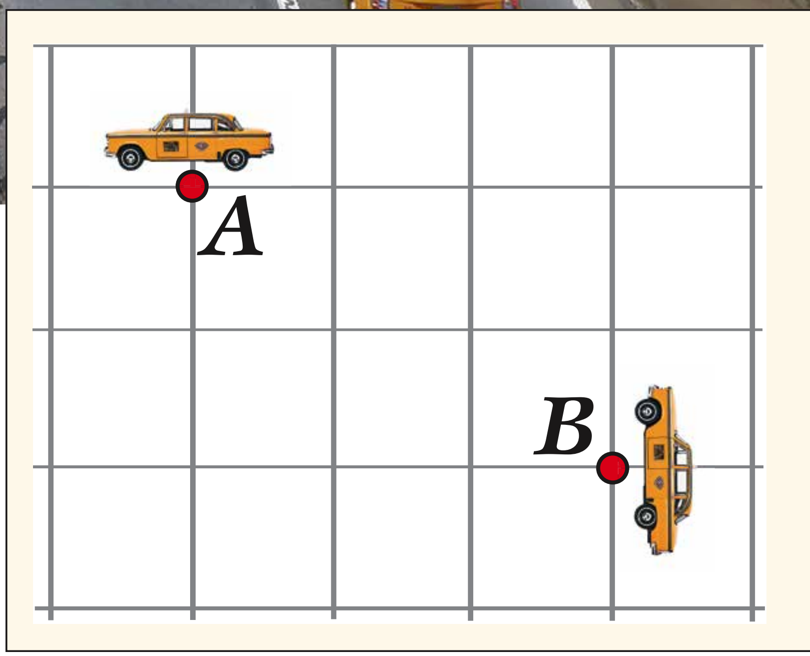

Fig. 1 In taxicab geometry, the distance between *A* and *B* is 5 units.

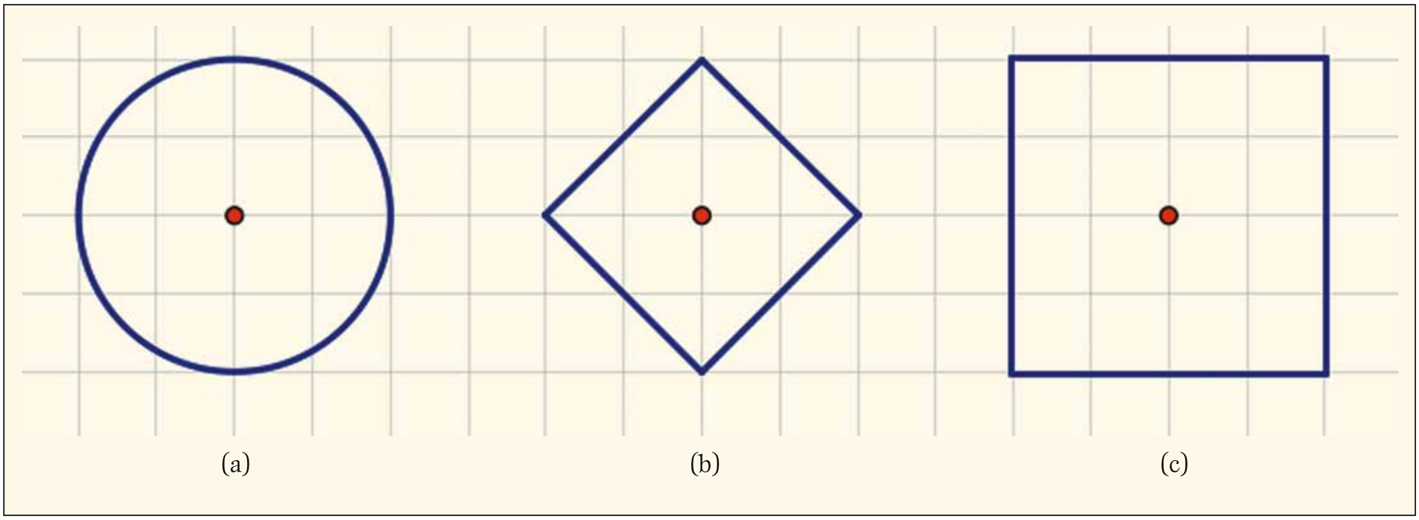

Fig. 2 Circles with radius of 2 appear differently in Euclidean (**a**), taxicab (**b**), and larger-distance (**c**) geometries.

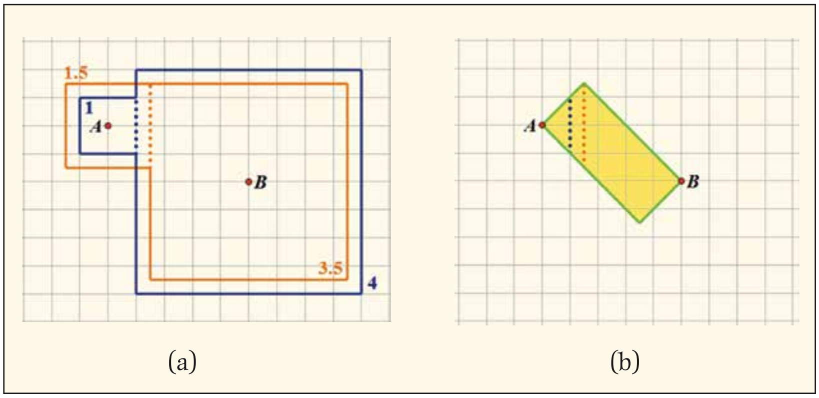

Fig. 3 Intersecting LDG circles are used to find points that are between *A* and *B* (**a**). The complete solution appears as a rectangle and its interior (**b**).

INTRODUCING LARGER-DISTANCE GEOMETRY (LDG)

Next, we defined the distance between two points as the larger of the horizontal or vertical distance between them (Krause 1986). If the two points are (x_1, y_1) and (x_2, y_2), this distance is written in mathematical notation as

$$d_L = \text{larger of } |x_2 - x_1| \text{ and } |y_2 - y_1|.$$

Assume that the horizontal separation between the two points is 2, and the vertical separation is 3 (see **fig. 1**); thus, the distance between the two points is 3. We called this new geometry *larger-distance geometry* (LDG).

My students completed a project using dynamic geometry software (DGS) to investigate how familiar geometric objects appear in this geometric world. This was an opportunity for them to apply what they had learned about sets of points to a more unstructured environment than guided exercises. Students were to create representative geometric objects for the previous list of five sets of points.

Students used The Geometer's Sketchpad® (GSP) to complete the project for two reasons. First, I created a custom tool in GSP to measure the larger distance between two points that students could use to explore and test their conjectures. (For the custom tool in GSP, see the online component for this article.) This use of GSP was dynamic because students could create a test point to drag while GSP automatically updated the distance from the test point to an initial point. Second, when students made mistakes, they could undo them quickly until they had created accurate drawings for each case.

Students worked in small groups for six class periods to produce a GSP file that contained representative objects of the five sets of points. When a set of points had multiple cases that needed to be considered, students made a separate page for each case. They also typed a description of how to generalize a solution for the different cases.

RESULTS GRAPHED FOR THE VARIOUS SETS

Circles, created in Euclidean geometry using the compass as a tool, are fundamental to making sense of sets of points. In LDG, a circle appears as a Euclidean square with sides parallel and perpendicular to the gridlines (see **fig. 2c**). Although students typically focused on points with integer coordinates, the coordinates need not be integers. For example, the LGD circle in **figure 2c** is the entire perimeter of the Euclidean square, not just the points with integer coordinates.

The following paragraphs elaborate on the interpretations for "betweenness" and equidistance as well as comparative distance to create ellipses and hyperbolas. The use of concentric circles is discussed only briefly to remind readers

of this method of construction. Instead, the focus is on ways that these objects are different from their Euclidean counterparts and on my students' successes and struggles. This development parallels what many of my students did for this project because initially they used circles, but then they looked for more efficient ways to construct the objects because of the tedium of drawing many circles and finding their intersection points.

Betweenness

The second set listed is "the set of all points P such that the sum of the distances from P to A and from P to B is the same as the distance from A to B." Said another way, P is the set of all points that are between A and B. **Figure 3a** shows points A and B separated by an LDG distance of 5, along with pairs of circles centered at A and B whose radii sum to 5. The pair of blue circles, with radii of 1 and 4, intersect in the dotted vertical line segment. The orange circles have radii of 1.5 and 3.5. The intersections are included in **figure 3b**, as part of the complete solution set. The solution set appears as a Euclidean rectangle along with its interior. The sides have slope of ±1, and the original points locate opposite vertices.

Two other possibilities for the position of points A and B should be considered. If the two original points align either horizontally or vertically (as in **fig. 4a**), then the set of points between them forms a Euclidean square and its interior with the original points as opposite vertices. When the original points lie on a line with a slope of ±1 (as in **fig. 4b**), the solution is a segment connecting them, as in Euclidean geometry. The square and segment are special cases of rectangles where the length equals the width or the width is zero.

When grading the students' final projects, I found that most students did not represent the set of points between A and B as a rectangle but instead as a smaller parallelogram contained within the rectangle. **Figure 5a** shows a student's solution in green for the set of points between A and B. **Figure 5b** shows the correct answer superimposed over the student's solution. Students did not include points (such as D) whose vertical positions are not between the vertical positions of A and B. Notice that, measured in LGD, D is 1 unit from A and 3 units from B—just as point C is 1 unit from A and 3 units from B. The distances are the same.

Equidistance

After analyzing the concept of betweenness, students looked at the set of all points that are equidistant from two given points—that is, points at

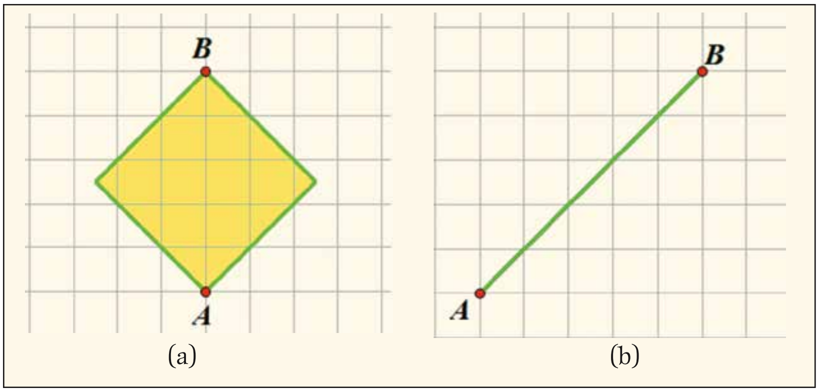

Fig. 4 Two variations (for particular alignments of the initial points A and B) of the solution rectangle appear as a square (**a**) or a line segment (**b**).

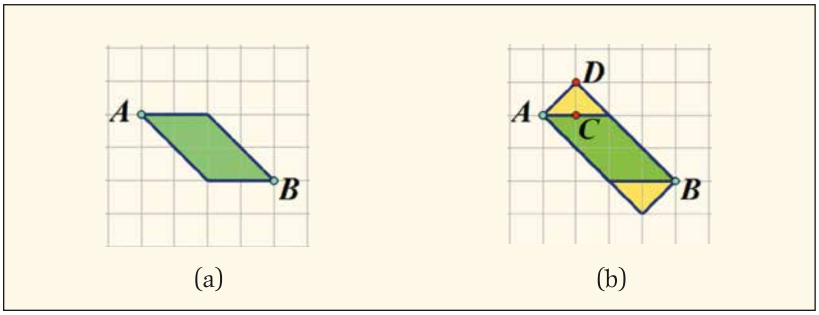

Fig. 5 A partially correct solution (**a**) fails to include some points that are between A and B when measured in LGD (**b**).

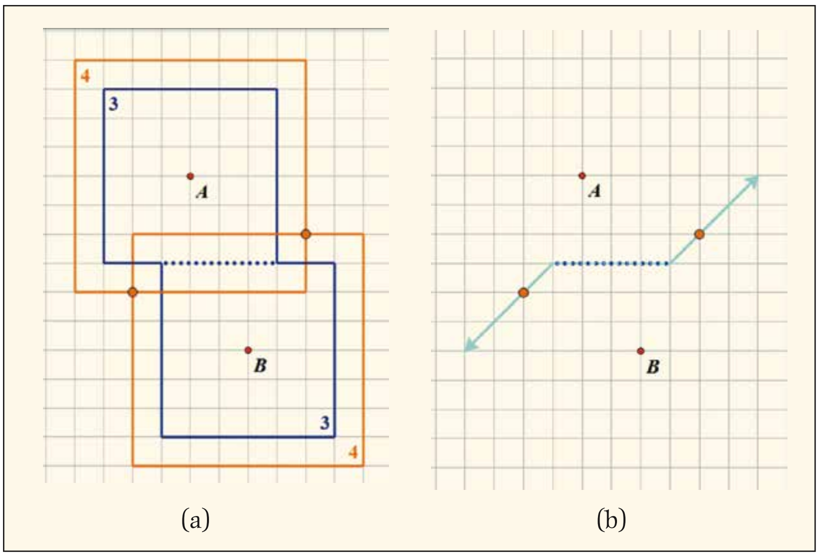

Fig. 6 Circles are used to find points that are equidistant from A and B (**a**). The complete solution is a segment connecting two rays (**b**).

the intersection of two equal-radii circles centered at the original points. **Figure 6a** shows a pair of circles (shown in blue) intersecting at a segment and another pair (shown in orange) intersecting at two points. The complete solution consists of a horizontal segment connected to two rays, each with a slope of 1. (See **fig. 6b**.) Slight variations of the graph of the solution depend on the position of the original points, A and B, and the slope of the line

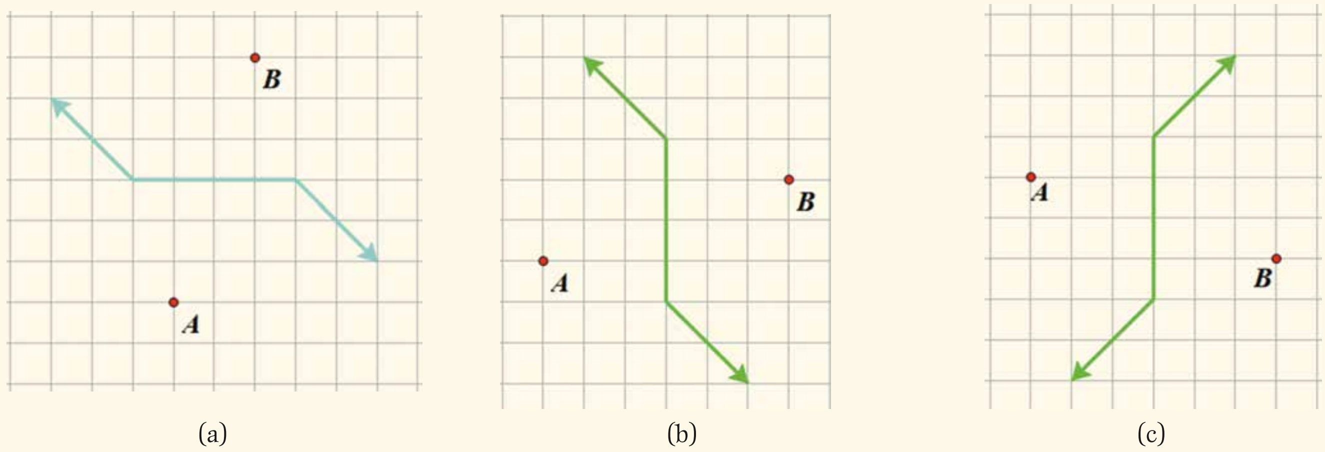

Fig. 7 The condition of equidistance to points *A* and *B* results in a solution set that depends on the slope of the line *AB*.

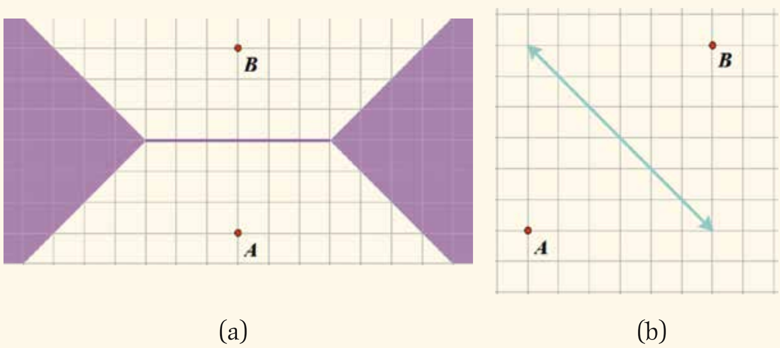

Fig. 8 The set of all points equidistant from *A* and *B* depends on the alignment of *A* and *B*.

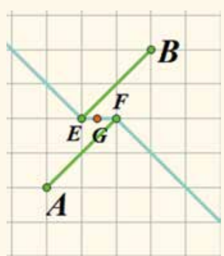

Fig. 9 A student procedure locates the set of all points equidistant from *A* and *B*.

connecting them. The rays of the solution set and the line joining *A* to *B* have slopes with opposite signs. For example, in **figures 7a** and **7b**, segment *AB* has a positive slope; therefore, the rays have a slope of –1. Also note that the segment connecting the rays is either horizontal (see **fig. 6b** and **7a**) or vertical (see **fig. 7b** and **7c**), depending on whether the absolute value of the slope of line *AB* is larger than or less than 1.

In addition, two other cases in LDG relate to equidistance. When the points *A* and *B* lie on a horizontal or vertical segment, the solution includes two infinite regions extending from a segment (see **fig. 8a**). When the two points lie on a line segment with a slope of ±1, the solution is the same as the Euclidean perpendicular bisector (see **fig. 8b**).

One student group devised an efficient method to find all the points that are equidistant from two given points for the first case, where the slope of the segment *AB* is not 0, ±1, or undefined (see **fig. 9**). As this group of students pointed out in their comments,

> You first must find the middle segment [*EF*]. This middle segment can be found by drawing a 45° line from *A* to the point [*F*], which is half the distance between *A* and *B*. Repeat that step for *B* to find the other point [*E*] . . . The diagonal [rays] will always be on a 45° line from the [endpoints of the] middle segment. These 45° lines must go away from both points at the same time.

To further clarify the students' explanation, we note that the slope of the auxiliary lines (*AF* and *BE*) must have the same sign as the slope of the segment connecting the two original points (*A* and *B*). In **figure 9**, the slopes of lines *AF*, *BE,* and *AB* are all positive.

Ellipses

The shape of an LDG ellipse is either a Euclidean octagon or a Euclidean hexagon. **Figure 10a** shows pairs of circles centered at *A* and *B* with radii that sum to 5, the focal constant. The two blue circles have sides that intersect on the dotted segment, and the two orange circles intersect at distinct points. **Figure 10b** shows the completed ellipse.

Just as we found with the condition of

equidistance, two more cases for the distance-sum ellipse exist. If A and B lie on a horizontal or vertical line segment, an octagon with horizontal and vertical symmetry is formed (see **fig. 11a**). When the points lie on a segment with a slope of ± 1, the solution set is a Euclidean hexagon (see **fig. 11b**).

A student group devised a procedure to draw families of solutions to the ellipse condition of requiring a constant sum of distances. **Figure 12** shows the students' work for the three cases. The students' explanation that follows refers to **figure 12a**:

> The dotted lines are created with making a line with a slope of 1 and –1 from A and B. The distance from A to B is 3 and is measured vertically. Take what k equals (for this example, we'll use 8) and subtract the distance from A to B, giving you 5. Divide that number by 2, and you get 2.5. Go that distance vertically (because the distance is measured along here) from both points, giving you points Y_a and Y_b. Add 0.5 to that, giving you 3. Go that distance horizontally, giving you X_a and X_b. Make a line that is horizontal through the Y points and extend this to the dotted lines. Do the same using the X points vertically. Connect the points to create an octagon [representing the LDG ellipse].

Even though the students did not explicitly mention it, the dotted lines in **figure 12a** intersect to form the betweenness rectangle. They extended these lines to use as a guide to assist in finding the sides of the LDG ellipse. In **figures 12b** and **12c**, they also used betweenness as a starting point.

Hyperbolas

As with the other sets, we could anticipate three cases for an LDG hyperbola based on the alignment

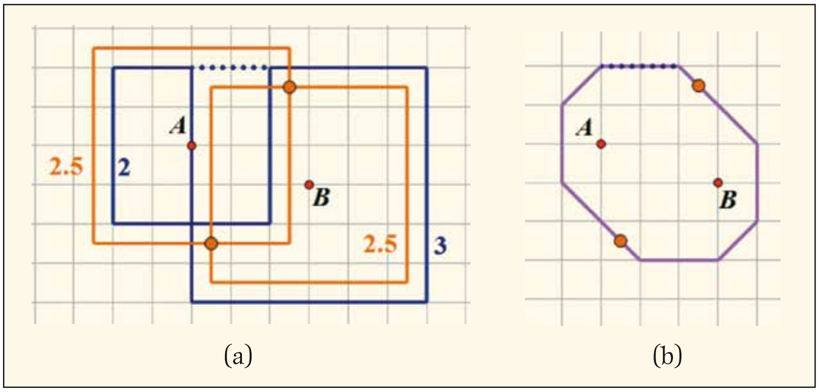

Fig. 10 Points located so that the sum of their distances to A and B is 5 (**a**) contribute to the complete solution in the shape of an octagon (**b**).

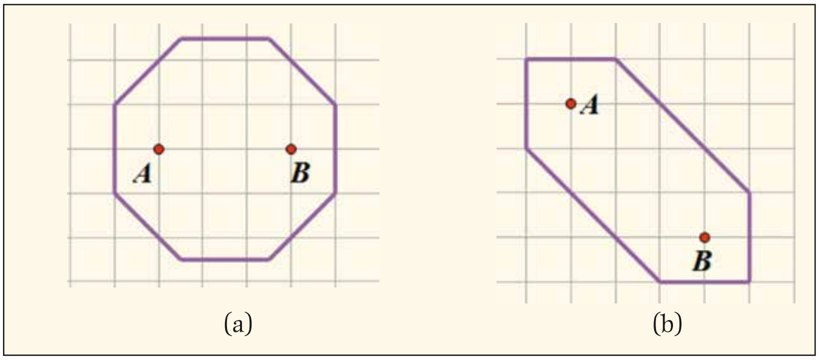

Fig. 11 The ellipse condition of a constant sum of distances creates an octagon with horizontal and vertical symmetry (**a**) or a hexagon (**b**), depending on the position of A and B.

Because they are related to equidistance, families of hyperbolas also have three cases, depending on the alignment of the original two points.

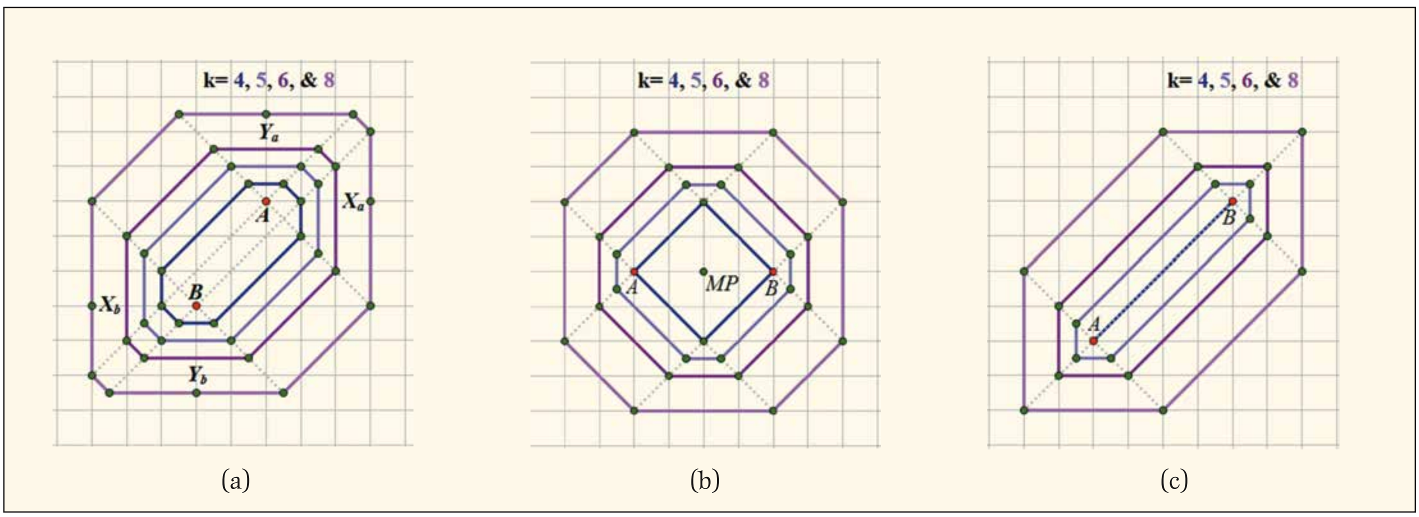

Fig. 12 A student group's graphs show ellipse families for three different pairs of points.

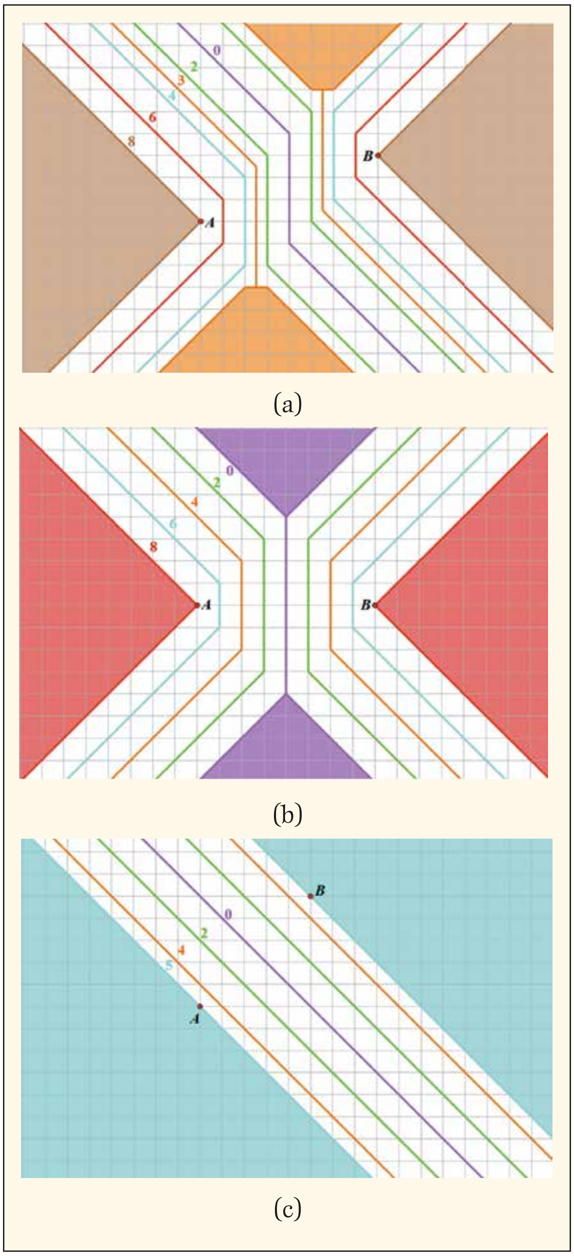

Fig. 13 Three different types of hyperbola families exist.

of the original two points. My students struggled with creating and describing the graph of a hyperbola because there are several different possibilities within each case. Constructing hyperbola families by varying the constant difference can help make sense of the graphs. Although students had completed a taxicab geometry exercise with hyperbola families, none did this for LDG. Consequently, most groups had only a partially correct answer.

This enrichment project gave my students a deeper understanding of Euclidean objects defined in terms of distance.

To start a hyperbola family, use a special case of a hyperbola's equation where the constant difference is zero, giving the condition of equidistance. Once the equidistance set is constructed, it serves as a landmark for constructing the two branches of a hyperbola with a constant close to zero. Then increase the constant until discovering that LDG hyperbolas cannot have a constant bigger than the distance between the two points.

Because they are related to equidistance, families of hyperbolas also have three cases, depending on the alignment of the original two points. For each case, when the constant difference in the hyperbola's equation equals the distance between the two points, the solution includes an unbounded region (see **fig. 13**). When the two original points lie on a line with a slope of ±1, the LDG distance between them can be measured horizontally or vertically. However, for the two cases when the points do not lie on a line with a slope of ±1, one of the horizontal or vertical separations is not used to measure the LDG distance between the points, and that value provides the constant for another hyperbola with an unbounded region. (Specifically, in **fig. 13a**, the orange hyperbola is associated with the constant 3; and in **fig. 13b**, the purple hyperbola is associated with the constant 0.)

DEEP THOUGHTS ABOUT DISTANCE

This enrichment project enhanced my students' understanding of distance. It gave them a deeper understanding of Euclidean objects defined in terms of distance, including ellipses and hyperbolas that they will analyze more thoroughly when they study conic sections next year. My students saw how results depend on the initial choice of definitions. Last, students had the opportunity to "look for and make use of structure" when comparing and contrasting LDG with Euclidean and taxicab geometry (CCSSI 2010, p. 8). Further, within LDG, students used the structure of three different cases for all the sets of points (except for the circle) based on the same alignment conditions—that is, the general case, then one case where the original points lie on a horizontal or vertical line, and the last case where the original points lie on a line with a slope of ±1.

Like my students, I too acquired a deeper understanding of mathematics as a result of this project. It reinforced for me the importance of the circle to create objects, using a compass in Euclidean geometry but requiring different construction tools for other geometries. Further, until this project, I had never made the connections that betweenness is a

special case of the ellipse and that equidistance is a special case of the hyperbola. Now I realize that a midpoint can be thought of as a point that satisfies both betweenness and equidistance from two other points. With this definition, in geometries other than Euclidean, it is possible for more than one midpoint to exist.

Even though this was an enrichment project with high school honors students, teachers could use it in other ways. They could implement it as an independent project for middle school or high school students or use parts of this project for stand-alone lessons. For example, after studying perpendicular bisectors in Euclidean geometry, students could explore equidistance in LDG. Last, preservice teachers in a geometry course could use this project to help strengthen their understanding of objects defined in terms of distance in Euclidean geometry.

Although this article explored some results of larger-distance geometry, still further investigations await in LDG. Some ideas for further exploration include—

- creating a method for measuring the distance of a point to a line and then exploring sets of points that are equidistant from (a) a point to a line (i.e., a parabola) and (b) between two lines (i.e., an angle bisector);
- investigating points of concurrency as well as inscribed and circumscribed circles for triangles;
- defining and exploring properties of other geometric concepts in LDG—such as parallel lines, area, polygons, chords, and tangents—or even in three dimensions, such as a sphere; and
- inventing individual definitions of distance and choosing different geometric objects to explore.

The spirit of the larger-distance geometry project embodies what I have adopted as a philosophy in my geometry course: Think deeply about simple things.

ACKNOWLEDGMENTS
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REFERENCES
Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://corestandards.org/assets/CCSSI_Math%20Standards.pdf.

Dreiling, K. M. 2012. "Triangle Construction in Taxicab Geometry." *Mathematics Teacher* 105 (6): 474–78.

Krause, Eugene, F. 1986. *Taxicab Geometry: An Adventure in Non-Euclidean Geometry*. New York: Dover Publications.

Smith, Christopher E. 2013. "Is That Square Really a Square?" *Mathematics Teacher* 106 (8): 614–19.

WAYNE NIRODE, Nirode-w@troy.k12.oh.us, teaches mathematics at Troy High School in Troy, Ohio. His interests include geometry, statistics, and technology.

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A Geometer's Sketchpad file with a custom larger-distance tool that students can use to explore and test their conjectures can be found with this article (at **www.nctm.org/mt**). The more4U content is a benefit for NCTM members only.