Investigating Integer

Restricting variables to integer values can

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Linear programming (LP) is an application of graphing linear systems that appears in many Algebra 2 textbooks. Although not explicitly mentioned in the Common Core State Standards for Mathematics, linear programming blends seamlessly into modeling with mathematics, the fourth Standard for Mathematical Practice (CCSSI 2010, p. 7).

In solving a linear programming problem, we always seek to find the optimal solution, which might be a maximum or a minimum depending on the nature of the problem. When the variables are restricted to integer values, as often happens in the real world, the problem is then an example of integer linear programming (ILP). In this article, we show why this distinction matters and how it might provide an interesting classroom investigation.

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Restrictions in Linear Programming

lead to interesting classroom dialogues.

THE MAXIMUM PROFIT PROBLEM

In a section entitled "Reteaching: Linear Programming," the *Prentice Hall Algebra 2* textbook includes the following problem (Charles et al. 2012, p. 40):

> Your school band is selling calendars as a fundraiser. Wall calendars cost $48 per case of 24. You sell them at $7 per calendar. Pocket calendars cost $30 per case of 40. You sell them at $3 per calendar. You make a profit of $120 per case of wall calendars and $90 per case of pocket calendars. If the band can buy no more than 1000 total calendars and spend no more than $1200, how can you maximize your profit if you sell every calendar? What is the maximum profit?

The text then goes on to formulate the problem algebraically and solve it by—

(1) graphing the constraints;
(2) shading the feasible region;
(3) finding the coordinates for each vertex of the feasible region;
(4) evaluating the objective function at each vertex; and then
(5) selecting the vertex that generates the most profit as the optimal solution.

The Corner Point Principle

The solution method outlined above uses the corner point principle, which states that if there is a unique optimal solution to a linear programming problem, it must lie at a vertex of the feasible region. However, the corner point principle also assumes continuous (i.e., real-number) variables (see, e.g., ISYE [1997]). The variables in this problem are not continuous; they are discrete. Even if we could purchase a fractional case of either type of calendar, the variables would still be discrete. Not every fraction would be possible; the only fractions possible are those whose denominators are divisors of 24 or 40, the number of calendars per case of each type. For example, we might be able to purchase a half case of either type but not a one-seventh of either type, because 7 divides neither 24 nor 40.

This distinction does not matter for this particular problem because the optimal corner point has integer coordinates. However, what if the optimal corner point

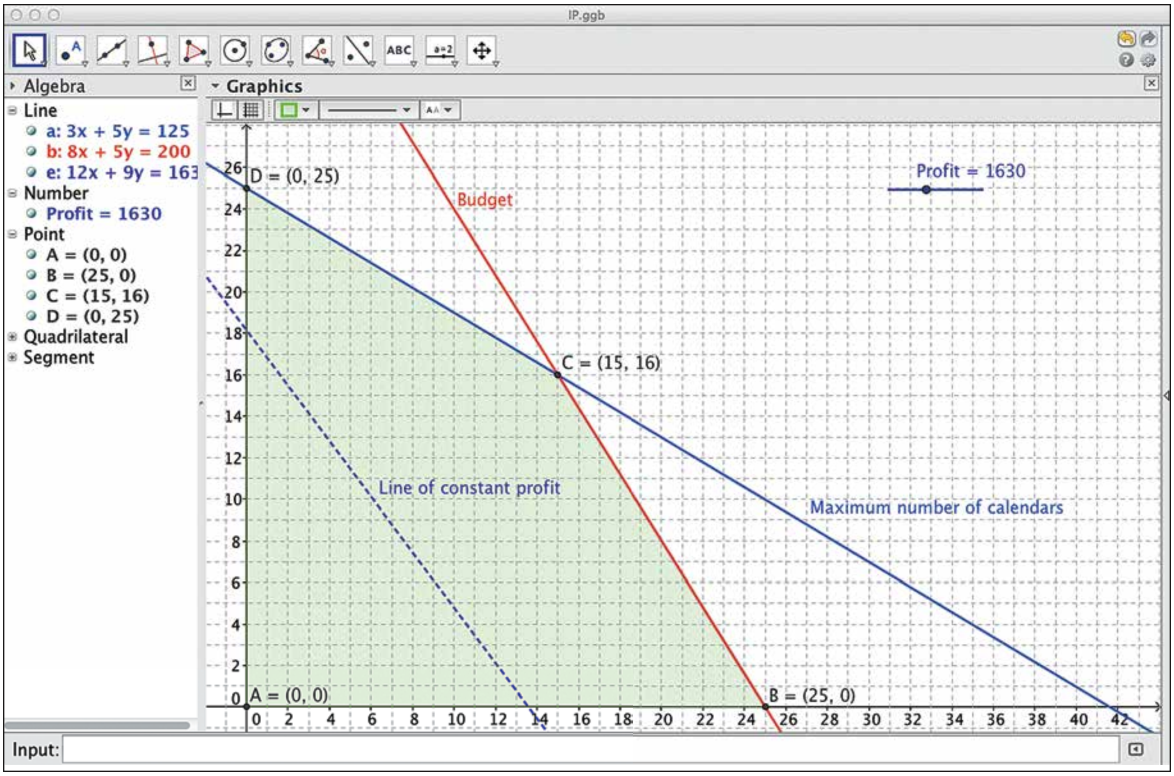

Fig. 1 A slider for profit changes the placement (but not the slope) of the line of constant profit. (For an interactive GeoGebra file of this figure, see the online component for this article.)

did not have integer coordinates? In that case, those coordinates would represent a solution that would not be feasible, let alone optimal. Even so, use of the corner point principle to solve a problem with integer restrictions is common in many textbook examples. The parameters of the problem are carefully chosen to guarantee an integer solution. However, neither a person solving a real-world problem with integer restrictions nor a student solving such a textbook problem knows in advance that the optimal solution generated by the corner point principle will have integer coordinates. Thus, we believe that the use of the corner point principle as the method of solution is inappropriate.

In fact, many of these problems can be reframed so that the solutions would no longer need to be integers. For example, one common class of problems is called "product mix." Such a problem involves finding the mixture of different products to produce using a company's limited resources that will maximize profit. Often this problem is framed in terms of the number of each product to be produced, implying integer solutions. However, if the problem is cast in terms of production rates, the rates at which products are produced need not be integers, and the question of what to do with a leftover fraction is easily answered. For example, an automobile unfinished on the assembly line at the end of one production period will be finished during the next production period.

INTEGER RESTRICTIONS

It is not difficult to alter the problem given earlier so that the optimal corner point, if solved as a linear programming problem, does not have integer coordinates. For example, suppose that as a result of a recent donation, the band members decided that they could spend no more than \$1250. As we shall see, this innocuous-seeming change creates a different set of corner points that greatly complicates the search for a solution.

Corners with Integer Coordinates

The first step in solving the original problem, with its \$1200 spending limit, is to formulate it algebraically:

> Let x represent the number of cases of wall calendars purchased for sale; let y represent the number of cases of pocket calendars purchased for sale; and let P represent the total profit from the sale of all the calendars purchased.
>
> Maximize: $P = 120x + 90y$
> Subject to: $24x + 40y \le 1{,}000$
> (maximum number of calendars constraint)
> $48x + 30y \le 1{,}200$
> (budget constraint)

Figure 1 shows a graph of the feasible region of this problem with a slider created to change

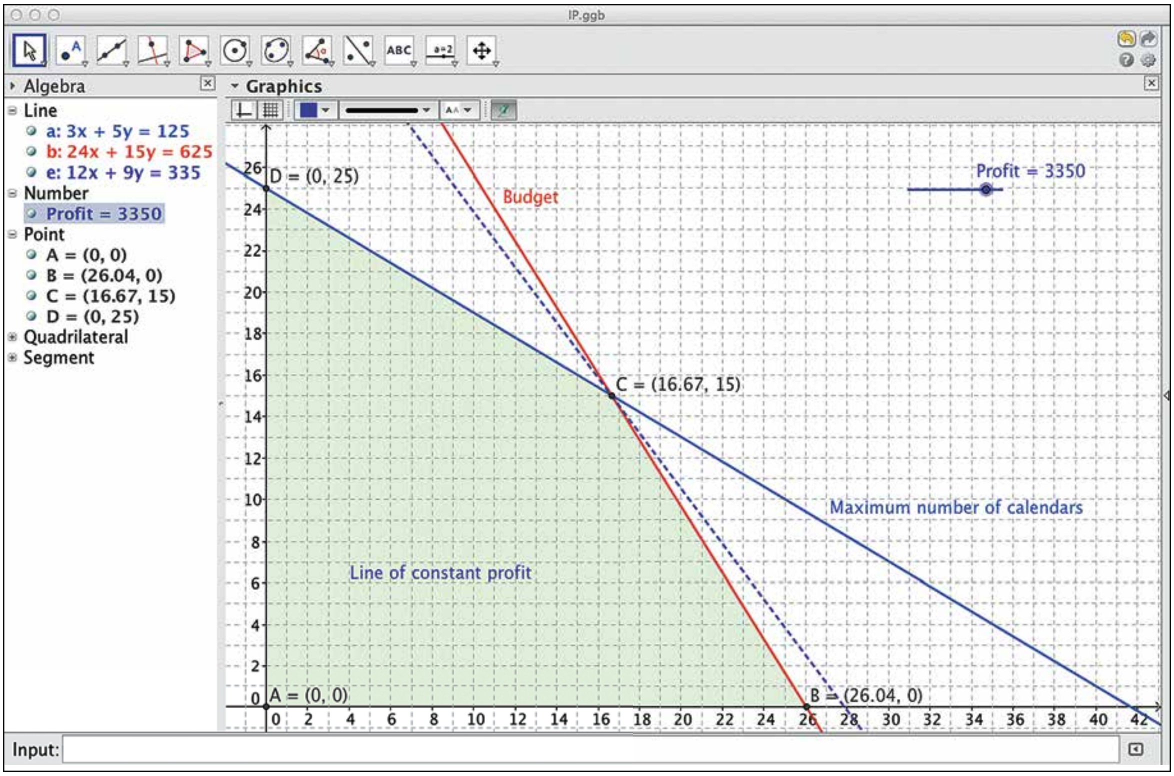

Fig. 2 The line of constant profit intersects the corner point, but *C* (16.67,15) has noninteger coordinates.

the value of the objective function—in this case, the profit. As the profit increases, the dashed line representing that profit "moves" upward and to the right. At a profit of 2250, the line contains the corner point *D* (0, 25); at 3000, *B* (25,0). The last point in the feasible region that the line touches is the corner *C* (15, 16). This point corresponds to purchasing 15 cases of wall calendars and 16 cases of pocket calendars, and this mixture generates the most profit—$3240. In this case, the solution to the integer linear programming problem coincides with the solution to the linear programming problem.

Corners with Noninteger Coordinates

Altering the problem slightly by increasing the amount that the band can spend from $1200 to $1250 changes the feasible region. (See **fig. 2**.) As before, the corner point that lies on neither coordinate axis generates the most profit—$3350. The largest profit is again generated at corner point *C*, but the coordinates of point *C* are now (16 2/3, 15). In this case and others that follow, the software has rounded to two decimal places of accuracy, resulting in (16.67, 15).

Now, if we may not purchase a fraction of a case of calendars, this solution is not feasible. In fact, when the decision variables x and y are restricted to integer values, the feasible region is no longer a continuous area. The integer restriction means that the feasible region is now the set of points with integer coordinates that satisfy all the constraints.

When students face this issue, they typically suggest two possible ways to avoid the dilemma. Both are reasonable suggestions, and either might generate the optimal point; however, neither of them will always do so. Thus, both suggestions provide an opportunity for a rich classroom discussion.

Noninteger coordinates might represent a solution that would not be feasible, let alone optimal.

The first suggestion is to round the optimal linear solution. Consider the following classroom dialogue:

***Student 1*:** We could just round the answer.
***Teacher*:** Then what would your optimal solution be?
***Student 1*:** (17, 15).
***Teacher*:** Can you point out that point on the graph? [*Student does so.*]
***Teacher*:** Now, do you see any problem with that? . . . Does anyone see a problem here?
***Student 2*:** Oh! It's outside that region . . . the feasible region.

The second suggestion is to use the feasible point nearest to point *C*. In this case, the nearest point is (16, 15), which yields a profit of $3270. The

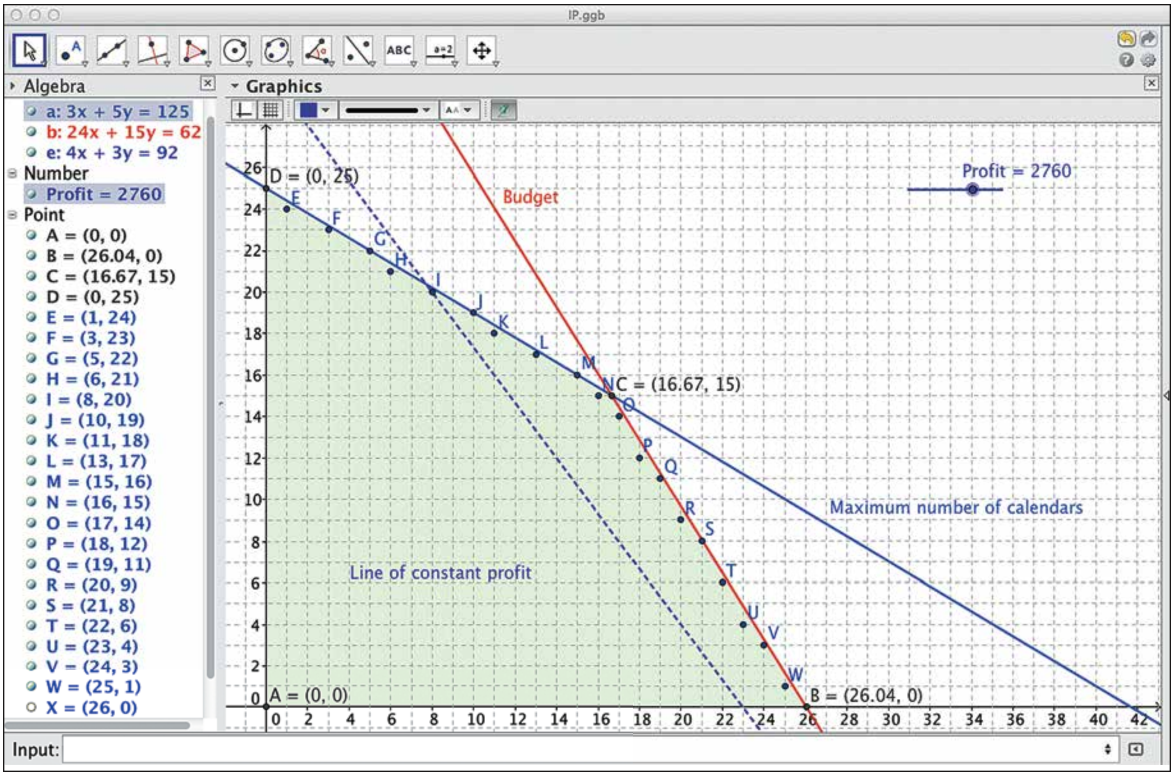

Fig. 3 Why were these 21 points shown?

question is whether any other feasible point generates more profit. As shown in **figure 3**, point *O* (17, 14) is now feasible. In a case such as this, we like to challenge students along the following lines:

***Teacher*:** Now, without doing the arithmetic, can anyone explain why (17, 14) must make more profit than (16, 15)?

***Student 1*:** Well, (16, 15) means they buy 16 [cases of] wall [calendars] and 15 [cases of] pocket calendars, and (17, 14) means they buy 1 more [case of] pocket calendars and 1 less case of wall calendars . . .

***Student 2*:** . . . and that's $90 less because of 1 less case of pocket calendars but $120 more because of the wall calendars, so . . .

***Student 3*:** . . . that's $30 more altogether!

When students see this, they spontaneously note that (16, 15) cannot be the optimal solution because there is at least one other feasible point, (17, 14), that yields more profit. Notice also how this interpretation of the situation brings students back to the context of the problem.

THE FEASIBLE REGION UNDER INTEGER RESTRICTIONS

We could find the optimal integer solution by examining every feasible point, but there are more than 400 lattice points in this feasible region. This observation provides good motivation to look for a more efficient way. **Figure 3** shows the feasible region for the integer problem and a line representing $2760 profit passing through point *I*. Some feasible lattice points are labeled on the graph and highlighted in the list to the left of the graph.

These points form what is called the *kernel* of the feasible region. To help students understand the significance of the kernel, we launch a discussion similar to what follows:

***Teacher*:** Why do you think the 21 points highlighted in the graph were chosen?

***Student 1*:** These points are closest to the boundary lines.

***Teacher*:** But why wouldn't we need to consider any other feasible points?

***Student 1*:** They're further from the boundary.

***Teacher*:** What does that mean in terms of the profit? [*Pauses . . .*] Let's see. Pick one of the highlighted points. [*Student 2 chooses point I with coordinates* (*8, 20*).]

***Teacher*:** Can you see from the graph how much profit point *I* makes?

***Student 2*:** $2760.

***Teacher*:** Can anyone explain how you know that the profit is $2760?

***Student 3*:** The slider says $2760 . . . isn't the profit supposed to be the same all along that dotted line?

***Teacher*:** What do others think about that?

***Student 2*:** That's why it's called a line of constant profit!

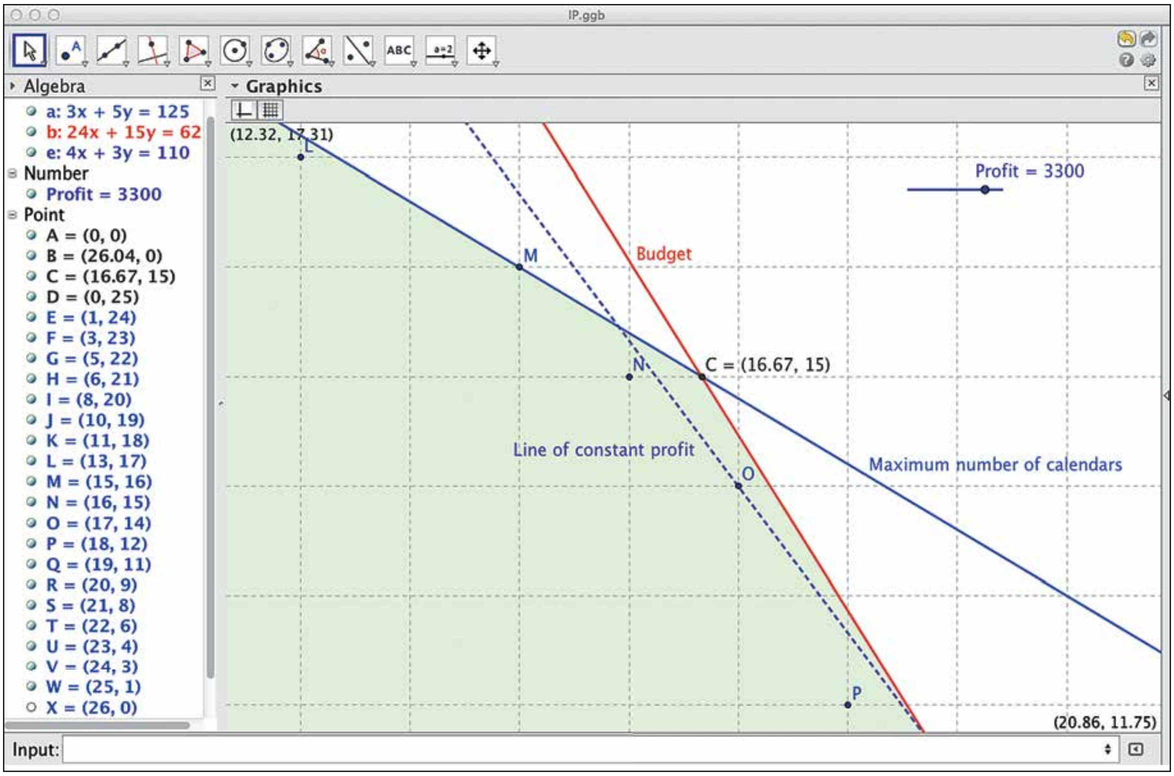

Fig. 4 Point *O* (17, 14) is the last feasible point touched by the "moving" line of profit.

Teacher: Okay, now don't calculate the total profit, but just tell me why the point (8, 19) must make less profit than (8, 20).

[*After a pause . . .*]

Student 4: Oh, there's the same number of wall calendars, but 1 less pocket calendar.

Teacher: You mean cases, right? [*Students agree.*] Okay, and where is (8, 19) with respect to (8, 20)?

Student 2: Right below it.

Teacher: Now, what about all the points below (8, 19). Will any of them . . .

Student 3: No, they'll all make less, because those points will always have fewer pocket calendars.

Teacher: So once we know the profit for (8, 20), we know that all of the points below (8, 20) make less profit. Does anyone see any other points that we can say make less profit for a similar reason?

[*Another pause . . .*]

Student 4: What about the ones beside . . .

Teacher: Which side?

Student 4: The left!

Teacher: And how do you know that all the points directly left of (8, 20) make less profit?

Student 4: They all have the same y-value but smaller x-values.

Teacher: Can someone say what that means in terms of calendars?

Student 5: They all have the same number of cases of pocket calendars but less wall calendars.

Teacher: Same as what? Less than what?

Student 2: (8, 20).

Teacher: Can someone summarize what all this means?

Student 6: We only need to test those marked points, because all the points below and to the left of them make less profit . . . because they have less of the pocket or wall calendars.

Teacher: So, count them up. How many points do we need to test?

Student 3: Wow; a lot—21 of them.

Teacher: Yes, but a lot fewer than over 400!

We could test all 21 points in the kernel, individually or in groups, but **figure 4** displays the result of zooming-in to show that (17, 14) is the optimal solution, generating a profit of $3300. It is the last feasible lattice point that the "moving" profit line touches before leaving the feasible region. We think this is a good time to interject to students the fact that software packages to handle both linear programming and integer linear programming exist. Lest students think that there is no need for algebra to solve such problems, we point out that the key to solving these problems,

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even when using appropriate software, is being able to represent the problem algebraically. Readers interested in some background in a method similar to the one we used for solving integer linear programming problems might be interested in Gomory's (1958) article.

CONTEXT AND CONSEQUENCES


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The Common Core's Standards for Mathematical Practice (CCSSI 2010, pp. 6–8) describe how students should engage with mathematics as they grow in maturity and expertise. The mathematical activity described here incorporates many of those Standards. In particular, when students realize that the solution (16.67, 15) does not fit with the integer restriction, they are making sense of the problem. When the students push a deeper understanding of the integer restriction and its consequences to a solution, they are persevering in solving the problem. When students write an algebraic formulation of the problem, they are abstracting a symbolic representation of a concrete situation. On the other side of the coin, in determining the kernel of the problem, they are necessarily returning to the problem context. The heart of solving the problem presented here is modeling it mathematically. Finally, applying technology to better understand and solve the problem is an example of using appropriate tools strategically.

Linear programming and its offshoot, integer linear programming, are rich sources of authentic real-world problems. Solving them necessarily involves representing them algebraically, and graphs of linear systems can be used to solve two-variable problems. Thus, in high school textbooks, linear programming has become a common application for graphing linear systems. However, the corner point principle, which is typically the core concept underlying the solution of linear programming problems, does not generalize to the solution of integer linear programming problems.

Thus, whenever the corner point principle is the vehicle used to solve such problems, great care should be taken to ensure that the problems being solved do not restrict one or more of the variables to integer values. That's the bad news. The good news is that such cases can be used to deepen and solidify students' understandings of problems of this nature and their solution. Moreover, solving an integer linear programming problem as shown here provides an opportunity for students to consider the effect of assumptions in mathematics in general and mathematical problem solving in particular.

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LIFETOUCH PORTRAIT STUDIOS, (PRINCIPATO)

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An interactive GeoGebra file of figure 1 can be found with this article online at **www.nctm.org/mt**. More4U content is a benefit for NCTM members only.