Step Up to Mathematics

Photograph 1
A graded pediment in Assisi, Italy

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In Assisi, Italy, built on a hill, all streets go up. One really cannot get lost there but need only find any of the many narrow winding streets, or pathways, to go down to the main parking lot, where tour bus guides invite their guests to explore the city. The Romans built steps and graded pediments (see **photograph 1**) to accommodate the different levels of terrain. And today restaurants and homes have installed well-designed structures to allow access without steps, such as the pyramidal platform outside Café La Selva (see **photograph 2**).

1. Positioned in **figures 1** and **2** is a black-covered sketchbook. Use the dimensions of the sketchbook, 9 × 14 cm, to estimate the lengths of the platform.

(*a*) Estimate the actual measurements of the pyramidal platform using the labeled points in **figures 1** and **2**.

(*b*) Assume that the platform's edge *AB* is horizontal and that edge *BD* is vertical. Can you find the angles of incline along *CB* (relating to the slope of the platform) and along *DA* (relating to the slope of the road)?

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Photograph 2
Ramp in front of restaurant, Assisi

Photograph 3
Entryway to an Assisi home

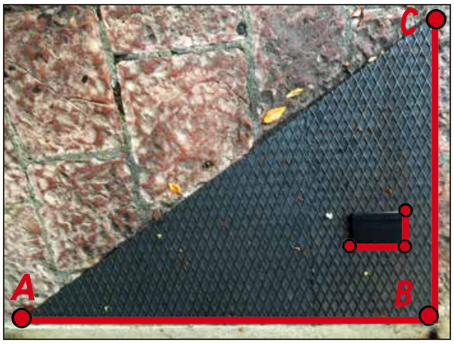

Fig. 1 The sketchbook serves as a reference to scale dimensions.

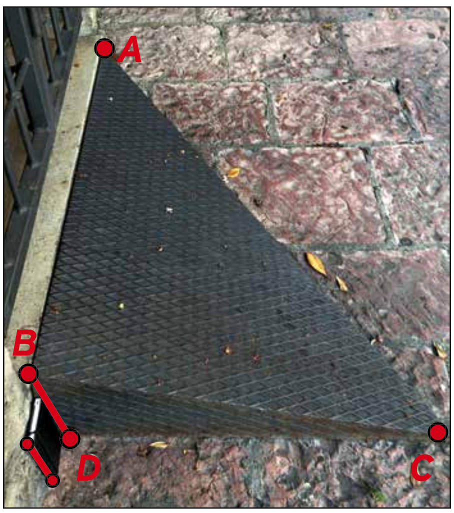

Fig. 2 The viewing angle distorts relative lengths.

Fig. 3 A rectangular ramp and a triangular apron give safe access to a doorway in Assisi.

2. The metal structure shown in **photograph 3** and **figure 3** was added to one of the homes to allow access without climbing steps. It consists of a horizontal, rectangular ramp and a triangular apron. Assume that the metal slats, 9 in the ramp and 21 in the apron, are all evenly spaced. Compared with the 9×14 cm sketchbook (at the left end of the ramp in **fig. 3**), the length and width of the ramp are estimated to be $GF = 137$ cm and $EG = 33$ cm.

   (*a*) Assume that $\angle HJK$ is a right angle and estimate the lengths JK and HK.

   (*b*) Assume that the width of the ramp, EG, is the same as the tread of each of the two lower stairs leading to the doorway. Use that assumption to estimate the angle of incline along JK.

   (*c*) If the roadway underneath the horizontal ramp slopes downward uniformly, estimate the angle of inclination of the road in the direction of the ramp.

Photograph 4
Wooden steps on Assisi street

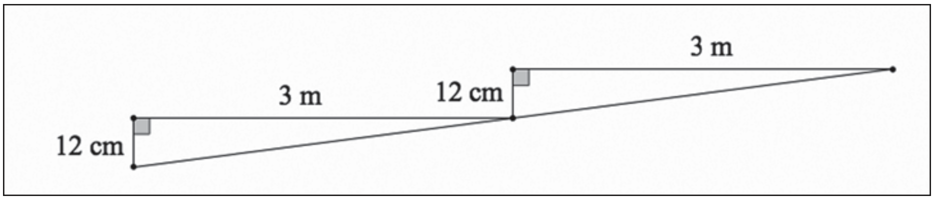

Fig. 4 Horizontal treads allow for the total rise to be divided over multiple steps.

3. At other places in the city, wooden steps were built to allow pedestrians to avoid walking on a steep incline (see **photograph 4**). **Figure 4** shows possible dimensions associated with these steps.

   (*a*) Suppose that 3 congruent steps were used instead of 2, covering the same distance along the road. How would the individual steps change?

   (*b*) Using the dimensions given in **figure 4**, create a table to show the lengths that change, as identified in part (*a*), when the number of steps increases from 2 to 6.

   (*c*) If you were designing such a staircase, how would you decide the number of steps to use?

4. In many parts of the city, steps provide shortcuts between levels as an alternative to the winding roads (see **photograph 5**).

   (*a*) Suppose that you are able to go down only 1 or 2 steps at a time. For example, to descend 3 steps, you have three options: 1 step at a time; 1 step followed by a 2-step leap; or a 2-step leap followed by 1 step. In how many ways could you go down 15 steps? Create a table showing the number of options as the number of steps increases from 1 to 15.

   (*b*) What is the pattern in the number of ways that you found? In what other contexts have you seen these numbers?

Photograph 5
A shortcut with many steps, Assisi

MATHEMATICAL LENS solutions

1. (*a*) An enlarged version of **figure 1** was used to determine the following measurements: $AB_{photo} = 18.3$; $BC_{photo} = 13.4$; sketchbook photo dimensions 2.4×1.5. Using ratios to find the actual lengths AB and BC,

$$\frac{AB}{18.3} = \frac{14}{2.4} \text{ and } \frac{BC}{13.4} = \frac{9}{1.5},$$

we find that $AB = 106.75$ cm and $BC = 80.4$ cm. (Notice that the relative dimensions of the sketchbook in the photograph measurements are close but not quite proportional to the given lengths of 14×9.)

In an enlarged version of **figure 2**, the height BD_{photo} was measured as 1.8 and the spine of the sketchbook was measured as 1.4. From these values, we estimate that $BD = (14/1.4)1.8 = 18$ cm.

It is interesting that similar measurements and calculations on the enlargement of **figure 2** give BC as approximately 135. This value is not close to the 106.75 determined in part (*a*). The discrepancy can be partially attributed to the position and angle of the camera as well as the placement of the sketchbook. Because measurement along the spine of the sketchbook is in the same direction as segment BD, we expect the scaling to be reasonably accurate. However, the reference object (the sketchbook) does not align well with the segment BC in **figure 2**, so the scaling is suspect.

(*b*) The angle of incline along segment CB cannot be determined without more information. If we suppose that segment BC lies in a horizontal plane, then the angle of incline along $\overline{BC}$ would be zero. If we suppose, on the other hand, that segment DC lies in a horizontal plane, then $\angle BDC$ would be a right angle and the trigonometric ratio $BD/BC = \sin\theta$ would give θ as the angle of incline from C to B.

However, as a consequence of specifications in the problem statement, we know that $\angle DBA$ is a right angle, so the angle of incline (or decline) along segment DA can be found using the trigonometric ratio in right triangle DBA. Thus, $DB/BA = \tan\theta$, so $\theta = \tan^{-1}(18/106.75) \approx 9.6°$.

2. (*a*) We assume that the ratio of lengths $JK:EG$ is approximately equal to the ratio of the number of slats in each part of the structure. Using $EG = 33$ cm, we estimate that $JK = 33(21/9) = 77$ cm. If we assume that $JH = FG$, and $m(\angle HJK) = 90°$, then the Pythagorean theorem gives $HK = \sqrt{137^2 + 77^2} \approx 157$ cm.

(*b*) Imagine a point, P, directly below F and J but on the same horizontal plane as K. The statement that the ramp width is the same as each tread depth means that the distance $KP = 2EG = 66$. In right triangle JPK, we have $KP/KJ = 66/77$, so the angle of incline along segment KJ is $\cos^{-1}(66/77) \approx 31.0°$.

(*c*) Our best estimate of the distance between point F and our imagined point P directly below would come from $\triangle JPK$ as $JP = \sqrt{77^2 - 66^2} \approx 40$ cm. Using $JP/GF = 40/137$, we would estimate that the road below the ramp is sloping downward at an angle of $\tan^{-1}(40/137) \approx 16.3°$.

Table 1 Tread and Rise of Steps

Number of Steps	Tread of Each Step (m)	Rise of Each Step (cm)
2	3	12
3	2	8
4	3/2	6
5	6/5	24/5
6	1	4
n	$3/n$	$24/n$

3. (*a*) The tread of each step would be shorter, and the rise would be less.

(*b*) **Table 1** shows the tread and the rise of each step as the number of steps increases.

(*c*) The number of steps must be large enough so that the rise is not too high. Also, the number must be low enough so that a person's foot fits safely on the tread.

4. (*a*) Suppose that you could go down n steps in x_n different ways and $(n + 1)$ steps in x_{n+1} different ways (see **table 2**). To find the number of ways to go down $(n + 2)$ steps, you might consider two options: go down n steps and then leap 2; or go down $n + 1$ steps and then 1 more. That reasoning suggests that $x_{n+2} = x_n + x_{n+1}$.

(*b*) These are the Fibonacci numbers, for which there are numerous resources.

Table 2 Ways to Go Down Steps

Number of Steps (n)	Number of Ways (x_n)
1	1
2	2
3	3
4	5
5	8
6	13
7	21
8	34
9	55
10	89
11	144
12	233
13	377
14	610
15	987