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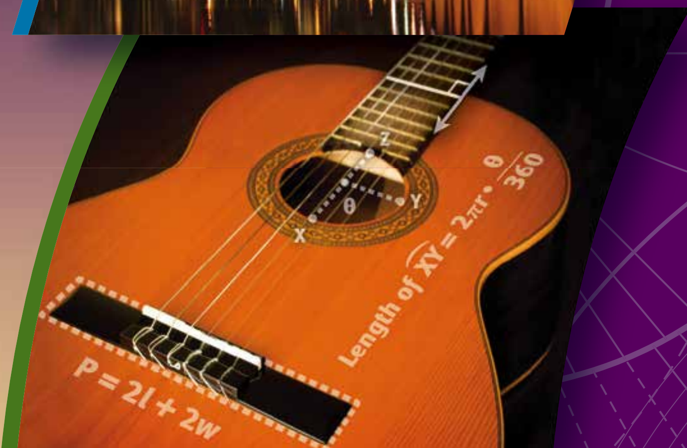

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READER reflections

REFRESHING A RATIONAL APPROACH TO THE CENTER OF MASS

Rational Functions and Balance in the Real World

Kudos to Theodore Hodgson and David Schultz ("A Refreshingly Rational Approach to the Center of Mass," Delving Deeper, *MT* May 2015, vol. 108, no. 9, pp. 710–15) for seeking real-world rational functions. Using a meterstick and small weights to explore center of mass does not generally work in the real world; whether the weights are placed at the ends of the stick or not, the stick balances at the center of mass of the dataset represented by the weights only when that point coincides with the meterstick's midpoint. (For more information about balance models for measures of center, see my article [*Journal of Statistics Education*, Nov. 2014].)

ACT in Algebra (1998, McGraw Hill), a textbook that I co-authored, offers additional real-world examples. Also, a rational function model of perspective drawing appears in "Rational Functions: A New Perspective," by Anna A. Davis, Ronald E. Zielke, and Jessica L. Lickeri (Activities for Students, *MT* March, vol. 104, no. 7, pp. 538–45). Finally, recall that the finite geometric series formula, $(1 - r^{n+1})/(1 - r)$, gives a rational function of r with a "hole" at $r = 1$.

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We appreciate the interest and value the views of those who write. Readers commenting on articles are encouraged to send copies of their correspondence to the authors. For publication: All letters for publication are acknowledged. Letters to be considered for publication should be in MS Word document format and sent to **mt@nctm.org**. Letters should not exceed 250 words and are subject to abridgment. At the end of the letter include your name and affiliation, if any, including email address, per the style of the section.

Tipping the Balance

It seems that Hodgson and Schultz's article contains an error. In attempting an intuitive explanation for the lowest position of the soda can's center of mass, the authors assert that, at this position, the mass above (on the left of) and below (on the right of) the center of mass (liquid level) are equal. But this is not necessarily—in fact, is never—the case.

As a counterexample, consider an extremely dense liquid for which, at the lowest position of the center of mass, the mass of the thin liquid layer at the bottom of the can has to balance practically the entire mass of the can. Clearly, this cannot happen if the two masses are equal; the liquid must be much more massive to compensate for its proximity to the balancing point. Notice that the graphs in figure 6 (p. 714) contradict the authors' very statement. For example, the green graph shows that the lowest position of the center of mass is attained at a height of 2.0 cm. According to the green inset, however, the mass below the liquid level is $2 + 2/8 = 2.25$ g, whereas the mass above is only $6/8 = 0.75$ g.

Moreover, the authors assert that removing some mass from the right side of the can shifts the center of mass to the left. Although the can's center of mass does shift to the left, the shift is not entirely due to the reduced mass. One can see that this logic is insufficient by considering the analogous situation in which the center of mass is not at its lowest position; then, dropping the liquid level would clearly shift the can's center of mass to the right to bring it closer to the lowest position. I know of no simple way that could explain this shift, other than by directly proving that changing the height h of the liquid on the right side by an arbitrary amount x always tips the balance to the left by an amount proportional to x^2.

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Hodgson and Schultz reply: We thank Mr. Rizcallah for noting our erroneous statement in the explanation of the soda can system. We should have stated that the *product* of the mass on either side of the balance point and the distance between the center of mass of each side and the center of mass of the system are equal. Seeking an intuitive understanding of the minimum height of the center of mass of the system, we offered an oversimplified and incorrect explanation.

Rather than the mass of each component, the inequalities in figure 7 (p. 714) should address the *moment*—that is, the product of the mass and the distance from the center of mass to the balancing point—of each component. When we consider the frozen soda can system, the moments, not the masses, to the left and right of the balancing point must equal one another.

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ASSESSMENT CRITERIA AFFECT EVALUATION

An old Cold War joke provides an alternative evaluation of an event. In an East-West race, in which the best runner from each side competed, the loser's headline reads, "We finished second, and they were next to last." Well, here's a numerical example of how rules for assessment matter.

In a multinational tournament, where only the two finalists in each event are given medals, country A's contestants win 5 gold medals and 4 silver medals, while country B's win 10 silver. Which country "did better"? The fact that B won more medals than A speaks in its favor. But if we value gold as 2 points and silver at only 1, then A has 14 points and B only 10. On the other hand, if gold is worth 7 and silver 6, then A has 59 points and B, with

60 points, has the superior performance. Obviously, the root of this disagreement is how we assign points.

Fortunately, elementary algebra provides the explanation. If we assign x points to gold and y to silver, then A has $5x + 4y$ points and B has $10y$. Therefore, A has more points whenever $5x > 6y$. (In the same way, we could change the result of a football game by assigning new point values to touchdowns and field goals.)

Similar analysis applies if we allow bronze medals, with value z points. Suppose that A had also won 3 bronze medals and B had won 4. In total, B has 14 medals compared with A's 12. So did B have the superior performance? Accordingly, let's examine the points: A has $5x + 4y + 3z$, and B has $10y + 4z$. Therefore, A would be ranked higher if $5x > 6y + z$. Of course, the same discussion applies if participants have quantities of gold, silver, and bronze medals different from those of our example. The weighting of the medals most certainly affects how we evaluate the quality of the overall performance.

The same principle applies to academic testing. The way points are distributed among the questions can drastically affect the test score.

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***Editor's note*:** Similar ideas are raised in "Ranking the Safest U.S. Cities," by Michael A. Jones and Jennifer M. Wilson (Media Clips, *MT* April 2015, vol. 108, no. 8, pp. 569, 571).

A SERIES EMERGES FROM REPEATED INTEGRATION BY PARTS

My AP Calculus teacher challenged us with the possibility of finding an antiderivative for e^{x^2}. I approached this by thinking about how I could create a function that, after taking the derivative with product rule, ends up as e^{x^2}. Realizing that this was the same logic behind integration by parts, I tried integrating systematically with the integration by parts formula. (See **fig. 1 [Arseneault].**)

The result,

$$\int e^{x^2}dx$$

$$\boxed{\begin{array}{ll} u = e^{x^2} & dv = dx \\ du = 2xe^{x^2}dx & v = x \end{array}}$$

$$\int e^{x^2}dx = xe^{x^2} - \int 2x^2e^{x^2}dx + C$$

$$\boxed{\begin{array}{ll} u = e^{x^2} & dv = 2x^2dx \\ du = 2xe^{x^2}dx & v = \frac{2}{3}x^3 \end{array}}$$

$$\int e^{x^2}dx = xe^{x^2} - \left[\frac{2}{3}x^3e^{x^2} - \int \frac{4}{3}x^4e^{x^2}dx\right] + C$$

Fig. 1 (Arseneault)

$$C + xe^{x^2} - \frac{2}{3}x^3e^{x^2} + \frac{4}{15}x^5e^{x^2} - \frac{8}{105}x^7e^{x^2} + \cdots,$$

is confirmed by taking the derivative. (See **fig. 2 [Arseneault].**)

Writing the series with an e^{x^2} term factored out, as

$$e^{x^2}\left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7 + \cdots\right) + C,$$

I noticed the odd exponents, the alternating sign, and increasing powers of 2. The pattern in the denominators was more difficult, but I realized that the numbers were all products of odd numbers in the pattern 1 • 3 • 5 • 7. . . . I thought of this as an "odd factorial" and researched it later, learning that it is called a "double factorial" and denoted as $n!!$

The embedded series can thus be written as

$$x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7 + \cdots = \sum_{n=0}^{\infty}\frac{(-2)^n x^{2n+1}}{(2n+1)!!},$$

leading to the finished result of

$$\int e^{x^2}dx = e^{x^2}\left(\sum_{n=0}^{\infty}\frac{(-2)^n x^{2n+1}}{(2n+1)!!}\right) + C.$$

Also, the double factorial can be removed with the substitution

$$(2n+1)!! = \frac{(2n+2)!}{2^{n+1}(n+1)!}.$$

Since this substitution creates a messier solution, the double factorial form of the power series may be preferable.

Several months later, our class learned about power series and the traditional way of obtaining an antiderivative for e^{x^2} (see **fig. 3 [Arseneault]**).

In addition to generalizing this approach to find antiderivatives of e to the power x^n, I have also tried applying this method of factoring with integration by parts on other cyclic functions. Readers might try to show that

$$\int \sin x dx = \cos x + C$$

$$\begin{aligned} &\frac{d}{dx}\left(C + xe^{x^2} - \frac{2}{3}x^3e^{x^2} + \frac{4}{15}x^5e^{x^2} - \frac{8}{105}x^7e^{x^2} + \cdots\right) \\ &= \left(e^{x^2} + 2x^2e^{x^2}\right) - \left(2x^2e^{x^2} + \frac{4}{3}x^4e^{x^2}\right) + \left(\frac{4}{3}x^4e^{x^2} + \frac{8}{15}x^6e^{x^2}\right) - \cdots \\ &= e^{x^2} \end{aligned}$$

Fig. 2 (Arseneault)

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots + \frac{x^{2n}}{n!} + \cdots = \sum_{n=0}^{\infty}\frac{x^{2n}}{n!}$$

$$\int e^{x^2}dx = x + \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42} + \frac{x^9}{216} + \cdots + \frac{x^{2n+1}}{n!(2n+1)} + \cdots = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{n!(2n+1)} + C$$

Fig. 3 (Arseneault)

using only the method shown above and the power series for the trigonometric functions.

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COMBINE CAREFULLY

Putting Essential Understanding of Functions into Practice, Grades 9–12, by Robert N. Ronau, Dan Meyer, and Terry Crites (NCTM 2014), a book likely to be purchased and read by many *Mathematics Teacher* readers, includes many good tasks and ideas. However, the discussion of students' learning about new functions formed from operations on functions (p. 54) contains a conceptual error. The authors discuss students' conceptual understanding of the domain and range of the product of functions: "$h(x) = \sqrt{x}$ and $k(x) = x\sqrt{x}$ are defined for all real numbers, yet their product is defined for all real numbers." Their example (see **fig. 1 [Coomes]**) makes no sense graphically because there are no function values to multiply for $x < 0$.

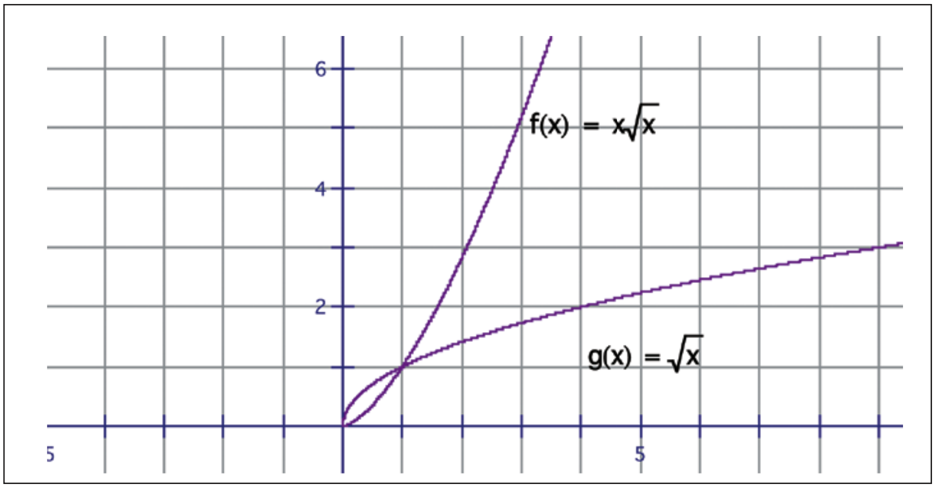

Fig. 1 (Coomes)

The authors' statement is incorrect in that the domain of the product fg must be the intersection of the domains of f and g. (By definition, any x in the domain of fg will be an input into f and into g before they are multiplied and so must be in both their domains.)

An example shows why this is important. Let $f(x) = \sqrt{x}$ and $g(x) = \sqrt{x^3}$, both of which have domain nonnegative real numbers. Then, by the reasoning in the book,

$$(fg)(x) = \sqrt{x}\sqrt{x^3} = \sqrt{x^4} = x^2.$$

Suppose that we evaluate using the definition given in the book: $(fg)(-4) = (-4)^2 = 16$. But using the definition of function product as $(fg)(x) = f(x)g(x)$, we have

$$\begin{aligned}(fg)(-4) &= f(-4) \bullet g(-4)\\ &= \sqrt{-4} \bullet \sqrt{(-4)^3}\\ &= 2i(8i)\\ &= -16.\end{aligned}$$

Thus, we have a contradiction: 16 does not equal −16.

This problem also highlights the fact that the product rule for square roots (i.e., $\sqrt{a}\sqrt{b} = \sqrt{ab}$), is true only when $\sqrt{a}$ and $\sqrt{b}$ are real numbers.

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***The volume editor responds*:** We thank Ms. Coomes for her comments. Being able to determine the domain of the combination of two functions (through composition, multiplication, etc.) is a very important skill, one that is sometimes difficult to master. We completely agree with her observations; in fact, they were the very points we were attempting to make.

Finding the domain of a combined function can sometimes be tricky. Therefore, students must pay careful attention to how the domains of the original functions influence the domain of the combined function. Unfortunately, our writing was not sufficiently clear to achieve our purpose. By raising this issue, Ms. Coomes has allowed us to edit and strengthen this section just in time for the book's second printing.

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Putting Essential Understanding
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