OCTOBER

NCTM® NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Theme: Lots of algebra

1

The graph of $y = x^2 + x - 30$ is a parabola with its vertex at (p, q). Find the sum $p + q$.

2

Given $f(x) = x^2$, find the equation of the line determined by the point $P\,(1, f(1))$ and $Q\,(5, f(5))$.

3

Find the maximum vertical distance between the parabola and the line described in problem 2 on the domain $[1, 5]$.

4

Factor the following:

$$18x^2 + 9x - 20$$

5

Find the three integer zeros for the polynomial

$$P(x) = x^3 + 2x^2 - 39x + 72.$$

6

Multiply each term of the polynomial defined in problem 5 by the corresponding number in the sequence $\{2, 3, 4, 5\}$ to get

$$P^*\,(x) = 2x^3 + 6x^2 - 156x + 360.$$

Find the one integer root of $P^*\,(x)$.

7

Refer to the result in problem 2 to determine the x-coordinates of the intersections of the function $f(x)$ with the perpendicular bisector of $\overline{PQ}$.

8

The reciprocal of $(2x^2 - 5x + 3)$ can be written in the form

$$\frac{m}{(x+p)} + \frac{n}{(x+q)}.$$

Find the sum

$$m + n + p + q.$$

9

The reciprocal of $[(ax + b)(cx + d)]$ can be written in the form

$$\frac{m}{(x+p)} + \frac{n}{(x+q)}.$$

Find the sum

$$m + n + p + q$$

in terms of a, b, c, and d.

10

Factor the following completely:

$$x^4 - 25x^2 + 144$$

11

The graph of the function

$$f(x) = ax^2 + bx + c,\ a \neq 0,$$

is symmetric about a vertical line. If a point on the graph has an x-coordinate of p, find the coordinates of its image in terms of a, b, c, and p, after reflecting the point over the line of symmetry.

12

The graph of the function

$$f(x) = ax^3 + bx^2 + cx + d,\ a \neq 0,$$

is symmetric about the point $(-b/3a, f(-b/3a))$. Find the coordinates of the point of symmetry when $a = 1$, $b = 6$, $c = -3$, and $d = -12$.

13

Use the function and values of a, b, c, and d as defined in problem 12 to find the coordinates of the image of $P\,(1, -8)$.

14

The function

$$g(x) = |x-2| + |x-1.5| + |x-1| + |x-0.5| + |x| + |x+0.5| + |x+1| + |x+1.5| + |x+2|$$

can be approximated on the interval $[-3, 3]$ by the quadratic function $y = ax^2 + b$, where a and b are both integers. Find a and b.

15

The graph of the function

$$f(x) = x^3 - 6x^2 - 3x + 18$$

can be translated horizontally t units to eliminate the quadratic term—that is, the result should be of the form $g(x) = x^3 + px + q$. Find the value and the direction for t.

16

A formula that is used to solve for one of the roots of a cubic equation of the form $x^3 + px + q = 0$ is

$$x = \sqrt[3]{\frac{-q}{2} + \sqrt{\left(\frac{-q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\left(\frac{-q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}.$$

Apply this formula to find the zeros of $f(x)$ in problem 15.

17

In problem 15, you were asked to eliminate the quadratic term. It is possible to eliminate the linear term instead. Determine the translation that would result in the form

$$h(x) = x^3 + px^2 + q.$$

18

A dish contains n strands of cooked spaghetti. Two ends of spaghetti are chosen at random and tied together. Two other ends are selected and tied together. If this process continues until there are no more loose ends, what is the probability that all n strands form one big loop?

19

Using the problem statement given in problem 18, find the probability that there will be n loops each consisting of one strand.

20

The numbers 1 to n are written on pieces of paper and placed in a box. The papers are removed one at a time, the number is recorded, and then they are destroyed. The first 5 recorded values are 7, 38, 25, 114, and 87. What is your best guess as to the value of n?

21

To estimate the number of fish in a small lake, researchers caught 200 fish, tagged them, and returned them to the lake. The next day they caught 100 fish. Of those caught, 25 had a tag. What would be a good guess for the total number of fish in the lake?

22

If there were 800 fish in the lake (see problem 21) and the same procedure were followed, what is the probability that 25 of the 100 fish in the second catch would have been tagged?

23

Polyominoes are shapes formed by arranging congruent squares with coincident sides. The domino has only 1 possible arrangement of two squares. Excluding rotations and reflections, determine the number of possible triominoes, tetrominoes, and pentominoes and use the results to find a quadratic polynomial function that satisfies your results.

24

The sum of two roots of a fourth-degree polynomial function with integer coefficients is $6 + \sqrt{5}$, and the product of the other two roots is $6 - 3\sqrt{5}$. If two of the four roots are integers, find the polynomial function.

25

One of the factors of

$$x^3 - 15x^2 + 73x - 115$$

is $x - 5$. Find the other two linear factors.

26

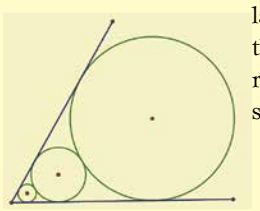
For all real a and b values, the graph of $y = ax^2$ is mathematically similar to the graph of $y = bx^2$. That is, a dilation will map the graph of the first equation onto the graph of the second equation. Define the dilation in terms of a and b.

27

As b varies from $-\infty$ to ∞ in the equation $y = ax^2 + bx + c$, the vertex of the parabola traces the path of another parabola. Find the equation of the resulting parabola.

28

Three circles are tangent to the sides of a 60° angle, and the middle-sized circle is tangent to the other two circles. Find the ratio of the length of the radius of the largest circle to the length of the radius of the smallest circle.



29

The functions f and g have polynomial approximations h and k, respectively:

$f(x) = e^x, g(x) = 1 + x + x^2/2 + x^3/6 + x^4/24 + x^5/120;$
$h(x) = \sin(x), k(x) = x - x^3/6 + x^5/120 - x^7/5040.$

Calculate $f(1) - g(1)$ and $h(1) - k(1)$ and determine whether g or k is the better approximation for their respective functions.

30

Find the cubic equation in the form

$$f(x) = x^3 + bx^2 + cx + d$$

given that its point of symmetry has coordinates $(2\ -38)$ and that one other point on the graph has coordinates $(4, -126)$.

31

The graphs of two quadratic functions are shown in the diagram. Sketch the graph of the sum of these two functions on the domain $[-4, 4]$.