CONNECTING RESEARCH to teaching

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Focusing on Mathematical Arguments

The importance of collective argumentation is highlighted in the Common Core's third Standard for Mathematical Practice, which states that students should be able to "construct viable arguments and critique the reasoning of others" (CCSSI 2010, p. 6). Researchers have described what productive mathematical argumentation might entail, including students participating in particular ways (Weber et al. 2008; White 2003); classroom environments where sense making is valued (Weber et al. 2008); and argumentation that progresses from intuition toward deductive reasoning (Prusak, Hershkowitz, and Schwarz 2011).

Collective argumentation occurs when a group works together to arrive at a conclusion (supporting it with evidence). Simplistically, this occurs when students give answers to questions and tell how they arrived at the answer, perhaps prompted by a teacher. But collective argumentation can be much richer, with a focus on the process of solving a problem and why that process is appropriate. Researchers have described focusing on argumentation in classrooms as contributing to developing students' mathematical autonomy (Yackel and Cobb 1996), giving learners opportunities to connect new ideas to those already known (Nussbaum 2008), and contributing to deeper understanding of concepts (Andriessen 2006). Choppin (2007) showed how teacher actions allow students' ideas to become building blocks of mathematical learning during arguments.

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Focusing on mathematical arguments directs attention to the reasons why a solution is viable or a process is appropriate rather than strictly on whether answers are correct. One way to keep our focus on important mathematical ideas is to use Toulmin's ([1958] 2003) conceptualization of arguments to reflect on our teaching. Toulmin provided a coherent way to think about the components of an argument by diagramming those components. His model has seven components, but Krummheuer (1995) suggested that for collective argumentation this model can be reduced to three main core components that help us think about the important content of a mathematical argument: *claim* (the result or statement being established); *data* (information provided that supports the claim); and *warrant* (reasons that link data to a claim). **Figure 1** presents an example of a core argument. We use these structures to organize thoughts about arguments in our classrooms, identifying key components without actually constructing the diagrams.

In addition to simple "core" argu-

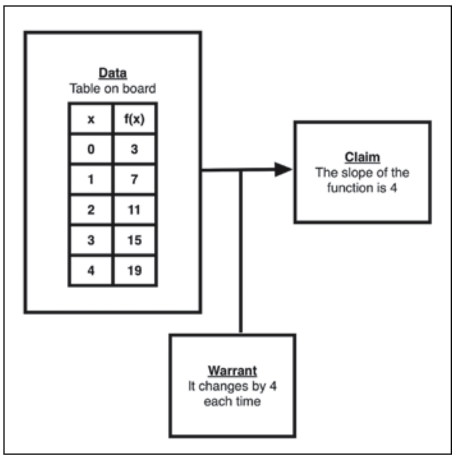

Fig. 1 The diagram of a hypothetical core argument identifies data, a warrant, and the claim.

ments like the one presented in **figure 1**, we can think of arguments as composed of chains of reasoning, wherein some statements act as data in one argument and a claim in another. For instance, imagine extending the previous core argument, which establishes that the slope of the function is 4. This initial claim can be used as data, along with the observation that the function is linear, to argue that the function must look like $f(x) = 4x + \_\_\_\_$ (see **fig. 2**). In this argument, the initial claim is used as data to support another claim, thus acting as both data and claim.

Arguments are not necessarily equivalent to proofs and can occur in any subject area. We view proof as a subset of argumentation, requiring deductive steps from one mathematical claim to another with mathematically appropriate data and warrants. Unlike a proof, an argument can depend on any data or warrants that students in the class find convincing, including statements that might not be mathematically correct or complete. For example, although the second warrant shown in **figure 2** is not, strictly speaking, mathematically appropriate—there are infinitely many different functions that could fit the data—it may nevertheless be justified by other, nonmathematical considerations (perhaps the class is currently studying linear functions or students are expected to find the simplest possible model to fit the data). One important role of a teacher is to help students contribute mathematically relevant data and warrants within collective argumentation.

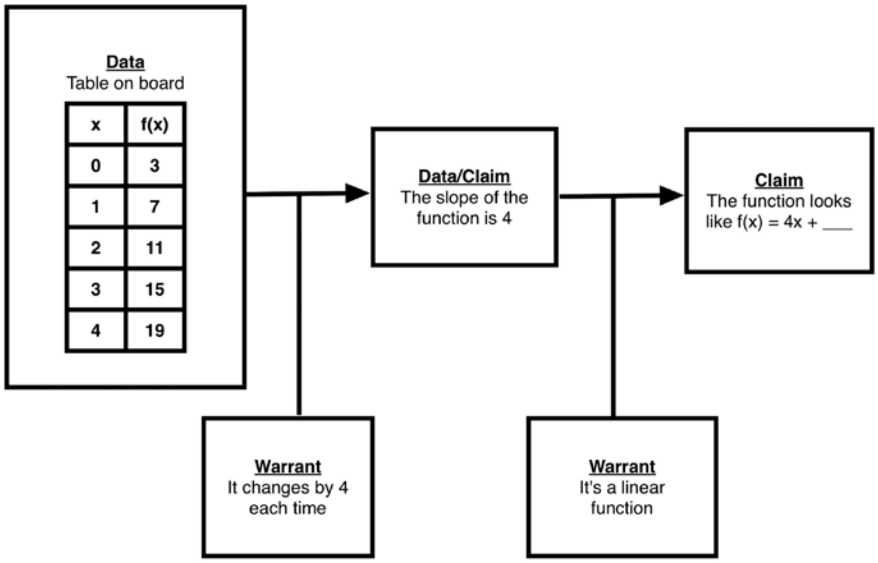

Fig. 2 The initial claim of **figure 1** is used as data to extend the argument.

USING TOULMIN'S MODEL

The findings and recommendations for practice that we discuss arise from part of a study in which we examined how teachers support collective argumentation in secondary school mathematics classes. One result of our research was a framework detailing three different kinds of actions that teachers use to support their students in making arguments (see Conner et al. 2014). Our primary analytic framework for the study was Krummheuer's (1995) adaptation of Toulmin's ([1958] 2003) model of argumentation to a classroom or collective situation.

In our study, we videotaped and analyzed units of instruction (for at least seven class days) by two teachers who exhibited a diverse range of teaching practices. We used 277 diagrams to represent the arguments that took place during our observations. We color-coded each argument component to indicate the contributor for that part of the argument—the teacher (red), the students (blue), or the teacher and students together (violet). The contributions are also distinguished by line style (solid, dotted, or dashed). Parts of arguments that were not explicitly stated but were inferred from the context of the argument we called *implicit parts*, and we indicated them with a surrounding cloud. Last, we included contributions by the teacher that are not themselves components of arguments but that prompted or responded to parts of arguments. We represented these by using red ovals that are connected to the argument components. **Figure 3** illustrates our modifications in the context of the previous argument.

The relevant questions from our research project were these:

- What parts of arguments were provided by the teacher and by the students?
- What kinds of questions from the teacher prompted students to contribute parts of arguments?
- What kinds of warrants were provided by the teacher and by the students?

In the following sections, we provide an example from Ms. Bell's classroom to exemplify the most crucial aspects of the teacher's role in collective argumentation.

ANALYZING CLASSROOM DIALOGUE

Ms. Bell's instruction regularly involved students engaging in mathematical exploration in small groups. Students then presented solution methods and connected mathematical ideas in whole-class discussions. The following teaching episode took place during a geometry unit. The students, with the support of Ms. Bell, had established a class definition for regular polygons as "all sides congruent and all angles congruent," and she posed the following problem to encourage them to continue to explore properties of regular polygons (see **fig. 4**).

Ms. Bell: [*Writes on the board, under the heading "Regular Polygon": All sides congruent, all angles congruent* (see data 1)] If I made another square [*draws a larger square on the board under the first;* (see data 2)], what is different between these two squares?
Angela: The side lengths. [see warrant 3]
Ms. Bell: So . . .

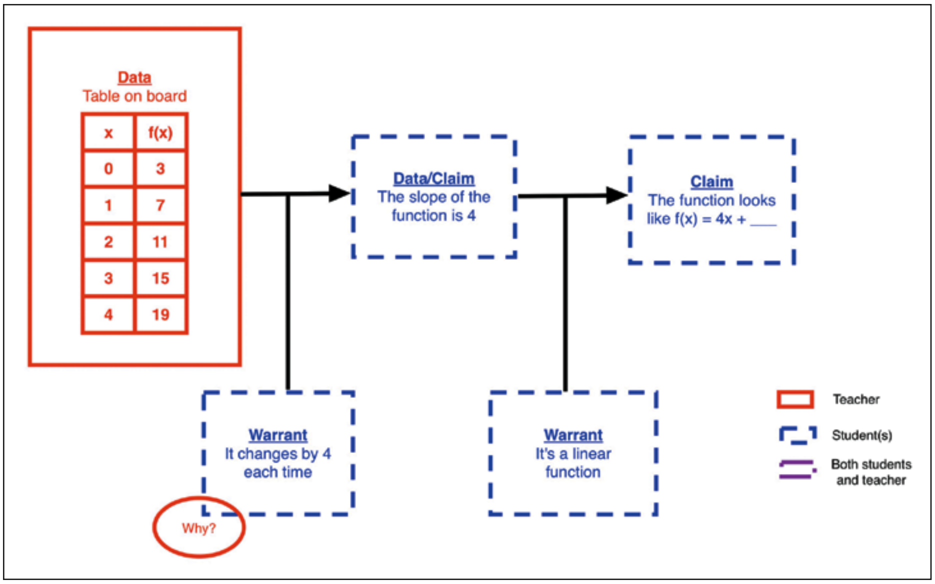

Fig. 3 Modifications to Toulmin's (1958, 2003) model lead to a diagram that includes a teacher prompt represented in a red oval.

Micah: No, just the side lengths.

Ms. Bell: So just the sides? All right, but according to our definition, all sides are congruent, all angles congruent. [see warrant 4]

Martin: It's regular because . . . [see data/claim 5]

Ms. Bell: Is it a regular polygon?

Martin: Yeah.

Ms. Bell: I guess what I'm trying to say—do you want to say it for me, Martin? You might say it better.

Martin: That you don't have to have, like, a certain size square for it to be a regular polygon. It can be large; it just has to be the same side lengths and the same angles. [see data/claim 6]

Ms. Bell: Do you guys agree?

Students: Yeah.

Ms. Bell: So my side lengths can change, as long as they're all the same, right? But what's going to stay the same? [see claim 7, with Adam's and Ms. Bell's contributions below]

Adam: The angles.

Ms. Bell: The angles. So my angles are never going to change with a regular polygon.

Within this episode, Ms. Bell and her students both contributed to the argument, with Ms. Bell providing support for some student contributions (see **fig. 4**). We use this example to illustrate some particularly relevant aspects of teachers' support for collective argumentation and then discuss how these aspects have informed our teaching of mathematics.

SUPPORTING STUDENT CONTRIBUTIONS

Student contributions are very important in collective argumentation. In our study, the arguments that were most productive often included students' contributions of warrants and claims, which made their reasoning public for the class. Teachers supported student contributions in various ways—they posed problems to be solved, contributed various parts of arguments, and asked questions to prompt student contributions of argument components.

Ms. Bell's problem posing provided the data to begin this argument. She also contributed the definition of regular polygons. She prompted students' contributions by asking questions of various kinds (see **table 1** for types of questions that teachers ask to support argumentation). Students responded to Ms. Bell's questions by contributing warrants and claims on their own or in collaboration with her. For instance, when Ms. Bell requested elaboration by asking, "Do you want to say it for me, Martin? You might say it better," Martin responded by contributing the data and claim, "That you don't have to have, like, a certain size square for it to be a regular polygon." Later, Ms. Bell requested an idea by asking, "But what's going to stay the same?" This prompted Adam to co-contribute the claim, clarifying that the angles were going to stay the same. In addition to Ms. Bell's contributions, her questions were important for making students' thinking public in the class.

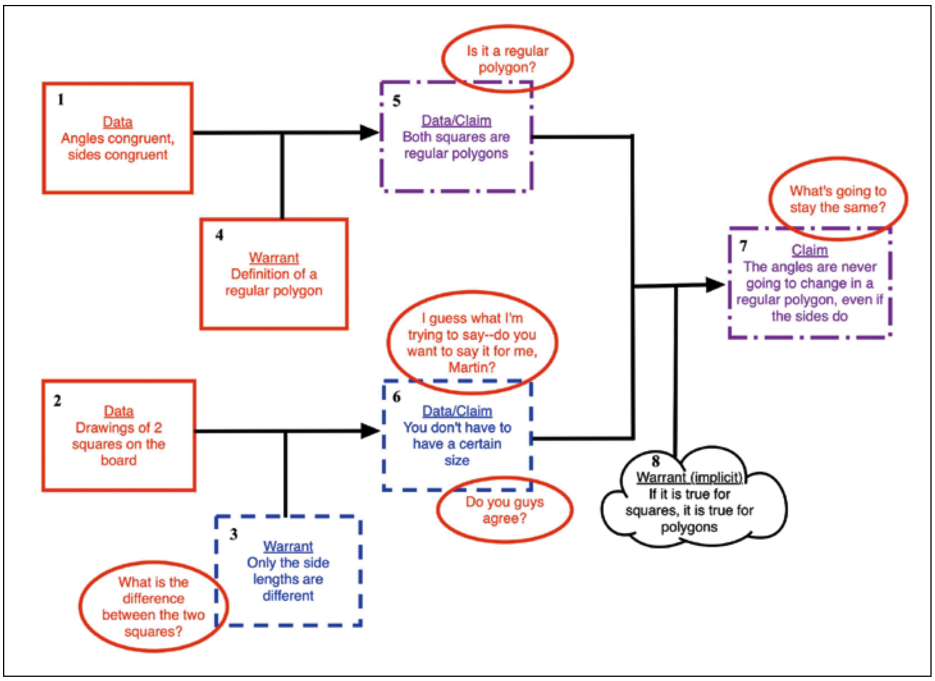

Fig. 4 An argument about regular polygons (a simplified version of an argument diagram from Conner et al. 2014) identifies seven explicit components, an implicit part, and teacher questions.

Table 1 Categories of Questions to Support Collective Argumentation

Category	Description	Example
Request a factual answer	Ask students to provide a mathematical fact	What is the square root of 16?
Request a method	Ask students to demonstrate or describe how they did or would do something	How did you solve this?
Request an idea	Ask students to compare, coordinate, or generate mathematical ideas	Looking at this table, what do you conclude?
Request an elaboration	Ask students to elaborate on some idea, statement, or diagram	Why is the answer 3 centimeters?
Request an evaluation	Ask students to evaluate a mathematical idea	Laura says that this is a right triangle. Do you agree?

Source: Adapted from Conner et al. (2014), p. 419

MAKING WARRANTS EXPLICIT

One of the most important aspects of arguments in mathematics classes is the warrant, because it is a mechanism by which reasoning can be shared. In our example, Ms. Bell provided support for a student's warrant by asking, "What is the difference between these two squares?" Angela responded to the question by saying, "the side lengths." This question requested an idea and prompted Angela to make a comparison between the two squares, thus providing the warrant.

When teachers pose appropriate questions, students are more likely to make their reasoning explicit by contributing a warrant. When a teacher does not ask a question, three outcomes are possible: (1) no warrant is given, so the reasoning remains implicit; (2) the teacher provides the warrant; or (3) the students provide the warrant unprompted. Sometimes, when students are working with a familiar idea or in a familiar way, in the teacher's judgment no warrant is necessary. For instance, the question "If it is true for squares, it is true for polygons?" is implicit in our example. We inferred that the students were reasoning inductively, given the content of their claim and the teacher's intention. An implicit warrant may result in reasoning being obscured for some students. Teachers should attend to which warrants should be made explicit in the class. Thus, sometimes the teacher will choose to contribute the warrant rather than allowing it to remain implicit.

Sometimes students provide warrants without explicit support from the teacher. Although not evident in this particular classroom episode, we observed several arguments in which students contributed warrants without a specific question from Ms. Bell. Experience in her classroom and many others suggests that asking students to provide warrants can establish norms (i.e., classroom practices that have become usual or expected) such that students provide warrants more autonomously. These norms support students engaging in collective argumentation.

The kinds of warrants along with the frequency of contributions depend on norms established in a classroom (Conner et al. 2011). Sometimes it is difficult to determine, in the midst of a conversaton, whether students are contributing warrants or other parts of arguments. Students use many different kinds of statements as warrants: mathematical concepts (e.g., definitions, theorems, and properties); mathematical procedures (e.g., calculations or solving equations); observations (e.g., recognizing congruence markings or patterns); and appeals to authority (e.g., referencing a book or teacher). These kinds of statements can be signals that a student is contributing a warrant within an argument. Some of these statements are the same kinds of statements that could be used in proofs. However, other less formal or even less mathematical warrants can be seen in collective argumentation.

As teachers, we need to be aware of the kinds of statements students are making and encourage them to contribute warrants that are appropriate and make sense in the situation. Students may give simplistic or authoritarian warrants (e.g., "That is what the book said"); however, providing theorems, properties, or previously established results may not come naturally, so teachers may need to support students in contributing these kinds of mathematically appropriate warrants. We need to work together with our students to establish the kinds of warrants that are acceptable given the context of a mathematical discussion.

IMPLICATIONS FOR THE CLASSROOM

From Ms. Bell's practice, we see examples of her support that encouraged students to contribute argument components. She asked questions that encouraged her students to share their reasoning with the class. In particular, she often asked her students "Why?" or to explain their reasoning, prompting her students to contribute warrants. Regularly expecting students to explain their reasoning translated into a classroom norm, because her students began contributing their reasoning without specific prompting from Ms. Bell. In addition, she regularly used mathematical tasks that valued sense making, allowing her students to progress from intuition to deductive reasoning as they developed understanding of mathematical concepts. Her choice of tasks allowed the students to contribute their mathematical thinking in the form of warrants and claims. These types of practices established norms in her classroom that supported her students' engaging in productive argumentation.

Our study of Ms. Bell's support for argumentation has informed our teaching practices as both mathematics teachers and mathematics teacher educators. Reflecting on Ms. Bell's support for argumentation and the norms in her classroom has led us to ask ourselves questions about the argumentation in our classes: To what extent did the students contribute to the argument? How did I support student contributions to

the argument? Did students provide warrants to make their reasoning public? What changes in classroom norms would highlight students' reasoning?

These questions are helpful in examining important aspects of collective argumentation, such as student contributions, contributions of warrants, and the establishment of norms. We encourage teachers to use these questions to continually refine the collective argumentation in their classrooms.

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