CORVETTES, CURVE FITTING, AND CALCULUS

A nonroutine task with three hallmarks of a good problem offers the flexibility

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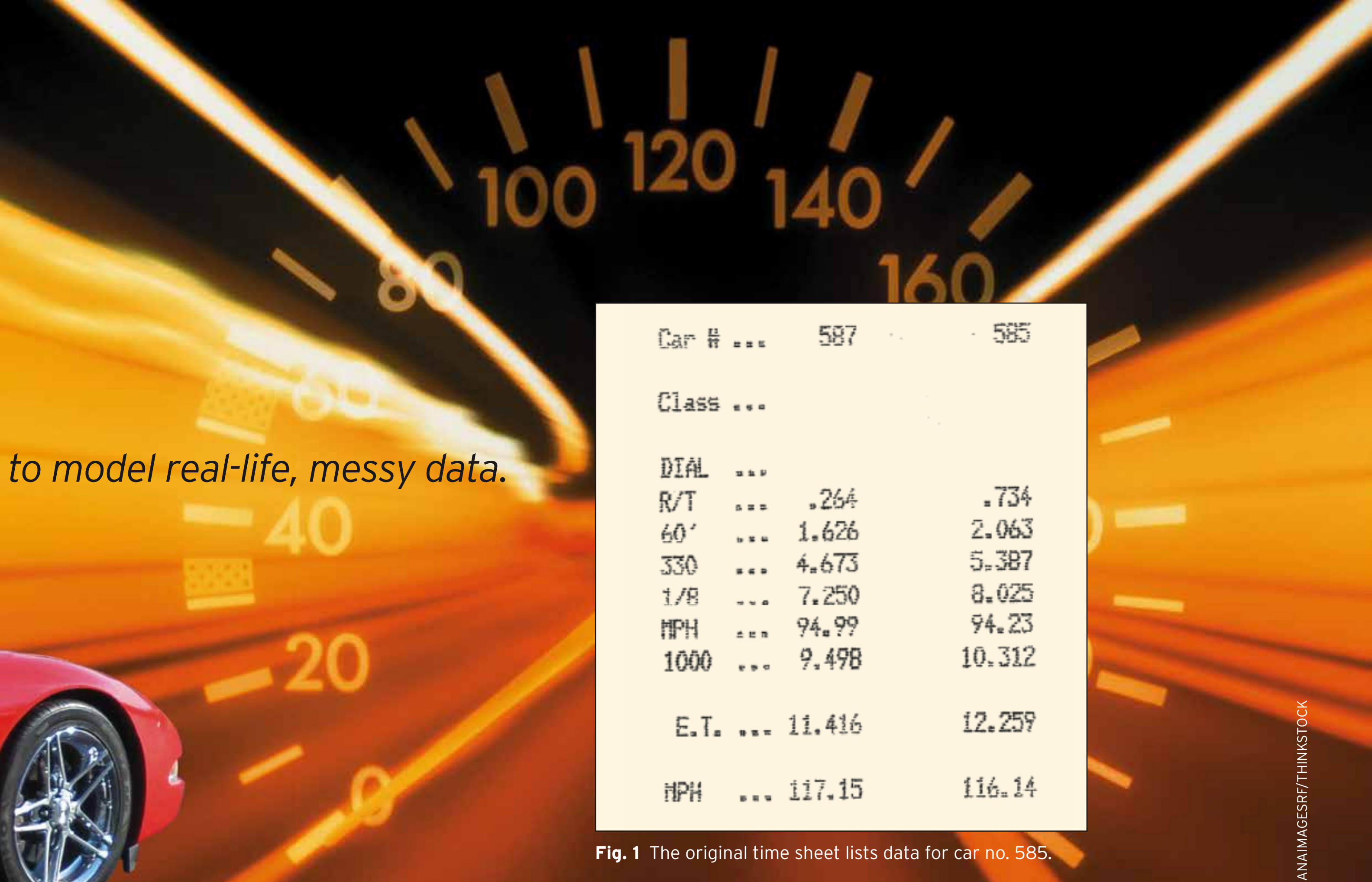
Sometimes the best mathematics problems come from the most unexpected situations. Last summer, a Corvette raced down our local quarter-mile drag strip. The driver, a family member, provided us with time and distance-traveled data from his time slip (see **fig. 1**) and asked us, "Can you calculate how many seconds it took me to go from 0 to 60 mph?" Although we initially thought that this question was a straightforward one, we soon discovered that, depending on the solution strategy and assumptions, different answers were possible. Thus began the ongoing discussions with our colleagues—and with high school mathematics teacher friends over pizza and with mechanical engineer family members at holiday dinners—to collectively decide on the "best" method. The mathematical discussions that arose on how to best solve the problem (**fig. 2**) prompted us to ask two questions: (1) What makes this problem so intriguing? and (2) What would students do?

HALLMARKS OF A GOOD PROBLEM

Any interesting mathematical task will likely encourage teachers to wonder what aspects of the task make it special. We wanted to know why this problem generated these mathematical conversations and how we could incorporate it into a calculus class. Insights from colleagues and students revealed several qualities of the problem that we believe contribute to its intrigue and worth. What we found illustrates what we consider to be three hallmarks of a good problem.

The Problem Solver Must Decide What Mathematics to Introduce.

In this nonroutine task, students must choose not only a strategy but also the appropriate mathematical ideas and tools to solve the problem. These choices arise here for two reasons. First, this task



to model real-life, messy data.

Car # ...	587	585
Class ...		
DIAL ...		
R/T ...	.264	.734
60′ ...	1.626	2.063
330 ...	4.673	5.387
1/8 ...	7.250	8.025
MPH ...	94.99	94.23
1000 ...	9.498	10.312
E.T. ...	11.416	12.259
MPH ...	117.15	116.14

Fig. 1 The original time sheet lists data for car no. 585.

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does not isolate a single high school content or practice standard; it cuts across many—functions, statistics, modeling with mathematics, using appropriate tools strategically, and others (CCSSI 2010, pp. 6–8). Second, the problem solver is not provided with a function rule, a graph, or an equation and thus is truly problem solving, meaning "engaging in a task for which the solution method is not known in advance" (NCTM 2000, p. 52). We believe that a worthwhile task is one in which the integration of multiple mathematical ideas may be necessary to finding a solution. Especially important is that these ideas need not be explicitly provided to the problem solver at the onset of the task.

The Task Uses Real-Life (Often Messy!) Data. Authentic problems do not often present themselves with neat data. For example, the distance data provided in the time sheet (**fig. 1**) are given in both feet and fractions of a mile. Also, the drag strip context of the problem is unfamiliar to most students, especially with regard to other data given on the original time sheet (i.e., *R/T* [reaction time]

In June 2013, a 1998 Corvette raced down the quarter-mile drag strip at the Route 66 Raceway in Joliet, Illinois. The following data were provided on a time slip after the race:

Distance	Time
0 feet	0.734 seconds
60 feet	2.063 seconds
330 feet	5.387 seconds
1/8 mile	8.025 seconds
1000 feet	10.312 seconds
$\frac{1}{4}$ mile	12.259 seconds

We also know that the car crossed the finish line at a speed of 116.14 mph. For sports cars, a common measure of performance is the number of seconds it takes the car to accelerate from 0 to 60 mph. The driver of this Corvette would like to know, according to these data, how many seconds it took him to reach a speed of 60 mph. Your task is to determine this time and support your claim mathematically. Include an explanation in words and a graph, if necessary.

Fig. 2 What makes this problem so intriguing?

	Students' Questions	Students' Chief Concerns
Hallmark 1 The problem solver must decide what mathematics to introduce.	What do I do, and what can I assume?	Selection of mathematical tools Problem-solving heuristics and emulation Classifying the problem as a specific type
	What statistics—if any—are relevant to this problem?	Adequacy of knowledge
Hallmark 2 The task uses real-life data.	What units should I use?	Equivalence of solution Teacher expectation
	Is the driver's delay important?	Task-specific knowledge (i.e., elapsed time and reaction time) Nonmathematical knowledge
Hallmark 3 The task requires mathematical modeling.	How do I plot the data and describe the situation?	Adequacy of knowledge Mathematical modeling Technology use
	How do I know whether my answer is reasonable?	Interpretation of model Reflection Metacognitive strategies

Fig. 3 Students' questions and concerns are aligned with each hallmark.

and *E.T.* [elapsed time]). Finally, the variable convention of "time as input" (usually in the left column) and "distance as output" (usually in the right column) is reversed. This problem combines measurement, calculus, and statistics in a real-life context, often lacking in the mathematics curriculum (Usiskin 2001). The problem solver must sift through the provided information and decide which information is most critical to devising a solution.

The Task Requires Mathematical Modeling. Although the term *mathematical modeling* can be defined in a variety of ways, a suitable definition here is "the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions" (CCSSI 2010, p. 72). Students have the flexibility to choose not only their mathematical solution strategy but also their preferred technology, such as a graphing calculator, spreadsheet, or computer algebra system. The ability to create and interpret a suitable mathematical model is integral to students' developing concept formation and problem-solving capabilities (CCSSI 2010; Lesh and Zawojewski 2007).

The acceleration column showed enough variation for the student to dismiss his earlier work and bring a different set of tools to solve the problem

These three hallmarks suggest a worthwhile problem for which the solution is not immediately obvious. In addition, the Corvette problem has the potential for students to engage in productive struggle, necessary in our mathematics classrooms (NCTM 2014). We assigned this task to our calculus students with little teacher intervention; only a few minutes were allotted in class to introduce the problem. If a student were stuck or expressed a misconception of some sort, we offered help in the form of asking a question or making suggestions. However, we were curious as to how the students themselves would approach the task and how they would support their findings. So we asked them to work outside class and bring back their solutions.

STUDENTS' SOLUTIONS

The Corvette problem was administered in two calculus courses at the community college level. Before discussing the students' solutions, we provide a list of questions that students asked us as they attempted to draw meaning from the problem (see **fig. 3**). These questions are categorized within the previously discussed hallmarks and were useful to us because they allowed us to identify and address students' chief concerns as they thought about the task. This, in turn, allowed us to anticipate future concerns and address them before they arose.

The depth and scope of these questions not only suggest a task rich in modeling, decision making, and sense making but also verify the depth of thinking necessary for students to attack such a problem. For example, students rarely voice concerns about assumptions if a task is a familiar one. However, in the case of the Corvette problem, some

students did not feel in complete control of the task; "I've got nothing to go off of" was a common cry. Moreover, students' questions hinted at their concern for a "proper" solution—one that would satisfy us as teachers. This was a useful concern for us to hear because we could reemphasize in the whole-class setting that the problem is receptive to whatever students choose to bring to it.

The students' questions (see **fig. 3**) and our specific handling of them provided students the comfort and space to think freely as they deemed appropriate. These qualities manifested themselves in the solutions that our students devised. We share these solutions and then consider the pedagogical implications of the students' work in the discussion.

Kenneth's Solution

Most of the students (19 of 35) followed what we now consider the conventional route: fitting a polynomial curve to the data, interpreting this curve as the Corvette's position, differentiating the function to find instantaneous velocity, and then algebraically or graphically finding the moment in time when the velocity is 88 feet per second or 60 miles per hour. Although this approach may appear straightforward, it was not easy for students. For example, Kenneth—as did four other students—initially found the laws of motion from kinematics to be an attractive tool. These students applied several equations—for example,

$$v_f = v_0 + a\Delta t \text{ and } d = \frac{v_f + v_0}{2}\Delta t$$

—by erroneously assuming a constant acceleration. The temptation of familiarity of principles relating position and velocity allowed the students to overlook this assumption. When Kenneth first used the equations, he realized that something was amiss, so he quickly made a table of values, establishing that acceleration is a variable quantity (see **fig. 4**).

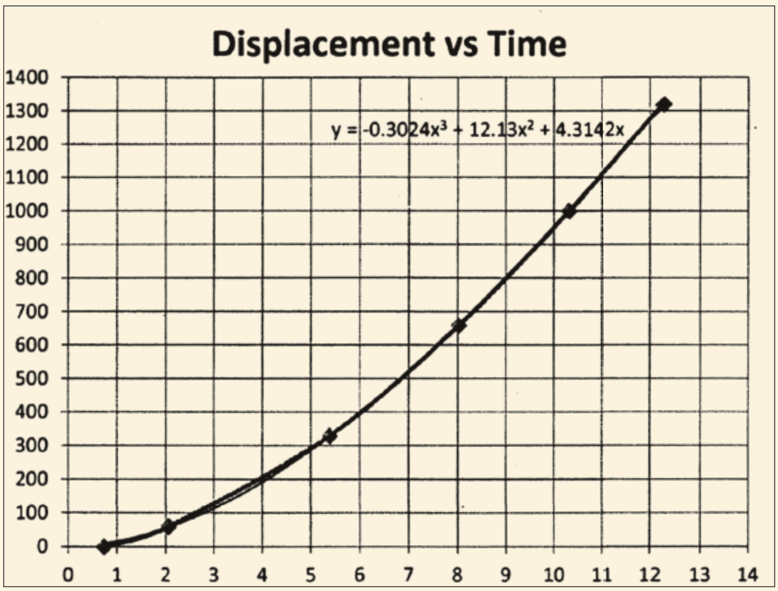
$$a_n = \frac{v_n - v_{n-1}}{t_n - t_{n-1}}$$

Acceleration (ft./sec.^{2})	Time (sec.)
0	0.734
33.97045	2.063
10.85461	5.387
16.62901	8.025
10.30678	10.312
8.05806	12.259

Fig. 4 Kenneth's table shows that the Corvette's acceleration is not constant.

Kenneth first calculated the average velocity on each of the subintervals and, from here, average accelerations. Although his calculations are rudimentary in the sense that they reflect only endpoint-to-endpoint averages, the acceleration column showed enough variation for him to dismiss his earlier work and bring a different set of tools to solve the problem. He then plotted the original data points, detected a predictable trend, and used spreadsheet software to obtain a regression equation. This approach provided him with the approximate position equation for the Corvette at any moment in time t (see **fig. 5**).

The remainder of Kenneth's solution consisted of finding the derivative of position, s, with respect to time, t, setting this expression equal to 88 feet per second, and solving for time. His answer was $t = 4.0685$ seconds. Other students used analogous ideas with only slight modifications—for example, using a different function, $s(t)$, or applying an intersect method to solve for time. Resulting answers were within a tenth of a second of Kenneth's result. We anticipated some differences in the answers because of variations in mathematical models and the degree of care used when rounding intermediate calculations.



We have fitted a third-order polynomial from the given data to model the position of the car with respect to time:

$$s = -0.3024t^3 + 12.13t^2 + 4.3142t$$

Fig. 5 Kenneth's regression curve is represented both graphically and algebraically.

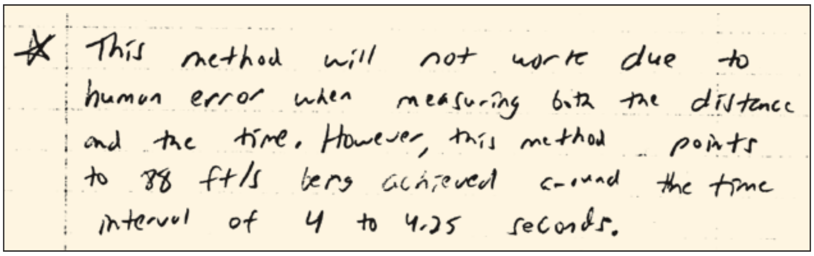

This method will not work due to human error when measuring both the distance and the time. However, this method points to 88 ft/s being achieved around the time interval of 4 to 4.25 seconds.

Fig. 6 Jeremy realized that his method was not sufficiently precise.

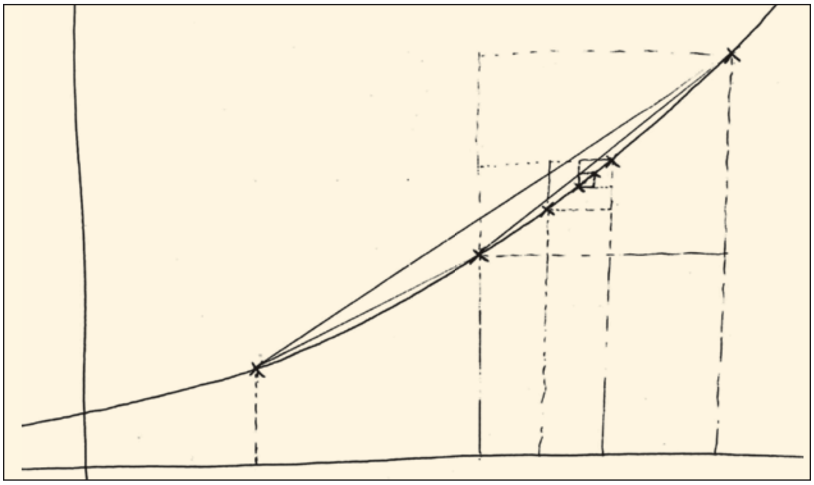


Fig. 7 Jeremy presented this graphical explanation when asked about his specific strategy.

Jeremy's Solution

Jeremy anchored his work in the concept of average rate of change, $\overline{v}$. Initially, he plotted the data and used line segments to calculate the average velocity for each of the five time intervals. First, he found average velocities that straddled 88 feet per second; that is, for $2.063 < t < 5.387$, $\overline{v} = 81.23$ feet per second, and for $5.387 < t < 8.025$, $\overline{v} = 125.09$ feet per second. He then "zoomed in" on these two intervals, constructed four subintervals having half the width of the original intervals, and recalculated. Although his technique is a viable one, Jeremy encountered difficulties with rounding error since he had not fit a curve to the data points; he was simply relying on his "best guess" to calculate refined average rates of change (see **fig. 6**). Even so, the care he exercised in measurement led him to an initial estimate that was remarkably close.

Jeremy decided at this point to find a polynomial regression equation. This way, he could still divide the intervals in half but determine the associated y-values (position) from the regression curve. With new information in hand, he modified his strategy by using t values that were easily halved. He knew the desired time was somewhere above four seconds, so he created the subintervals 4.0-4.25-4.5

Students learn value in coping with temporary setbacks yet sticking with a method unless evidence suggests that it is time to reboot.

and, refining further, 4.0-4.125-4.25-4.375-4.5, and so on. Since his approach to solving the problem was not initially obvious to us, he explained his strategy graphically in a cobweb graph (see **fig. 7**).

The continuation of this process led to refinements in Jeremy's tables as he zoomed in closer and closer toward the target of 88 feet per second. This process led to a final result between 4.175 and 4.18125 seconds (see **fig. 8**).

After submitting his solution, Jeremy admitted that he took pride in "not needing calculus" to solve the problem. However, we found his empirical versions of limit definition and limit convergence to be sufficiently close to textbook calculus. His solution is noteworthy.

Dan's Solution

Dan's solution relied on the principle of local linearity—any smooth curve, no matter how it bends, looks linear on a small scale. Using this as the seed of his idea, he sought two time values, t_1 and t_2, sufficiently close so that

$$\frac{f(t_2)-f(t_1)}{t_2-t_1}\approx 88 \text{ ft./sec.}$$

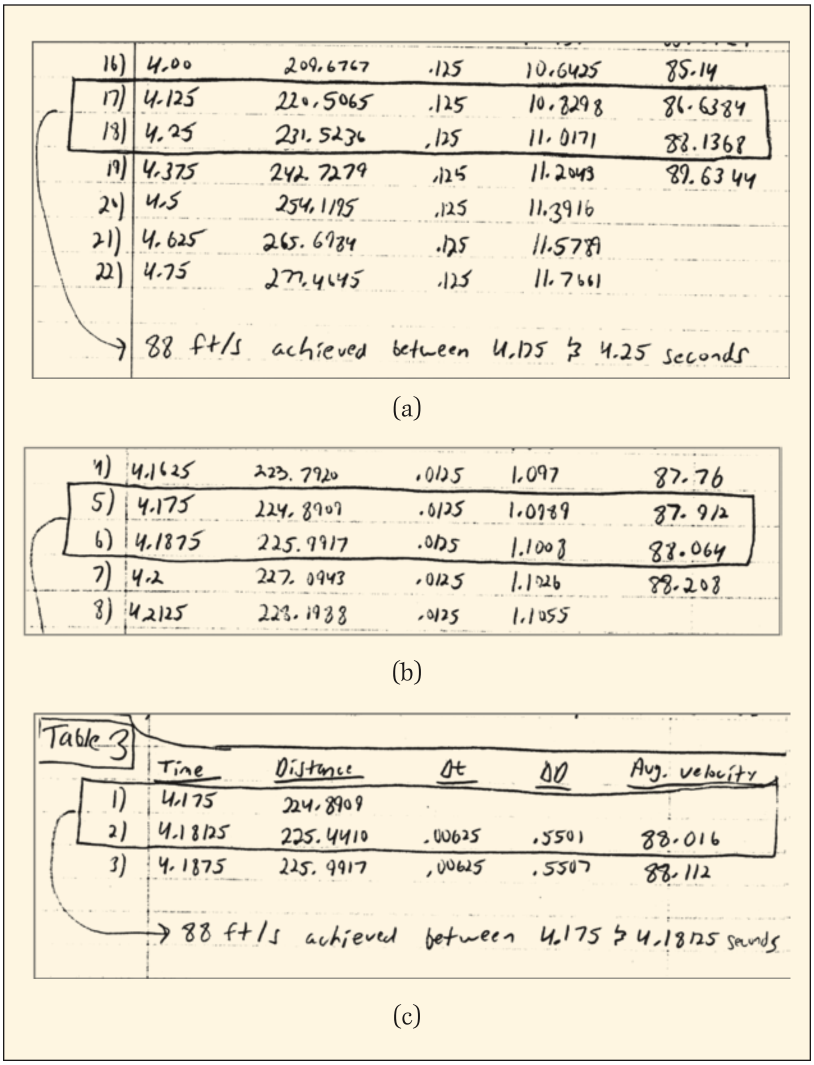
For a beginning student, this is a profound realization and is reflected in Dan's graphical work (see **fig. 9**).

His technique was haphazard at first (in his own words, "I had to play around a bit since I didn't know where the time would be"), but eventual fine-tuning near 4.1 seconds resulted in readings close to 88 feet per second. **Fig. 10** shows a Maple™ log of some of Dan's work.

Dan admitted to us that he initially had six pages of calculations. In his final project, however, he found it appropriate to simply provide a subset of this information. Although his solution has similarities to Jeremy's work, he let computer software do the calculations. Also, on closer examination, the points of focus of Dan's and Jeremy's work were quite different even if mathematically equivalent—"flatter curves" and "shrinking intervals," respectively.

Our Solution

We agree with the solutions obtained by Kenneth, Jeremy, and Dan and found that a reasonable



16)	4.00	209.6767	.125	10.6425	85.14
17)	4.125	220.5065	.125	10.8298	86.6384
18)	4.25	231.5236	.125	11.0171	88.1368
19)	4.375	242.7279	.125	11.2043	89.6344
20)	4.5	254.1195	.125	11.3916	
21)	4.625	265.6984	.125	11.5789	
22)	4.75	277.4645	.125	11.7661	

88 ft/s achieved between 4.125 & 4.25 seconds

(a)

4)	4.1625	223.7920	.0125	1.097	87.76
5)	4.175	224.8909	.0125	1.0989	87.912
6)	4.1875	225.9917	.0125	1.1008	88.064
7)	4.2	227.0943	.0125	1.1026	88.208
8)	4.2125	228.1938	.0125	1.1055	

(b)

Table 3

	Time	Distance	Δt	ΔD	Avg. velocity
1)	4.175	224.8909			
2)	4.18125	225.4410	.00625	.5501	88.016
3)	4.1875	225.9917	.00625	.5507	88.112

88 ft/s achieved between 4.175 & 4.18125 seconds

(c)

Fig. 8 Snapshots show Jeremy's calculations.

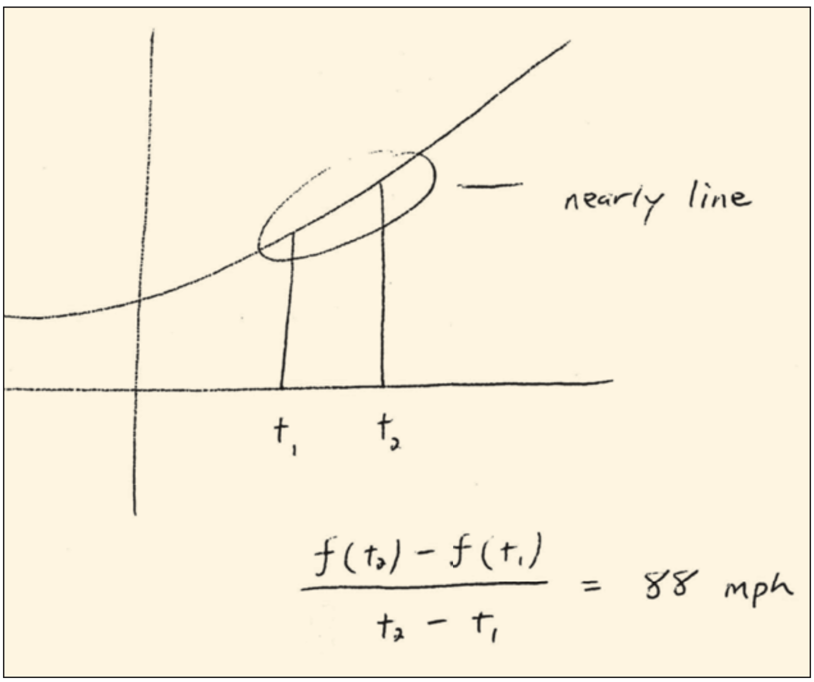


Fig. 9 Dan's graph shows local linearity.

```
> f:=5.99*t^2+37.95*t-38:
> t1:=5: t2:=6:
> f1:=eval(f,t=t1): f2:=eval(f,t=t2):
> (f2-f1)/(t2-t1);
                                                    103.84
> f:=5.99*t^2+37.95*t-38:
> t1:=4.5: t2:=5:
> f1:=eval(f,t=t1): f2:=eval(f,t=t2):
> (f2-f1)/(t2-t1);
                                                    94.85500000
> f:=5.99*t^2+37.95*t-38:
> t1:=4: t2:=4.5:
> f1:=eval(f,t=t1): f2:=eval(f,t=t2):
> (f2-f1)/(t2-t1);
                                                    88.86500000
> f:=5.99*t^2+37.95*t-38:
> t1:=4.2: t2:=4.3:
> f1:=eval(f,t=t1): f2:=eval(f,t=t2):
> (f2-f1)/(t2-t1);
                                                    88.86500000
> f:=5.99*t^2+37.95*t-38:
> t1:=4.1: t2:=4.2:
> f1:=eval(f,t=t1): f2:=eval(f,t=t2):
> (f2-f1)/(t2-t1);
                                                    87.66700000
> f:=5.99*t^2+37.95*t-38:
> t1:=4.17: t2:=4.18:
> f1:=eval(f,t=t1): f2:=eval(f,t=t2):
> (f2-f1)/(t2-t1);
                                                    87.96650000
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Fig. 10 Dan worked in a computer algebra system.

answer should be approximately 4.1 seconds. In fact, performance times for Corvettes with comparable horsepower and model years fall between 4.1 and 4.7 seconds (Auto Rooster 2014). Further, reaction time plus elapsed time equals total time. This means that the reaction time of 0.734 seconds for the Corvette was not included in the elapsed time of 12.259 seconds, as we originally thought. Thus, it was not necessary to shift the position function horizontally to account for the driver's reaction time. So when we choose data points for the position function, replacing the data point of (0, 0.734) with the point (0, 0) would provide a higher level of accuracy.

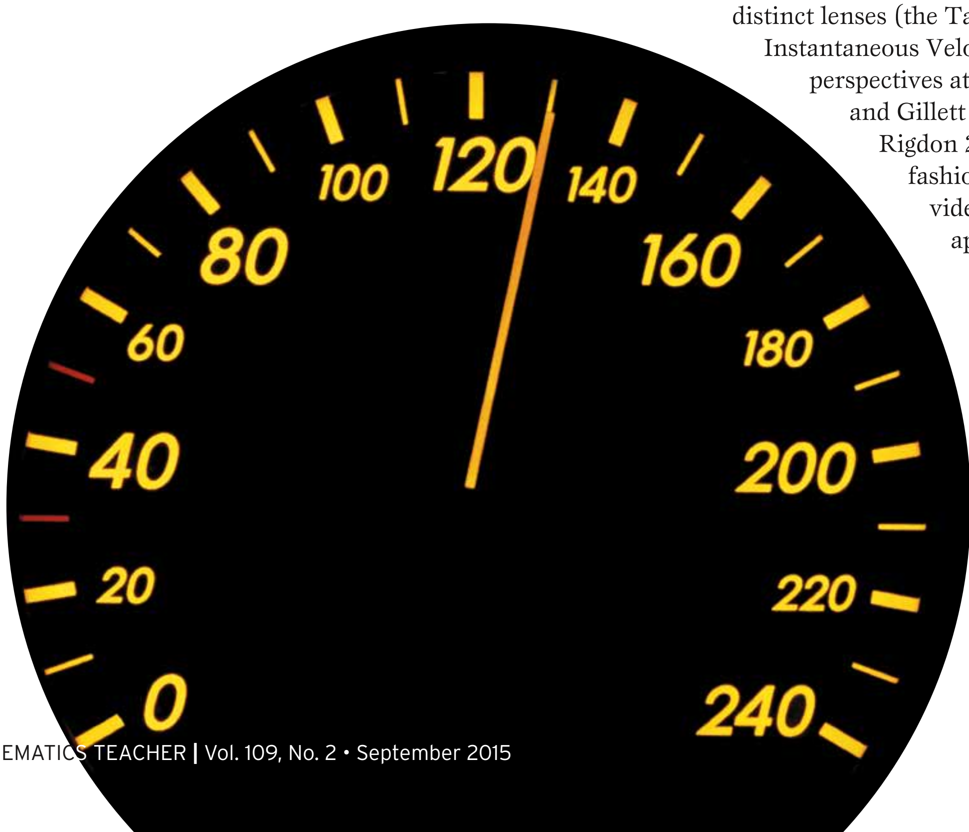
LEARNING TRAJECTORIES

Students experiencing calculus for the first time can gain valuable insights from the solutions described here, and teachers can use these different approaches to facilitate a whole-class discussion to help reinforce important calculus concepts. The benefit is both in the mathematics used and the behaviors observed en route to a solution.

First, Kenneth's work displays the balancing act between assumption validity and contextual factors. He found—through old-fashioned thinking—that the equations he used in physics were inappropriate for the Corvette problem. This realization was an epiphany for him: The problem needed to be studied for what it was and not for what he thought it should be (a kinematics problem).

Second, we learn from both Kenneth and Jeremy that first instincts may not be the best; flexibility and adaptability are important. In Jeremy's case, rounding errors nearly jeopardized his solution plan, but once he acquired additional information (the regression equation), his initial strategy was greatly facilitated. Students learn value in coping with temporary setbacks yet sticking with a method unless evidence suggests that it is time to reboot (e.g., Kenneth's first attempt).

Third, the solutions of Jeremy and Dan suggest pedagogical alternatives to introducing the derivative concept. For example, calculus textbooks commonly introduce differentiation through two distinct lenses (the Tangent Line problem and the Instantaneous Velocity problem) and unite these perspectives at a later time (Briggs, Cochran, and Gillett 2015; Varberg, Purcell, and Rigdon 2007). In a more seamless fashion, the Corvette problem provides a data-driven, authentic approach (e.g., Freudenthal 1973; Rasmussen et al. 2005; Treffers 1987) to teaching derivative as velocity—one in which the oft-segregated secant-tangent line geometry co-emerges. Moreover, Dan's observations emphasize the importance of understanding the derivative by way of "zooming in" on a smooth graph and observing its linear qualities (Tall 1987, 1991). Such learning



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trajectories are in harmony with reforms on the definite integral (Jones 2013) and support the view of calculus as the study of motion.

Finally, the solutions of Jeremy and Dan highlight the relevance of the mean value theorem. Too often, students view the mean value concept as detached from anything meaningful. (Arguing that the mean value theorem is "used to prove other theorems," we believe, does little to support students' curiosity or intellectual need.) However, both Jeremy and Dan provide solutions that accentuate conceptually why the theorem is so important. Jeremy blended both graphical and tabular methods to support the feasibility of "average" and "instantaneous" being arbitrarily close, whereas Dan used analogous reasoning while working on a computer. They both tacitly use the mean value theorem in their quest to find the average rates of change that march closer and closer to 88 feet per second.

Each of these points illustrates a potential learning opportunity for students. After the students have had the opportunity to tackle the problem individually, the teacher can use these students' approaches to examine the integration of multiple mathematical concepts. For example, Kenneth's approach can be used to review instantaneous velocity, Jeremy's approach can be used as a reinforcement of the limit concept, and Dan's approach can be used to highlight an application of the mean value theorem.

What makes the Corvette problem unique is its appeal and its perceived simplicity—it has a low entry point for engagement. However, the tools, reasoning, and solution paths are not initially obvious to the problem solver, so a high bar is set for success. Further, the three hallmarks of a good problem—the problem solver must decide what mathematics to introduce, the task uses real-life data, and the task requires mathematical modeling—underscore the problem's effectiveness and appeal. Much can be gained from similar tasks in which students make sense of mathematical situations, test their assumptions, stumble along the way, and validate their solutions.

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