READER reflections

LANGUAGE AND MATHEMATICS

I thoroughly enjoyed the recent article by Tena L. Roepke and Debra K. Gallagher, "Using Literacy Strategies to Teach Precalculus and Calculus" (*MT* May 2015, vol. 108, no. 9, pp. 672–78). As a licensed teacher in both English language arts and mathematics, I see a lot of overlap between literacy strategies and the Common Core Standards for Mathematical Practice. What Stanford professor Jo Boaler calls "valuing different ways of seeing math" feels a lot like teaching students how to engage in critical reading. I hope these small suggestions add to the conversation started by Ms. Roepke and Ms. Gallagher:

- Frayer model variations sometimes include a visual representation—an opportunity to link models to key mathematical concepts.
- Sentence frames such as those promoted in the popular book *They Say/I Say: The Moves That Matter in Academic Writing* (2014) can be recast in a mathematics class as a "formula skeleton" or used verbatim to verbalize understanding. My favorite sentence frame is extremely simple and requires students to justify a claim with two reasons: "I believe that ________ because ________ and ________."
- Purposeful annotations and close reading are already used by precocious students everywhere; sometimes these strategies are worth teaching explicitly.
- Finally, mathematicians make inferences all the time to solve problems. Teachers can call attention to the way we read between the lines to restrict the domain, determine what is extraneous or assumed information in a given context, or predict the solution through estimation.

Using the strategies and also sharing this same academic language from other disciplines reinforce critical thinking and give our students more practice in seeing patterns.

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PRECISION IN THOUGHT, WORD, AND MATHEMATICS

Consider the following problem:

> An experienced carpenter can detail a new room 3 times faster than an apprentice can. Working together, they can detail the room in 12 hours. How long would it take each one working alone to do the job?

Let x room/hour be the work speed of the apprentice. Then $3x$ room/hour is the work speed of the experienced carpenter. We compute $3x \cdot 12 + x \cdot 12 = 1$ so that $x = 1/48$ and $3x = 1/16$. Thus, the answer is 16 hours for the experienced carpenter and 48 hours for the apprentice, right? Wrong. The answer is 15 hours for the experienced carpenter and 60 for the apprentice.

The error comes from misinterpreting "3 times faster than" to mean "3 times as fast as." In comparisons, the word "times" or any of its specifics ("half," "twice," "triple," etc.) mandates multiplication in its mathematical expression. Further, just as "50% faster than" precisely means to add on half of a given speed, "300% faster than" precisely means to add on 3 times a given speed. With the word "than" expressed or implied, a modifying phrase that means "more" mandates addition, and a similar phrase that means "less" mandates subtraction.

Therefore, in the stated problem, both multiplication and addition are mandated. To be precise, we must add on 3 times the apprentice's speed. If x is the apprentice's speed, then $(x + 3x)$, or $4x$, is the experienced carpenter's speed. Then we have $4x \cdot 12 + x \cdot 12 = 1$, so $x = 1/60$ and $4x = 1/15$. Thus, the answer is indeed 15 hours for the experienced carpenter and 60 for the apprentice.

If 16 hours and 48 hours were the intended answer, the problem should begin, "An experienced carpenter can detail . . . 3 times as fast as an apprentice can." An alternative but equivalent statement is, "An apprentice carpenter can detail . . . one-third as fast as an experienced carpenter can." Let x be the experienced carpenter's speed. Then $(1/3)x$ is the apprentice's speed. Thus, $x \cdot 12 + (1/3)x \cdot 12 = 1$, so $x = 1/16$ and $(1/3)x = 1/48$. Again, the answer would be 16 and 48.

If the problem began, "An apprentice carpenter can detail . . . one-third slower than an experienced carpenter can," multiplication and subtraction would be mandated. To be precise, we would subtract off 1/3 of the experienced carpenter's speed. If x is the experienced carpenter's speed, then $(x - (1/3)x)$, or $(2/3)x$, is the apprentice's speed. That is, $x \cdot 12 + (2/3)x \cdot 12 = 1$, so $x = 1/20$ and $(2/3)x = 1/30$. This time the answer would be 20 hours for the experienced carpenter and 30 hours for the apprentice.

Finally, precision abandoned, sheer gibberish might have been intended to convey real meaning—for example, "3 times slower than" to mean "one-third as fast as." The first phrase would have mandated the physical impossibility of subtracting 3 times the given speed, and the mathematical sentence would have

read $x - 3x = (1/3)x$.

Precision in thought, word, and mathematics is essential to a proper academic curriculum. The implications of such precision are important in commercial advertising, for example. A skilled but devious advertiser can use an expression that strictly meets a truth-in-advertising standard but that at the same time appears falsely more appealing to an ignorant audience that misinterprets the words. Even worse is the situation in which an ignorant advertiser unknowingly uses an expression that comes across as utterly confusing to the potential client who relies on the kind of precision discussed in this presentation.

Let's say what we mean and mean what we say.

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VISUALIZING THE SUM OF SQUARES

Most mathematicians are familiar with visualizing the sum of the first n counting numbers: $1 + 2 + 3 + \cdots + (n - 1) + n = n(n + 1)/2$. And many of us are also familiar with the formula for the sum of the first n squares,

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$

which can be proved using mathematical induction.

A geometric visualization for this sum formula in the form

$$6\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)$$

The sum $1^2 + 2^2 + 3^2$ could be interpreted this way:

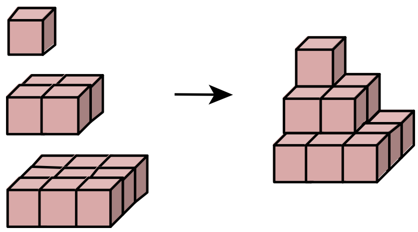
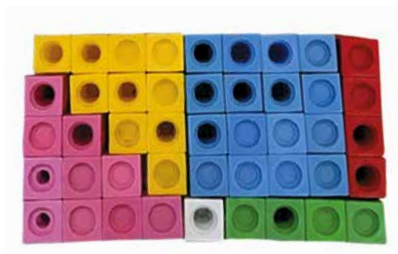


Fig. 1 (Haws)

$6(1^2) = 1(2)(3)$

$6(1^2 + 2^2) = 2(3)(5)$

$6(1^2 + 2^2 + 3^2) = 3(4)(7)$



$6(1 + 2^2 + 3^2 + 4^2) = 4(5)(9)$

Fig. 2 (Haws)

interprets the right side as a brick with dimensions $n \times (n + 1) \times (2n + 1)$. A sum of squares, represented on the left, can be visualized in tiers (see **fig. 1 [Haws]**).

So 6 copies of the sum should comprise the brick. Models show how 6 copies of the sum of squares fit together in a brick for values of n from 1 up to 4. (See **fig. 2 [Haws]**.)

The floor plan (see **fig. 3 [Haws]**) for the model when $n = 3$ shows the bottom, middle, and top layers to help see how the pieces fit together.

This construction is easily generalized to larger n and reveals the sum formula using geometry.

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LINEAR COMBINATIONS OF POWERS OF COS *X*

It is well known that $\cos 2x = 2\cos^2 x - 1$ and $\sin 2x/\sin x = 2\cos x$. These formulas lead us to wonder if similar formulas exist for $\cos nx$ and $\sin nx/\sin x$, where n is a natural number. Accordingly, we generate **table 1 (Siegel)** by use of formulas

$$\cos nx = \cos(n-1)x \cdot \cos x - \sin(n-1)x \cdot \sin x$$

and

$$\sin nx = \sin(n-1)x \cdot \cos x + \sin x \cdot \cos(n-1)x.$$

Note that the process for $\cos nx$ requires an extra step and uses the identity $\sin^2 x = 1 - \cos^2 x$.

Making sense of **table 1 (Siegel)**, which lists the results only up to $n =$ 10, requires some explanation because

Bottom Level

Y	Y	Y	B	R	R	R
P	W	W	W	G	R	R
P	W	W	W	G	G	R
P	W	W	W	G	G	G

Middle Level

Y	Y	Y	B	B	R	R
P	Y	Y	B	B	R	R
P	P	W	W	G	G	R
P	P	W	W	G	G	G

Top Level

Y	Y	Y	B	B	B	R
P	Y	Y	B	B	B	R
P	P	Y	B	B	B	R
P	P	P	W	G	G	G

Fig. 3 (Haws)

Table 1 (Siegel)

n	1	$\cos x$	$\cos^2 x$	$\cos^3 x$	$\cos^4 x$	$\cos^5 x$	$\cos^6 x$	$\cos^7 x$	$\cos^8 x$	$\cos^9 x$	$\cos^{10} x$
0	1										
1	**1**	1									
2	–1	**2**	2								
3	**–1**	–3	**4**	4							
4	1	**–4**	–8	**8**	8						
5	**1**	5	**–12**	–20	**16**	16					
6	–1	**6**	18	**–32**	–48	**32**	32				
7	**–1**	–7	**24**	56	**–80**	–112	**64**	64			
8	1	**–8**	–32	**80**	160	**–192**	–256	**128**	128		
9	**1**	9	**–40**	–120	**240**	432	**–448**	–576	**256**	256	
10	–1	**10**	50	**–160**	–400	**672**	1120	**–1024**	–1280	**512**	512

it is actually two tables in one. The red entries are the coefficients of the powers of cosine for the $\sin nx/\sin x$ series, whereas the black entries are those of the $\cos nx$ series. For example,

$$\cos 5x = 5\cos x - 20\cos^3 x + 16\cos^5 x$$

and

$$\sin 6x/\sin x = 6\cos x - 32\cos^3 x + 32\cos^5 x.$$

Just as Pascal's triangle follows a pattern, so does this "Siegel triangle." (We consider "missing" entries to be zero.) Red entries are the sum of the entry above them and the entry to the immediate left of that. The others are the sum of the opposite of (or negative one times) the entry immediately above it and the two entries to the left of that. For example, from the bottom row, we have $-160 = -120 + (-40)$, and $1120 = -(-448) + 432 + 240$.

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THE SCREAM IN CONICS

I have always been fascinated by Edvard Munch's painting *The Scream* (1893). In short, I could see that the simplicity of the underlying drawing would be relatively easy to translate into conics equations. However, it was not until I discovered Desmos® that I was able to find the appropriate tool.

Creating the graph took about two weeks and was done in sections (see **fig. 1 [Ewan]**).

The process was amazingly revealing. I learned much more about the nature of conics as I adjusted the variables. One discovery was that for the most part parabolas are the most useful for rendering curves. Another was that keeping the equations in standard form—namely, $y = a(x - h)^2 + k$ or $x = a(y - k)^2 + h$—is clearly the best approach.

My honors students were able to produce some truly impressive renderings based on, for example, cartoons, sketches of a Picasso, and sports symbols.

DAVID EWAN

Fig. 1 (Ewan)

The next goal is to use Desmos to mimic movement. For example, use the following:

$$(x - 4)^2 + y^2(y - 5e^{-0.1a}|\operatorname{sinc}|)^2 < d,$$

where d varies from 1 to 10, a from 0 to 10, and c from –10 to 10.

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PROBLEM 31, MAY 2015 CALENDAR

Problem 31 of the May 2015 Calendar (*MT* May 2015, vol. 108, no. 9, p. 681) was corrected online to read:

> At Pizza Pi Restaurant, every sixth customer receives a free drink, and every eighth customer receives a free slice of pizza. How many of the day's 79 customers received a free drink or a free pizza slice?

Two readers noticed that the solution to the problem as it appeared in the print journal was not 19. To fit the puzzle, we changed the question rather than the solution. Thanks to Carol Detweiler and Louis Freese for pointing out the error.