DELVING deeper

Charles F. Marion

The Coffee-Milk Mixture Problem Revisited

Recently, I had occasion to re-read George Pólya's (1957) classic work on problem solving in mathematics, *How to Solve It*. Early in the book, Pólya describes the different phases of problem solving, including the fourth and final phase—"looking back." Describing that phase, he writes:

> Even fairly good students, when they have obtained the solution of the problem and written down neatly the argument, shut their books and look for something else. Doing so, they miss an important and instructive phase of the work. By looking back at the completed solution, by reconsidering and reexamining the path that led to it, they could consolidate their knowledge and develop their ability to solve problems. A good teacher should understand and impress on his students the view that no problem whatever is completely exhausted. There remains always something to do; we could improve any solution and, in any case, we can always improve our understanding of the solution. (Pólya 1957, pp. 14–15)

Pólya's words seem to suggest that looking back at the solution of a problem should occur shortly after the problem has been solved. I took Pólya's advice some years after first encountering a problem that is frequently posed at professional development workshops, in print (Gamow and Stern 1958; Gardner 1959), and on the Web (QED Infinity 2014; Wikipedia). Although my look back was long overdue according to Pólya's guidelines, I was surprised and delighted to find just how far my revisit took me.

Edited by Theodore Hodgson

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Department editors
Brian M. Dean, *bdean@pasco.k12.fl.us, District School Board of Pasco County, FL;* **Daniel Ness,** *nessd@stjohns.edu, St. John's University, Jamaica, New York; and* **Nick Wasserman,** *wasserman@tc.columbia.edu, Teachers College, Columbia University, New York*

THE PROBLEM AND INITIAL SOLUTIONS

The problem that I revisited starts with two cups, each containing exactly the same amount of different liquids. Some versions of the problem mix water and wine, others cream and coffee. The version I used mixes milk and coffee:

> Suppose that you have a white cup of milk and a black cup of coffee, neither filled to the brim and each containing 8 ounces of liquid. Suppose further that you remove an ounce of milk from the white cup, place it in the black cup, and thoroughly mix the resulting contents. Then remove an ounce of the coffee-milk mixture and place it in the white cup. What can you say about the amount of milk in the black cup in relation to the amount of coffee in the white cup? Are the two amounts equal, or is one amount greater than the other?

After the initial transfer of an ounce of milk to the black cup, the white cup contains 7 ounces of milk and the black cup contains 1 ounce of milk and 8 ounces of coffee. Assuming a thorough mixing of the contents of the coffee-milk mixture, the black cup now contains 9 ounces of liquid, 1 part milk and 8 parts coffee. Thus, 1/9 of the ounce transferred back to the white cup will be milk and 8/9 will be coffee. **Table 1** presents the results of these two transfers; as the table illustrates, the amount of coffee in the white cup is the same as the amount of milk in the black cup.

Table 1 The Results of Six Transfers between the White and Black Cups

	White Cup		Black Cup	
	Ounces of Milk	Ounces of Coffee	Ounces of Milk	Ounces of Coffee
Initial Condition	8	0	0	8
After One Transfer	$8-\frac{1}{8}\bullet 8=7$	0	$0+\frac{1}{8}\bullet 8=1$	8
After Two Transfers	$7+\frac{1}{9}\bullet 1=7\frac{1}{9}$	$0+\frac{1}{9}\bullet 8=\frac{8}{9}$	$1-\frac{1}{9}\bullet 1=\frac{8}{9}$	$8-\frac{1}{9}\bullet 8=7\frac{1}{9}$
Percentage of milk in the white cup is $\frac{7\frac{1}{9}}{8}\approx 88.9\%$				
After Three Transfers	$7\frac{1}{9}-\frac{1}{8}\bullet 7\frac{1}{9}=6\frac{2}{9}$	$\frac{8}{9}-\frac{1}{8}\bullet\frac{8}{9}=\frac{7}{9}$	$\frac{8}{9}+\frac{1}{8}\bullet 7\frac{1}{9}=1\frac{7}{9}$	$7\frac{1}{9}+\frac{1}{8}\bullet\frac{8}{9}=7\frac{2}{9}$
After Four Transfers	$6\frac{2}{9}+\frac{1}{9}\bullet 1\frac{7}{9}=6\frac{34}{81}$	$\frac{7}{9}+\frac{1}{9}\bullet 7\frac{2}{9}=1\frac{47}{81}$	$1\frac{7}{9}-\frac{1}{9}\bullet 1\frac{7}{9}=1\frac{47}{81}$	$7\frac{2}{9}-\frac{1}{9}\bullet 7\frac{2}{9}=6\frac{34}{81}$
Percentage of milk in the white cup is $\frac{6\frac{34}{81}}{8}\approx 80.2\%$				
After Five Transfers	$6\frac{34}{81}-\frac{1}{8}\bullet 6\frac{34}{81}=5\frac{50}{81}$	$1\frac{47}{81}-\frac{1}{8}\bullet 1\frac{47}{81}=1\frac{31}{81}$	$1\frac{47}{81}+\frac{1}{8}\bullet 6\frac{34}{81}=2\frac{31}{81}$	$6\frac{34}{81}+\frac{1}{8}\bullet 1\frac{47}{81}=6\frac{50}{81}$
After Six Transfers	$5\frac{50}{81}+\frac{1}{9}\bullet 2\frac{31}{81}=5\frac{643}{729}$	$1\frac{31}{81}+\frac{1}{9}\bullet 6\frac{50}{81}=2\frac{86}{729}$	$2\frac{31}{81}-\frac{1}{9}\bullet 2\frac{31}{81}=2\frac{86}{729}$	$6\frac{50}{81}-\frac{1}{9}\bullet 6\frac{50}{81}=5\frac{643}{729}$
Percentage of milk in the white cup is $\frac{5\frac{643}{729}}{8}\approx 73.5\%$				

With this result in hand and Pólya's admonition that "there remains always something to do" in mind, I considered how I might build on my initial work on this problem. In particular, I wondered what would be the result of continuing the process of transferring an ounce of the contents of the two cups from one to the other? Intuitively, it seems evident that the contents of each cup would trend toward some steady state—perhaps an even distribution of milk and coffee in each cup—since each transfer from the white cup to the black cup would increase the percentage of milk in that cup. Likewise, each transfer from the black to the white cup would increase the percentage of coffee in the milk, at least until the contents of the two cups were identical. However, questions about the number of transfers needed to reach a steady state or a particular mixture—say, a 75%-25% mixture of the two liquids in each cup—still remained. I opted to explore the latter of these questions.

To determine the number of transfers required to reach a 75%-25% mix in each cup, I first used an arithmetic strategy. In particular, after two transfers, the number of ounces of milk in the white cup, 7 1/9, divided by the number of ounces of liquid in the cup, 8, rounded to the nearest integer percentage point is 89%. To find the number of transfers needed for that percentage to be 75, we can extend the work of **table 1**, retaining exact percentages after each transfer to avoid round-off errors. The percentage of milk in the black cup and of coffee in the white cup first exceeds 25% after six transfers of the liquids (or three complete back-and-forth transfers).

I now had the answer to the question, How many transfers of one ounce of liquid from one cup to the other would it take to reach a distribution of at most 75% milk in the white cup? Aside from having an idea of the calculations that I needed to perform, however, I did not have an efficient way to answer more general and complex questions, such as these:

- What percentage of the liquid in the white cup would be milk after 10 transfers if, once again, each cup initially contained 8 ounces of liquid and each transfer moved 1 ounce?

- How many transfers would it take to reach a distribution that left at most 60% milk in the white cup if there were initially 10 ounces of the two liquids in each cup and 3 ounces of liquid were moved with each transfer?

To avoid the onerous arithmetic calculations needed to answer these and other more complex questions, I knew that I had to adopt a more comprehensive methodology to generalized coffee-milk mixture problems. As is often the case when considering several variables and seeking generalized descriptions of numeric patterns, the language of algebra is particularly useful.

A GENERALIZATION OF THE COFFEE-MILK MIXTURE PROBLEM

The generalized problem may be stated this way:

> Suppose that we have a white cup of milk and a black cup of coffee, neither filled to the brim and each containing x ounces of liquid. Suppose further that y ounces of milk are removed from the first white cup and placed in the second black cup and the contents are thoroughly mixed. Then y ounces of the mixture are transferred from the second black cup to the first white cup. How much milk will be in the white cup after $2n$ such transfers?

To solve this generalized problem, I first assumed that each cup had a capacity of at least $x + y$ ounces; otherwise, the first transfer would exceed the capacity of the cup, a situation practically impossible to model. Guided by the calculations in **table 1**, I developed expressions to represent the results for the first six transfers (see **table 2**).

As I reviewed the expressions in **table 2** after an even number of transfers had been completed, a pattern seemed to emerge. In an attempt to make that pattern more discernible, I rewrote those expressions with the variable x factored from each, as in **table 3**. I knew now that I was dealing with alternating terms of binomial expansions.

In particular, the numerators of the fractional expressions were the sums of alternating terms of the denominators after they had been expanded. For example, consider the expansion of $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.

The sum of the first and third terms, those with odd powers of x, forms the numerator of the fractional expression $(x^3 + 3xy^2)/(x + y)^3$, used to represent the amount of coffee in the black cup. Similarly, the sum of the second and the fourth expressions forms the numerator of $(3x^2y + y^3)/(x + y)^3$, used to represent the amount of coffee in the white milk cup.

When we compare the expansions of $(x + y)^3$ and $(x - y)^3$, it becomes apparent that the sum of the expansions has only odd powers of x, whereas the difference has only even powers of x. Specifically,

$$\frac{\left(x+y\right)^3+\left(x-y\right)^3}{2}=x^3+3xy^2$$

and

$$\frac{\left(x+y\right)^3-\left(x-y\right)^3}{2}=3x^2y+y^3.$$

With these two identities at my disposal, I conjectured that the amounts of liquid in the two cups after $2n$ transfers could be represented in terms of the binomial expressions $(x + y)^n$ and $(x - y)^n$, as in the last row of **table 3**.

This conjecture can be verified using mathematical induction. To start, when $n = 1$,

$$\begin{aligned}\frac{x}{2}\bullet\frac{\left(x+y\right)^n+\left(x-y\right)^n}{\left(x+y\right)^n}&=\frac{x}{2}\bullet\frac{\left(x+y\right)^1+\left(x-y\right)^1}{\left(x+y\right)^1}\\&=\frac{x}{2}\bullet\frac{2x}{x+y}\\&=x\bullet\frac{x}{x+y}.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{x}{2}\bullet\frac{\left(x+y\right)^n-\left(x-y\right)^n}{\left(x+y\right)^n}&=\frac{x}{2}\bullet\frac{\left(x+y\right)^1-\left(x-y\right)^1}{\left(x+y\right)^1}\\&=\frac{x}{2}\bullet\frac{2y}{x+y}\\&=x\bullet\frac{y}{x+y}.\end{aligned}$$

What remains to be shown is that if the amount of coffee in the black cup after $2n$ transfers (or n complete back-and-forth transfers) is represented as

$$B_{2n}=\frac{x}{2}\bullet\frac{\left(x+y\right)^n+\left(x-y\right)^n}{\left(x+y\right)^n}$$

then

$$B_{2n+2}=\frac{x}{2}\bullet\frac{\left(x+y\right)^{n+1}+\left(x-y\right)^{n+1}}{\left(x+y\right)^{n+1}}$$

would be that amount after $2n + 2$ such transfers.

After $2n$ transfers, the amount of coffee in the white cup is

$$W_{2n}=\frac{x}{2}\bullet\frac{\left(x+y\right)^n-\left(x-y\right)^n}{\left(x+y\right)^n}.$$

Table 2 Amounts of Liquid for the Generalized Problem

	White Cup		Black Cup	
	Amount of Milk	Amount of Coffee	Amount of Milk	Amount of Coffee
Initial Conditions	x	0	0	x
After One Transfer	$x-y$	0	y	x
After Two Transfers	$x-y+\frac{y}{x+y}\cdot y=\frac{x^2}{x+y}$	$0+\frac{y}{x+y}\cdot x=\frac{xy}{x+y}$	$y-\frac{y}{x+y}\cdot y=\frac{xy}{x+y}$	$x-\frac{y}{x+y}\cdot x=\frac{x^2}{x+y}$
After Three Transfers	$\frac{x^2}{x+y}-\frac{y}{x}\cdot\frac{x^2}{x+y}=\frac{x^2-xy}{x+y}$	$\frac{xy}{x+y}-\frac{y}{x}\cdot\frac{xy}{x+y}=\frac{xy-y^2}{x+y}$	$\frac{xy}{x+y}+\frac{y}{x}\cdot\frac{x^2}{x+y}=\frac{2xy}{x+y}$	$\frac{x^2}{x+y}+\frac{y}{x}\cdot\frac{xy}{x+y}=\frac{x^2+y^2}{x+y}$
After Four Transfers	$\frac{x^2-xy}{x+y}+\frac{y}{x+y}\cdot\frac{2xy}{x+y}=\frac{x^3+xy^2}{(x+y)^2}$	$\frac{xy-y^2}{x+y}+\frac{y}{x+y}\cdot\frac{x^2+y^2}{x+y}=\frac{2x^2y}{(x+y)^2}$	$\frac{2xy}{x+y}-\frac{y}{x+y}\cdot\frac{2xy}{x+y}=\frac{2x^2y}{(x+y)^2}$	$\frac{x^2+y^2}{x+y}-\frac{y}{x+y}\cdot\frac{x^2+y^2}{x+y}=\frac{x^3+xy^2}{(x+y)^2}$
After Five Transfers	$\frac{x^3+xy^2}{(x+y)^2}-\frac{y}{x}\cdot\frac{x^3+xy^2}{(x+y)^2}=\frac{x^3+xy^2-x^2y-y^3}{(x+y)^2}$	$\frac{2x^2y}{(x+y)^2}-\frac{y}{x}\cdot\frac{2x^2y}{(x+y)^2}=\frac{2x^2y-2xy^2}{(x+y)^2}$	$\frac{2x^2y}{(x+y)^2}+\frac{y}{x}\cdot\frac{x^3+xy^2}{(x+y)^2}=\frac{3x^2y+y^3}{(x+y)^2}$	$\frac{x^3+xy^2}{(x+y)^2}+\frac{y}{x}\cdot\frac{2x^2y}{(x+y)^2}=\frac{x^3+3xy^2}{(x+y)^2}$
After Six Transfers	$\frac{x^3+xy^2-x^2y-y^3}{(x+y)^2}+\frac{y}{x+y}\cdot\frac{3x^2y+y^3}{(x+y)^2}=\frac{x^4+3x^2y^2}{\left(x+y\right)^3}$	$\frac{2x^2y-2xy^2}{(x+y)^2}+\frac{y}{x+y}\cdot\frac{x^3+3xy^2}{(x+y)^2}=\frac{3x^3y+xy^3}{\left(x+y\right)^3}$	$\frac{3x^2y+y^3}{(x+y)^2}-\frac{y}{x+y}\cdot\frac{3x^2y+y^3}{(x+y)^2}=\frac{3x^3y+xy^3}{\left(x+y\right)^3}$	$\frac{x^3+3xy^2}{(x+y)^2}-\frac{y}{x+y}\cdot\frac{x^3+3xy^2}{(x+y)^2}=\frac{x^4+3x^2y^2}{\left(x+y\right)^3}$

During transfer $2n+1$, a fraction, y/x, of the liquid is removed from the white cup and added to the black cup, which now contains coffee in the amount of $B_{2n+1}=B_{2n}+(y/x)W_{2n}$. Then, during transfer $2n+2$, a fraction, $y/(x+y)$, of the liquid in the black cup is transferred back to the white cup. Thus, it remains only to prove that $B_{2n}+(y/x)\cdot W_{2n}-y/(x+y)\cdot B_{2n+1}=B_{2n+2}$. An equivalent equation, after we multiply both sides by $2(x+y)^{n+1}/x$, is this:

$$
\begin{aligned}
&\left(x+y\right)\cdot\left(\left(x+y\right)^n+\left(x-y\right)^n\right)\\
&\quad+\frac{y}{x}\left(x+y\right)\cdot\left(\left(x+y\right)^n-\left(x-y\right)^n\right)\\
&\quad-y\cdot\left(\left(x+y\right)^n+\left(x-y\right)^n\right)\\
&\quad-\frac{y^2}{x}\cdot\left(\left(x+y\right)^n-\left(x-y\right)^n\right)\\
&\quad=\left(x+y\right)^{n+1}+\left(x-y\right)^{n+1}
\end{aligned}
$$

The first and third lines of the previous equation combine to give

$$x\cdot\left(\left(x+y\right)^n+\left(x-y\right)^n\right).$$

The second and fourth lines may be combined as

$$y\cdot\left(\left(x+y\right)^n-\left(x-y\right)^n\right).$$

Expanding these, adding, and regrouping yields our desired outcome:

$$\left(x+y\right)^{n+1}+\left(x-y\right)^{n+1}$$

Returning to our original context by dividing this result by $2(x+y)^{n+1}/x$, we recover the expression we sought for the amount of coffee in the black

Table 3 A Modified Version of Table 2

	White Cup		Black Cup	
	Amount of Milk	Amount of Coffee	Amount of Milk	Amount of Coffee
Initial Condition	x	0	0	x
After 2 Transfers	$x \bullet \frac{x}{x+y}$	$x \bullet \frac{y}{x+y}$	$x \bullet \frac{y}{x+y}$	$x \bullet \frac{x}{x+y}$
After 4 Transfers	$x \bullet \frac{x^2+y^2}{(x+y)^2}$	$x \bullet \frac{2xy}{(x+y)^2}$	$x \bullet \frac{2xy}{(x+y)^2}$	$x \bullet \frac{x^2+y^2}{(x+y)^2}$
After 6 Transfers	$x \bullet \frac{x^3+3xy^2}{(x+y)^3}$	$x \bullet \frac{3x^2y+y^3}{(x+y)^3}$	$x \bullet \frac{3x^2y+y^3}{(x+y)^3}$	$x \bullet \frac{x^3+3xy^2}{(x+y)^3}$
⋮	⋮	⋮	⋮	⋮
After $2n$ Transfers	$\frac{x}{2} \bullet \frac{(x+y)^n+(x-y)^n}{(x+y)^n}$	$\frac{x}{2} \bullet \frac{(x+y)^n-(x-y)^n}{(x+y)^n}$	$\frac{x}{2} \bullet \frac{(x+y)^n-(x-y)^n}{(x+y)^n}$	$\frac{x}{2} \bullet \frac{(x+y)^n+(x-y)^n}{(x+y)^n}$

cup after $2n + 2$ transfers. This completes the induction process and proves that our conjecture was true.

Further, we can rewrite the formula as

$$(1)\quad B_{2n} = \frac{x}{2} \bullet \frac{\left(x+y\right)^n + \left(x-y\right)^n}{\left(x+y\right)^n} = \frac{x}{2} \bullet \left(1 + \left(\frac{x-y}{x+y}\right)^n\right).$$

Similarly,

$$(2)\quad W_{2n} = \frac{x}{2} \bullet \frac{\left(x+y\right)^n - \left(x-y\right)^n}{\left(x+y\right)^n} = \frac{x}{2} \bullet \left(1 - \left(\frac{x-y}{x+y}\right)^n\right).$$

In his description of the "looking back" component of problem solving, Pólya suggests that, once a problem has been solved, "if there is some rapid and intuitive procedure to test either the result or the argument, it should not be overlooked" (Pólya 1957, p. 15).

To test the reasonableness of equation (1), examine specific cases for each variable. When $y = x$ and $n = 1$, for instance, the expressions in both equations (1) and (2) reduce to $x/2$. The assumptions $y = x$ and $n = 1$ correspond to initially transferring all the milk in the white cup into the black cup; accordingly, after the second transfer, each cup would contain equal amounts, $x/2$, of milk and coffee.

A USEFUL SUBSTITUTION

The fraction $(x - y)/(x + y)$ is central to the mathematics in this problem. For convenience, we will use $X = (x - y)/(x + y)$. Now, suppose that y is less than x so that the fraction X is a positive number less than 1. In that case, if n is greater than m, then $X^n < X^m$. This corresponds to what we would expect to happen as the number of transfers is increased; namely, the amount of milk in the white cup decreases. On the other hand, no matter how large n becomes, the value of X^n is always greater than zero, indicating that there will always be more milk than coffee in the white cup. Even though the limit as n approaches infinity of $x/2 \cdot (1 + X)^n$ is $x/2$, for finite values of n it is true that $x/2 \cdot (1 + X)^n$ is always greater than $x/2 \cdot (1 - X)^n$.

In keeping with Pólya's suggestion that we examine the solution in multiple ways, I decided to use equation (1) to solve three additional problems that start with the conditions assumed in my generalization of the coffee-milk mixture problem.

Problem 1: Suppose that we start with 10 ounces of liquid in each cup and 3 ounces are shifted with each transfer. After 8 transfers, what percentage of liquid in the white milk cup will then be milk?

Solution: Remembering that the cup contains x ounces of liquid and using the variable $X = (x - y)/(x + y)$, we know that the fraction of milk in the white cup after $2n$ transfers is $x/2 \cdot (1 + X^n)$ divided by x, or $(1 + X^n)/2$. With $x = 10$, $y = 3$, and $n = 4$, we have $X = (10 - 3)/(10 + 3) = 7/13$ and $(1 + X^n)/2 = (1 + (7/13)^4)/2 \approx 54\%$.

Problem 2: Suppose that we start with 10 ounces of

the two liquids in each cup and that 3 ounces of liquid are shifted with each transfer. After how many transfers will the percentage of milk in the white cup be at most 51%?

Solution: Defining p_{2n} to be the percentage of milk in the white cup after $2n$ transfers, we have $p_{2n} = (1 + X^n)/2$. Our initial conditions correspond to $x = 10$, $y = 3$, and $X = (10 - 3)/(10 + 3) = 7/13$. Thus, we solve $0.51 = (1 + (7/13)^n)/2$. Equivalently, we can write $(7/13)^n = 0.02$. Taking the logarithm of both sides gives us $n \log(7/13) = \log(0.02)$, or $n \approx 6.32$. Since we cannot have a fractional number of transfers, this value can be interpreted as telling us that it will take 14 transfers (or $n = 7$ complete back-and-forth transfers) before the white cup contains less than 51% milk.

Problem 3: Suppose that we start with 20 ounces of liquid in each cup and wish to have at most 60% milk in the white cup after 10 transfers. How much liquid should be transferred at each step to obtain these conditions?

Solution: The conditions in this problem are reflected in the equation $p_{2n} = (1 + X^n)/2$ with $p_{2n} = 0.6$, $n = 10/2 = 5$, and $x = 20$, leading to $X = (20 - y)/(20 + y)$. Thus, we start with $0.6 = (1 + X^5)/2$ or, equivalently, $5\log(X) = \log(0.2)$. From this, we have $X = 10^{\log(0.2)/5} = \sqrt[5]{0.2} \approx 0.725$. Returning to the variable y, we find that

$$\frac{20 - y}{20 + y} = \sqrt[5]{0.2},$$

so $y = 20(1 - \sqrt[5]{0.2})/(1 + \sqrt[5]{0.2}) \approx 3.19$. This value tells us that if we execute 10 transfers of about 3.2 ounces, the white cup will contain about 60% milk.

TIMELESS ADVICE

We have come a long way from solving the well-known coffee-milk mixture problem. Taking Pólya's counsel to heart, we first solved a slight extension of that problem and then posed a more ambitious generalization. A long journey through an algebraic thicket uncovered a formula that gave us, in turn, the ability to answer a multitude of other questions. The mathematics needed to answer those questions ranged from the evaluation of algebraic expressions to the use of logarithms to solve equations.

Our analysis of the coffee-milk mixture riddle also illustrates the timeless advice of Pólya's masterpiece *How to Solve It*, a work that has enriched the lives of generations of problem solvers. Continuously in print since 1945, it will likely serve future generations of mathematics teachers and their students equally well. We can only imagine the mathematical explorations and discoveries that will come to light when, as Pólya recommends, other problems previously solved and put aside are revisited.

REFERENCES

Gamow, George, and Marvin Stern. 1958. *Puzzle-Math*. New York: Viking Press.

Gardner, Martin. 1959. *The Scientific American Book of Mathematical Puzzles and Diversions*. New York: Simon and Schuster.

Pólya, George. 1945 (1957). *How to Solve It: A New Aspect of Mathematical Method.* Princeton, NJ: Princeton University Press.

QED Infinity. "Problem 1: Mixing Milk and Coffee." Problem of the Day: January 28, 2013. http://www.qedinfinity.com/problem-1-mixing-milk- and-coffee/

Wikipedia: The Free Encyclopedia. "Wine/water mixing problem." http://en.wikipedia.org/wiki/Wine/water_mixing_problem

CHARLES F. MARION, charliemath@optonline.net, taught mathematics at Lakeland High School in Shrub Oak, New York, for thirty-two years. In retirement, he has enjoyed searching for and writing about undiscovered patterns in elementary number theory and sharing his enthusiasm for mathematics with his five grandchildren.