SEPTEMBER

NCTM® NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Consecutive squares differ by consecutive odd integers—for example, $2^2 - 1^2 = 4 - 1 = 3$; $3^2 - 2^2 = 9 - 4 = 5$; and $4^2 - 3^2 = 16 - 9 = 7$. Find the consecutive positive integers whose squares differ by 2015.

1

The consecutive integers 1, 2, and 3 can replace a, b, and c in the expression $a + b\sqrt{c}$ in six different ways, assuming that each integer appears exactly once in the expression. Determine, without a calculator, which arrangement yields the largest value, which yields the smallest value, and which two expressions are equal.

2

The consecutive angles of a quadrilateral inscribed in a circle have measures $2a$, $4b - 8$, $2a$, and $3b + 20$. What is the measure of the smallest angle?

3

The Fibonacci sequence begins with two 1s. Every term after these two is the sum of the previous two terms. Thus, the sequence begins 1, 1, 2, 3, 5, 8, . . . Let a, b, c, and d be four consecutive terms of the Fibonacci sequence. If $a + b + c = 3194$ and $b + c + d = 5168$, find the value of a.

4

Consider the integers from 2 to 20, inclusive. You can write many of these as a sum of two or more consecutive positive integers. Are there any integers that cannot be written as such a sum? If so, what do the "holdouts" have in common?

5

Two fair dice are tossed. If at least one of the dice shows the number 4, what is the probability that the sum of the two dice is greater than 7?

6

Bert and Ernie are paid by the hour. Bert earns \$4.60 more in 3 hours than Ernie earns in 2. When Bert works 15 hours and Ernie works 12 hours, they earn the same amount. How much does Ernie earn per hour?

(September 7 is Labor Day.)

7

Assume that the sum of k consecutive Fibonacci numbers is divisible by 11. Find the smallest value for k.

8

One of the two diagonals of a square in the xy-coordinate plane has a slope of 2. What are the slopes of the sides of the square?

9

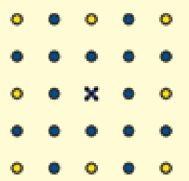
Here is a procedure for generating a Pythagorean triple such that the hypotenuse and long leg are consecutive integers: Select an odd integer for the shorter leg. Square it and rewrite the square as a sum of consecutive integers. Confirm that the procedure works for an odd number of your choice. Then prove that the procedure works for all odd integers.

10

Triangles with integral side lengths and integral areas are called *Heronian triangles*, in honor of the geometer Heron of Alexandria (d. ca. 75 CE). The Indian mathematician Brahmagupta (b. 598 CE) studied a special subset of Heronian triangles, those whose sides are consecutive integers. The two smallest such triangles have sides 3-4-5 and 13-14-15. Find the area of each.

11

The 5×5 array of dots represents trees in an orchard. If you were standing at the central spot marked with an X, you would not be able to see 8 of the 24 trees (shown in yellow). If you were standing at the center of a 9×9 array of trees, how many of the 80 trees would be hidden?

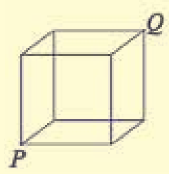


12

Consider an $n \times n$ orchard with n odd, such that the center, C, is located at the origin and each tree is located at a lattice point. (See the problem for September 12.) Devise a method to determine—without looking at the array—whether a given location (x_1, y_1) is visible or hidden from view if you are located at C.

13

Construct a cube from narrow but sturdy straws. The two labeled points, P and Q, are opposite vertices. Suppose that you hold the cube in such a way that you see P but point Q is hidden from your sight. Make a sketch of the figure formed by the straws that you see from this point of view.



14

We have three triangles: one with sides 3-4-5, one with sides 13-14-15, and one with sides 51-52-53. (See the problem for September 11.) In each case, let the even-length side be the base. For each triangle, find the distance between the foot of the altitude to the base and the midpoint of the base.

15

The roots of the given cubic equation are distinct consecutive integers:

$$x^3 - 18x^2 + 107x - d = 0$$

Find the value of d.

16

Consider the product of four consecutive integers—for example, $8 \cdot 9 \cdot 10 \cdot 11 = 7920$. If we add 1, we get 7921, which is 89^2. Try this with four consecutive integers of your own choosing. Then prove that this conjecture is true in general: The product of any four consecutive integers is always 1 less than a perfect square.

17

Consider the set of integers from 100 to 200, inclusive. Some pairs of consecutive integers, as in 100 + 101, can be added without regrouping; other pairs, as in 108 + 109, require carrying a 1 to the tens place. How many pairs of consecutive integers in the set can be added without regrouping? Note: 108 + 109 uses the same pair as 109 + 108.

18

A triangle has consecutive integers as side lengths, such that two side lengths are odd and one side, which will serve as the base, is even. Let M be the midpoint of the base, and let P be the foot of the altitude to the base. Prove that $PM = 2$.

19

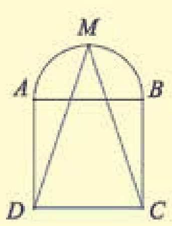
The squares of all two-digit numbers ending in 1 have an even digit in the tens place, and the squares of all two-digit numbers ending in 6 have an odd digit in the tens place. Explain why this is true.

20

An *automorphic number* with n digits has a perfect square that ends in the same n digits as the number itself. For example, 0^2 ends in 0, and 1^2 ends in 1; perhaps more interestingly, $5^2 = 25$ ends in 5, and $6^2 = 36$ ends in 6. There are only two two-digit automorphic numbers. Can you find them?

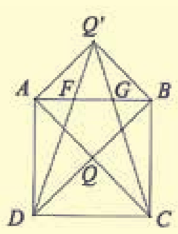
21

Square $ABCD$ has side length 6, and semicircle $\widehat{AMB}$ has midpoint M, such that M lies in the exterior of $ABCD$. Find the perimeter of $\triangle DMC$.



22

Square $ABCD$ has diagonals that intersect at Q. Triangle $AQ'B$ is the reflection of $\triangle AQB$ in $\overline{AB}$. $\overline{Q'D}$ intersects $\overline{AB}$ at F, and $\overline{Q'C}$ intersects $\overline{AB}$ at G. Prove that points F and G trisect $\overline{AB}$.



23

The two-digit prime numbers a, b, and c have the following relationship:

$$\frac{c+1}{2} = b \text{ and } \frac{b+1}{2} = a.$$

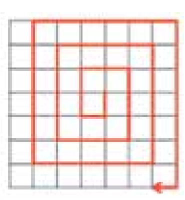
Find a, b, and c.

24

Determine how many integers satisfy the following condition: When twice the integer is subtracted from twice the square of the integer, the result is at most 12.

25

David's morning walk follows a precise pattern—a right-angled spiral—as shown in the figure (drawn to scale but not complete). If David walks 50 ft. before his first turn, how many turns has he made when he has walked 4 miles?



26

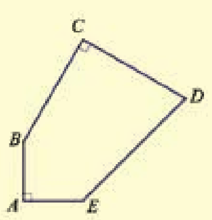
Divya has posed a number puzzle: "I'm thinking of three positive integers—a, b, and c—all between 1 and 20, inclusive. The three integers form a geometric sequence. Each integer has an odd number of divisors, and the number of divisors—$d(a)$, $d(b)$, $d(c)$—form an increasing arithmetic progression. Find my three numbers and their divisors."

27

A square is inscribed in a circle with radius 5. P is a point on the circumference but not on the square. Find the sum of the squares of the distances from P to each of the vertices of the square.

28

Convex polygon $ABCDE$ has $AB = AE$, $CB = CD$, and right angles at A and C. Diagonal BE is perpendicular to side DE, and $BD = 2BE$. If the perimeter of the polygon can be written as $a + a\sqrt{a}$, find the value of a.



29

Observe the first three rows of a number pattern: $1^3 = 1^2$; $1^3 + 2^3 = (1 + 2)^2$; $1^3 + 2^3 + 3^3 = (1 + 2 + 3)^2$. Write the next two rows to the pattern and confirm that your equations are true. What is the smallest integer n such that the sum of the first n cubes is at least 10^{12}?

30