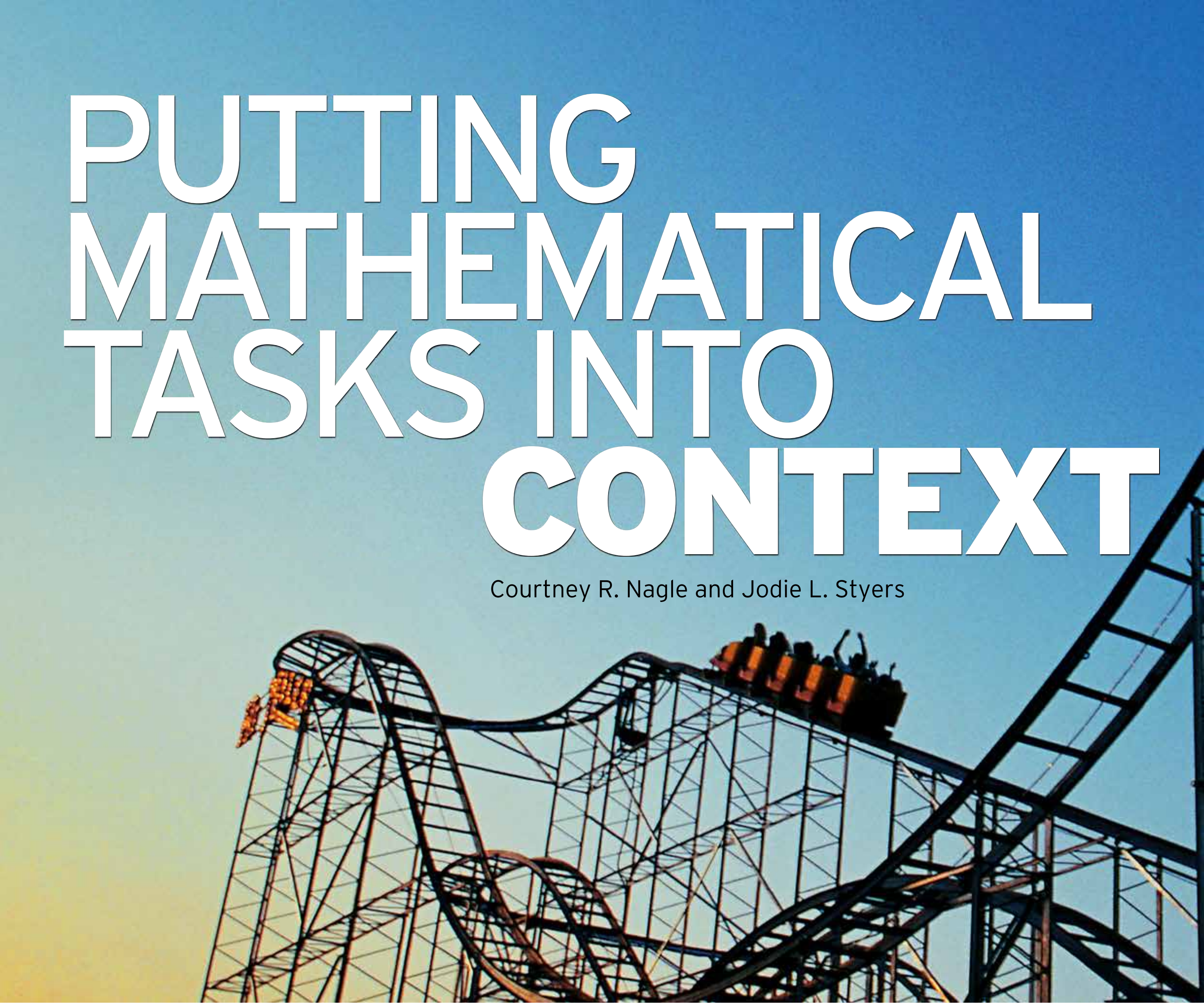


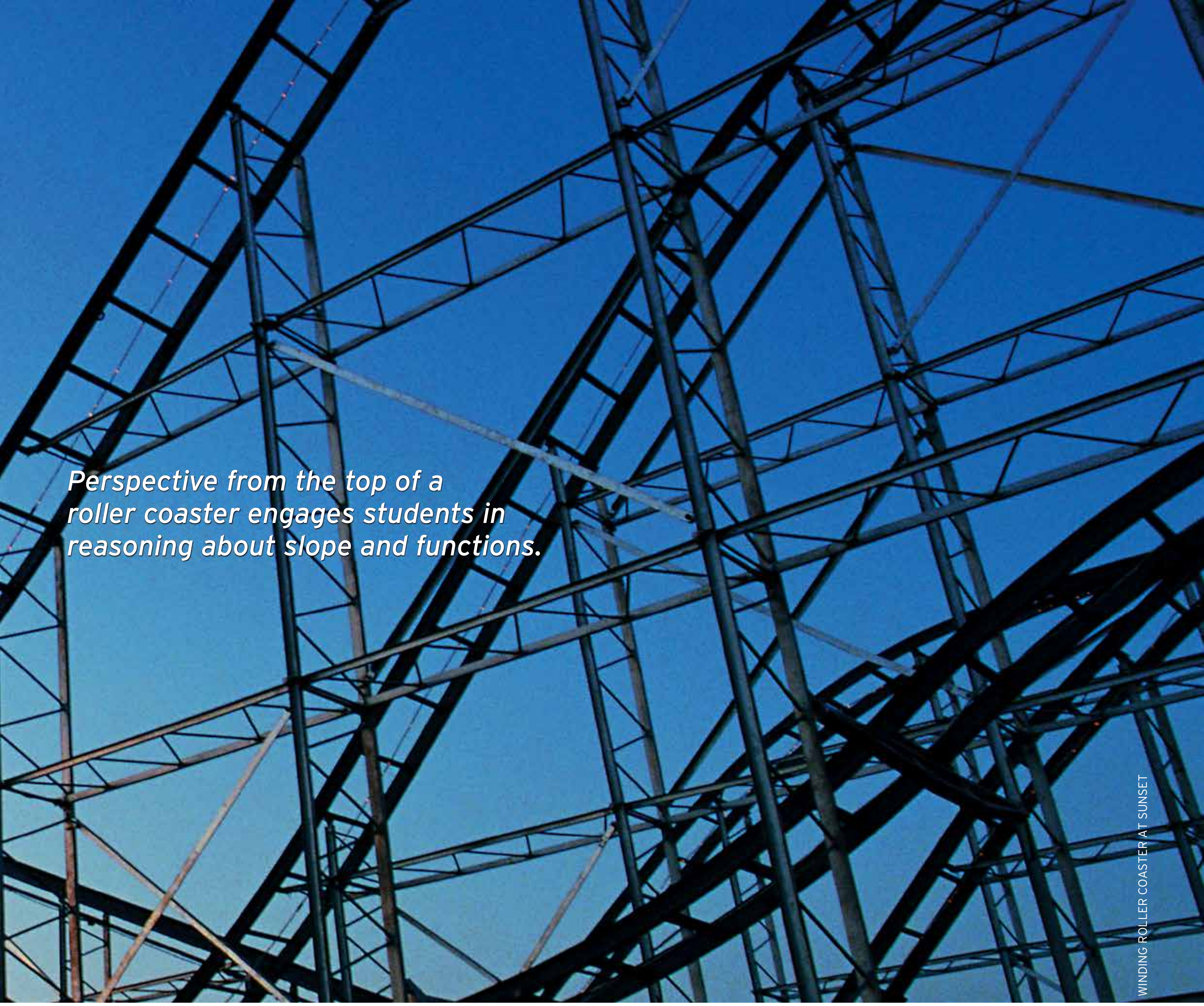
PUTTING MATHEMATICAL TASKS INTO CONTEXT

Courtney R. Nagle and Jodie L. Styers

Although many factors affect students' mathematical activity during a lesson, the teacher's selection and implementation of tasks is arguably the most influential in determining the level of student engagement. Mathematical tasks are classroom activities intended to focus students' attention on a particular mathematical concept (Stein, Grover, and Henningsen 1996). Doerr and English (2003) emphasize the importance of carefully developing and sequencing tasks to encourage students' independent exploration of mathematical ideas. Lappan (1993, p. 524) stated, "No other decision that a teacher makes has a greater impact on students' opportunity to learn and on their perceptions about what mathematics is than the selection or creation of tasks with which the teacher engages the students in studying mathematics."

Why are tasks so important? Stein, Grover, and Henningsen (1996) explain that tasks "determine not only what substance [students] learn but also how they come to think about, develop, use, and make sense of mathematics" (p. 459). Nagle and Moore-Russo (2014) distinguish between a student's mathematical understanding and mathematical engagement, explaining that it is possible for a student to have a rich, deep understanding of a mathematical topic but to engage with the content in a trivial or superficial manner.

For example, a student might conceptually understand the formula for calculating the distance between two points in the coordinate plane. She may make connections between the distance formula and the Pythagorean theorem, relating the straight-line distance between the two points to the length of the hypotenuse of a right triangle with



Perspective from the top of a roller coaster engages students in reasoning about slope and functions.

WINDING ROLLER COASTER AT SUNSET

side lengths $|x_2 - x_1|$ and $|y_2 - y_1|$. However, if the same student is given a set of two ordered pairs and asked only to find the distance between the points, her engagement will likely be limited to carrying out a memorized procedure. Despite the student's robust understanding of the distance formula and the potential for richer engagement, the task provided restricts her engagement and limits opportunities for learning.

Alternatively, rich tasks may encourage students to engage conceptually, seeking understanding for how and why procedures hold, even if that conceptual understanding is not yet firmly in place. To return to our previous example, a student with procedural understanding may recite and apply the distance formula but not understand why this equation yields the distance between two points. However, if she is presented with two points and given an opportunity to plot the points on graph paper and explore how to determine the distance between them, she may engage conceptually by trying to make sense of the rule through the graphical representation. Although the student may not yet demonstrate conceptual knowledge, she is engaged conceptually while asking questions that seek to expand and deepen her current understanding. Thus, tasks have the potential to engage students in mathematical activity below or beyond their present level of mathematical understanding, influencing their learning in the process.

Much work has been done to classify mathematical tasks and describe features of tasks that offer promise for engaging students deeply in the mathematical content (Boston and Smith 2009; Smith and Stein 1998; Stein and Smith 1998). Rich mathematical tasks may have many features, but

we have chosen to focus on one: tasks that promote contextual reasoning. What is contextual reasoning? Why is it important? And how do we get students to engage in contextual reasoning?

CONTEXT AND REASONING

To begin to address these questions, consider task 1 and task 2 (see **figs. 1** and **2**, respectively). Which of these tasks would you describe as encouraging contextual reasoning and why?

This question was posed to a group of twenty middle school and high school mathematics teachers during a professional development workshop. The teachers unanimously agreed that task 2 was a contextual task. When pressed to explain their reasoning, most teachers acknowledged that the task included a physical context through the real-world situation presented—in this case, roller coasters. Others mentioned that the task also included a human context through the opportunity for personal interpretation. An individual's personal preferences affect the "correct answer," even though the underlying mathematics remains unchanged. Interestingly, most of the participants agreed that they would rather ride roller coaster 2. However, although some chose roller coaster 2 because they were looking for a thrill ride and it appeared to have the steepest lift hill, others chose it for a different reason—they were afraid of roller coasters and thought the ride would be over sooner on roller coaster 2. Some thought that the track looked smoother and the car looked safer. Still others, citing their fear of heights, chose roller coaster 1. Despite their different attitudes toward roller coasters, the teachers agreed that this task included opportunities for students to reason about the personal or human context, which influenced the mathematical interpretation and therefore the resulting answer.

The teachers' discussion highlights two classifications of contextual reasoning: (1) situational or physical context (i.e., the roller coasters' lift hill) and (2) human or social context (i.e., personal feelings about height, personal preferences for thrill rides, etc.). Engaging with the physical context includes reasoning about the real-world setting of a task as well as reasoning about and choosing between any available tools for solving the task (Nagle and Moore-Russo 2014). In task 2, the physical setting of the roller coaster's lift hill provided a means for physical contextual reasoning. Students engage with social context when they consider personal preferences (their own or others') while interpreting mathematical results or making decisions about how to communicate those results

Task 1: The graphs of two lines are shown. How do the lines' slopes compare? Explain.

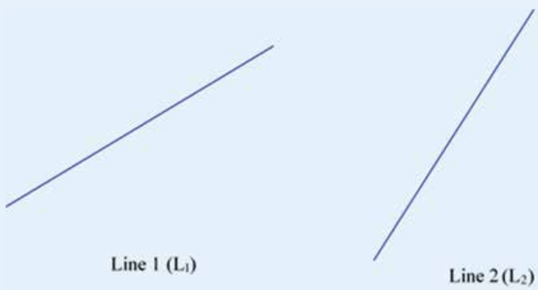


Fig. 1 This task involves comparing the slope of two lines.

Task 2: The first hill's incline for two roller coasters are shown. Write a paragraph explaining which roller coaster you would rather ride and explain why.

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ADAPTED FROM A PHOTOGRAPH BY ADAM AHMED

Roller Coaster 1 Roller Coaster 2

Fig. 2 Task 2 involves comparing the incline of two roller coasters.

to others (Nagle and Moore-Russo 2014). Task 2 affords opportunities for both these categories of contextual reasoning, prompting the teachers to identify task 2 as a contextual task.

Nagle and Moore-Russo (2014) describe a third category of contextual reasoning—mathematical context—which was notably absent in the teachers' discussion of contextual features of the tasks. We discuss mathematical context with an illustrative example followed by an explicit definition in the following section.

REASONING AND ASSUMPTIONS

To illustrate reasoning about mathematical context, consider alternative task 1 (see **fig. 3**). After the teachers discussed the opportunities for contextual reasoning afforded by task 2, they were presented with alternative task 1. The presenter emphasized that the lines, line 1 (L_1) and line 2 (L_2), in alternative task 1 are the same lines as in the original task 1. The question remained the same: Which line has the greater slope? A visual inspection of the lines indicates that L_1 appears to be steeper than L_2, contradicting the result from task 1. The teachers participating in the workshop were not too concerned with the apparent contradiction. Instead, they pointed out that the axes' scaling must have been changed between tasks, resulting in a change in the appearance of the lines. To further illuminate the importance of mathematical contextual reasoning in these examples, the presenter next showed the teachers how each set of lines was generated.

Alternative task 1: The graphs below also represent line 1 and line 2. How do the lines' slopes compare? Explain.

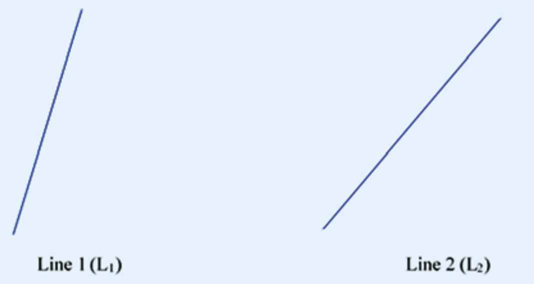

Fig. 3 The same two lines are represented graphically, prompting explanation of the apparent contradiction.

The images in **figure 4** show how The Geometer's Sketchpad® was used to superimpose a line on top of the lift hill of two roller coaster images. The lines from task 1 were created using a rectangular grid (where horizontal and vertical axis were scaled using different increments), whereas the lines from alternative task 1 were created using a square grid (where a set distance along either the horizontal or the vertical axis represented the same constant quantity). The important note is that the roller coasters themselves did not change, and neither did the slope of the line representing the lift hill. However, the mathematical context—the scaling of the coordinate system—affected the appearance of the lift hill and therefore the appearance of the line

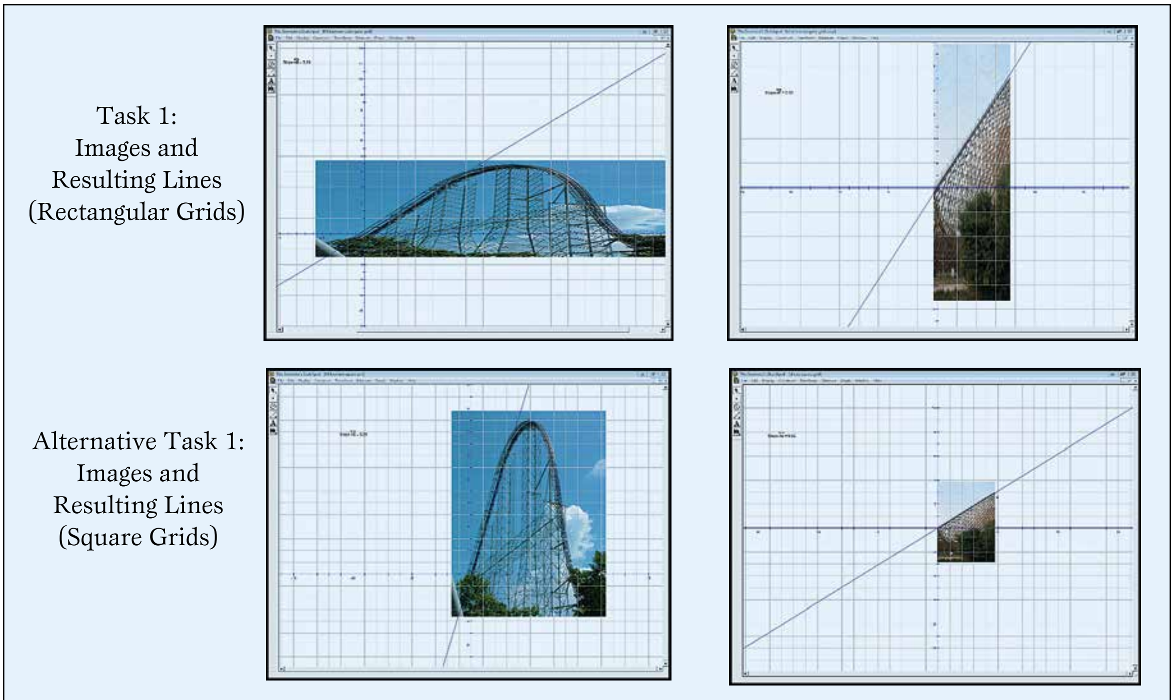

Fig. 4 These images of The Geometer's Sketchpad sketches were used to generate lines L_1 and L_2 in the original (rectangular grid) and alternative (square grid) tasks.

representing the hill. Comparing the images in **figure 4**, the teachers also recognized that in task 2, the hill on the right appeared to be steeper than that on the left but was actually less steep. Thus, both task 1 and task 2 included a potential layer of contextual engagement—namely, mathematical context—not initially considered by the teachers.

Reasoning about mathematical context involves recognizing the mathematical conditions and assumptions that validate a mathematical procedure, with consideration of mathematical contexts under which the results would not be valid. In the example above, recognizing the important context of the scaling of the axis requires students to attend to precision (Standard for Mathematical Practice 6, CCSSI 2010, p. 7), while clarifying the correspondence of the scaling with the quantities represented. Although the two problems are mathematically identical, in the second the discussion includes interpretation and consideration of alternatives and definitions and personal reaction because the visual representation now gives an incentive for contextual reasoning (Gresalfi and Williams 2009). As seen from the response to alternative task 1, the teachers in this professional development workshop were capable of reasoning about the mathematical context underlying the statement that a steeper line has a greater slope. However, the design of the original task 1 was such that the teachers did not engage in such reasoning.

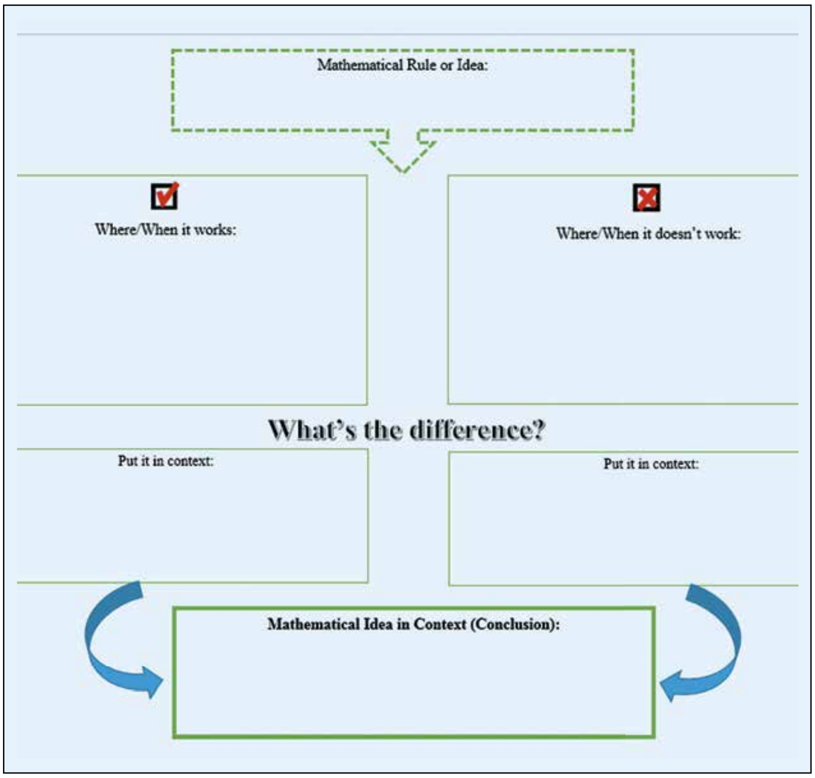

Fig. 5 A graphic organizer draws students' attention to the role of mathematical context.

ASSUMPTIONS AND RESPONSES

The claim that "given two increasing lines, the steeper line has the larger slope" implicitly assumes that the lines are graphed on a square coordinate system. Although the teachers never considered this assumption when initially responding to task 1, they did not hesitate to state that the condition had been violated when they were presented with alternative task 1.

Interestingly, some teachers reported feeling "tricked" by the task and mentioned that they would not feel comfortable "tricking" their students in the same manner. Comments such as these highlight the need for teachers and students to be more aware of mathematical contexts and to explicitly consider the mathematical context that validates (or violates) particular mathematical rules and procedures. Teachers' feelings that students would feel tricked by task 1 suggests that students would assume that the lines were graphed on homogenous coordinate systems.

Establishing classroom norms that agree on implied mathematical context in particular situations is acceptable, but such discussions between students and teachers are critically important. Explicit discussions about mathematical context help students validate the mathematical rules and procedures they use as well as recognize limitations on when and how they can be used. In the case of comparing the slopes of two lines on different, unmarked axes, the absence of a discussion about assumptions would indicate a lack of attention to the critical mathematical context. When tasks that bring the mathematical context to the forefront are avoided, students miss out on opportunities to understand the context that validates common procedures, possibly leading to misuse of these results in inappropriate contexts.

Examples of students using mathematical results in inappropriate mathematical contexts are easy to find. For example, consider the procedure of cross-multiplying to solve a proportional relationship (e.g., $a/b = c/d$ implies $ad = bc$). A favorite technique of students, this approach is often used in mathematical contexts where it is not valid, such as when multiplying fractions (e.g., $a/b \cdot c/d = ad/bc$).

As a second example, consider the zero product property, which can be used to solve equations such as $x^3 - 5x^2 - 6x = 0$ by factoring and setting each factor equal to zero so that $x(x - 6)(x + 1) = 0$ leads to the solution $x = -1, 0, 6$. However, when students learn this property without attention to the mathematical context that validates the procedures (namely, a product of factors set equal to zero), they likely will use this technique in situations where it does not apply—for example, $(x - 1)^2 + x^2 = 0$, so $(x - 1)^2 = 0$ or $x^2 = 0$.

RESPONSES AND FOCUS

How can we provide opportunities for students to reason about mathematical context? One possibility is to engage students in reasoning using the graphic organizer displayed in **figure 5**. This tool draws students' attention to mathematical context by having them consider a mathematical statement that is purposely void of context (and, hence, only sometimes applicable). Next, students are asked to find a situation or context in which the statement is valid as well as a counterexample in which the statement is not valid. By comparing and contrasting the mathematical contexts of the example and the nonexample, students can identify the implied mathematical context and then rewrite the mathematical statement to include the appropriate mathematical context.

Consider the work displayed in **figure 6**, done by college calculus students who were asked to reason about the mathematical context of the intermediate value theorem. The initial mathematical rule was provided: If $f(a) < 0$ and $f(b) > 0$, then $f(c) = 0$ for some c in $[a, b]$. Students were then instructed to complete the rest of the graphic organizer. Students were able to generate an example context in which a statement is true and an example in which it is false. By comparing the mathematical context of the two examples, students turned their attention to the idea that the function in the working example was continuous on $[a, b]$, whereas the function in the counterexample was discontinuous somewhere on the interval $[a, b]$. As a result, students were able to rewrite the mathematical rule (here, the intermediate value theorem), this time focusing attention on the mathematical context required to make the statement true.

Although the second student's response indicates that she may be confused about the difference between c and $f(c)$, she recognizes the necessity of function f being continuous as a precondition for the intermediate value theorem. By having students' explicitly think about what might go wrong in the case when f is not continuous, the students are less likely to misuse the intermediate value theorem for functions such as $f(x) = 1/x$ on $[-1, 1]$.

Reasoning about context can also be a powerful impetus for introducing new mathematical topics. The example in **figure 7** shows how the same graphic organizer was used to identify the need to describe a new mathematical concept (i.e., intervals of change) while making connections with previous mathematical ideas (i.e., slope). This example was used in a college algebra class with students who were already familiar with the concept of slope for a linear function.

To anticipate the upcoming topic of intervals of change, the instructor asked students to think

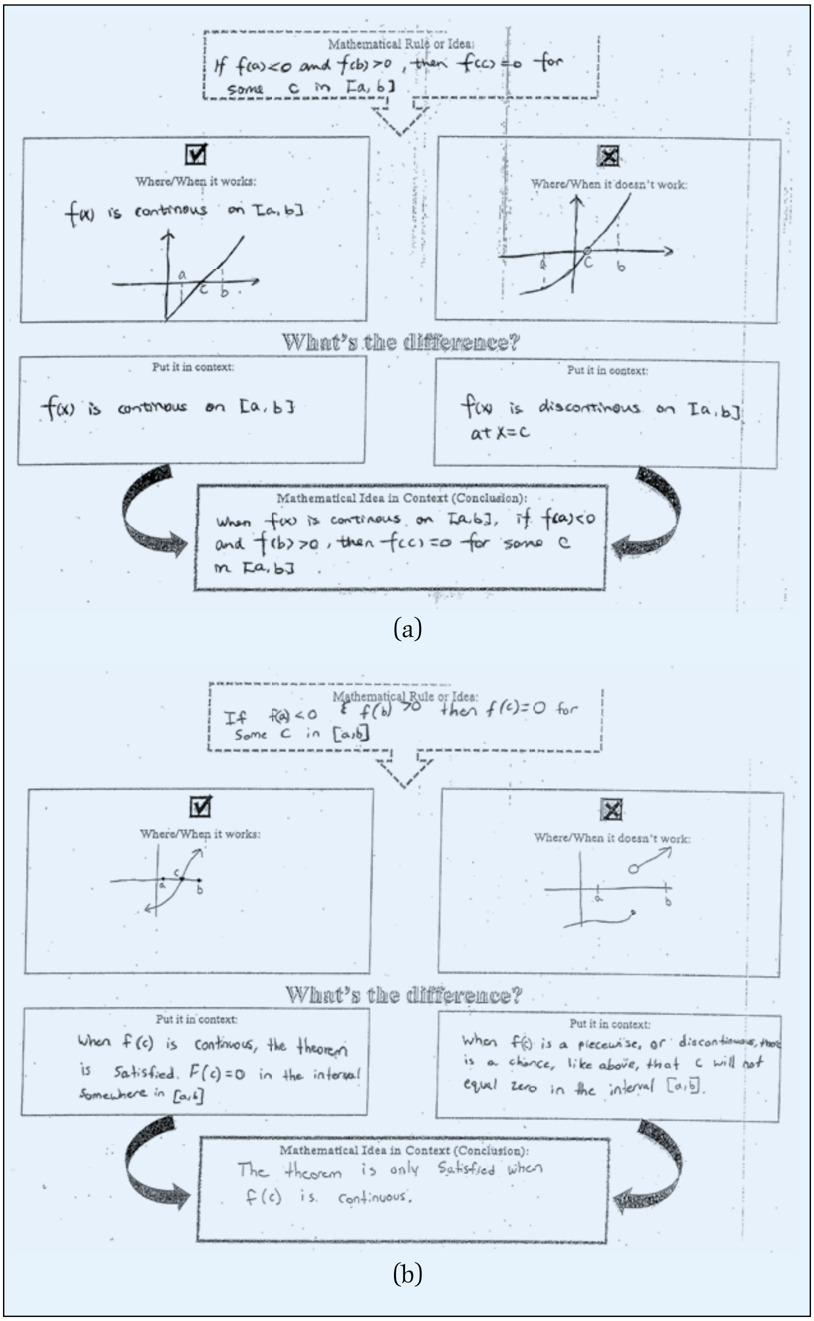

Fig. 6 Calculus students' work shows reasoning about the mathematical context for the intermediate value theorem.

about slope and to provide examples of functions where the notion of a constant slope did or did not apply. This reflection allowed the class to begin to discuss using intervals of change to describe the behavior of nonlinear functions with nonconstant rates of change.

The work in **figure 7** shows a student's attention to viewing windows along the graph of the function. In the case of the linear function, each viewing window shows the same slope. The two viewing windows sketched on the nonexample depict how the slope is not always the same along the absolute value function. Without a rich

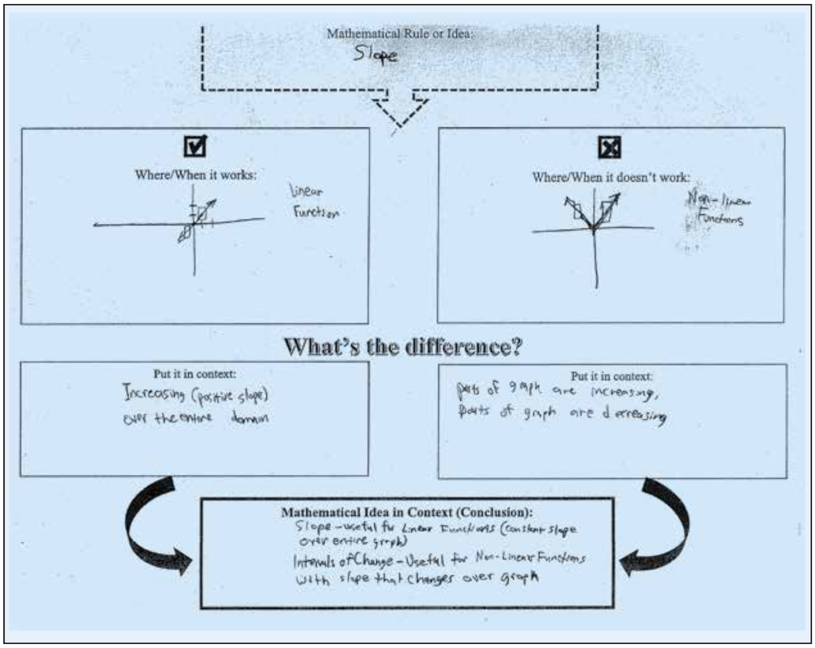

Fig. 7 An algebra student reasons about the mathematical context for slope and intervals of change for nonlinear functions.

classroom discussion of this new context, this student might be focusing only on the idea that part of the graph is increasing and part of it is decreasing. However, in this college algebra class, other students used monotonic functions, such as $f(x) = x^3$, to show that although the average rate of change was always positive, the rate of change was not constant across all viewing windows. In such instances, it is the role of the teacher as facilitator to be sure that students are considering all relevant and appropriate features of the model.

In this particular example, classroom discussion extended to monotonic functions, although the absolute value functions sufficed as an example because the goal was to support a discussion of functions that both increased and decreased. However, if the goal was to introduce calculus students to variable rate of change as a prelude to the derivative function, then encouraging students to think about functions with constantly varying rate of change would be extremely important. Thus, it is critical that teachers design mathematical tasks and classroom discussions appropriate for the learners with whom they are working.

FOCUS AND ENGAGEMENT

Physical and social contexts are frequently considered when writing mathematical tasks and are easily recognizable contextual features of tasks. For instance, opportunities to reason about physical contexts are usually couched in a real-world situation, whereas tasks that promote social context reasoning typically ask for the students' personal preferences or require them to communicate an answer to others (e.g., writing a letter summarizing their findings). In this article, we have suggested that mathematical context is an equally important layer of context that can help students understand the conceptual underpinnings of mathematical procedures and the mathematical contexts in which those procedures are valid.

Students and teachers alike should engage in discussions about explicit and implied mathematical contexts to prevent misuse or overgeneralization of mathematical results. Incorporating opportunities to reason about the mathematical context in everyday tasks may promote student engagement in verifying and validating mathematical rules or ideas. This high-cognitive-demand analysis is likely to help students build a deep and robust understanding of the underlying mathematical concepts.

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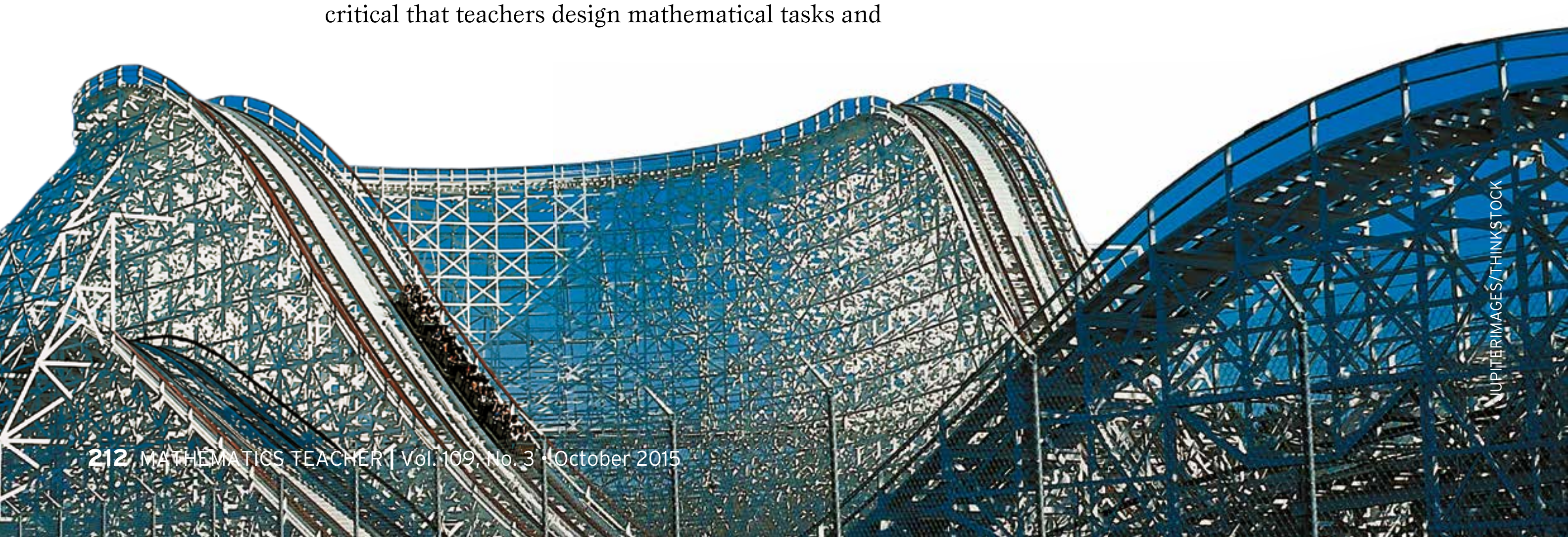

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For a graphic organizer to help students explain mathematical ideas in context, see the online article at **http://www.nctm.org/mt**.