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From Circle to Hyperbola in Taxicab Geometry

How far is it from Grand Central Station to the Empire State building? A pigeon could fly there in a straight line, but a person is restricted by the street grid. The geometry measuring the distance between points using the shortest path traveled along a square grid is known as taxicab geometry. This topic can engage students at all levels with tasks from plotting points and observing surprising shapes, to examining the underlying reasons for the appearance of these figures. Having to work with a new distance measurement takes everyone out of their comfort zone of routine memorization and makes them think, even about definitions and facts that seemed obvious before. Group discussions can be lively!

Many good resources on taxicab geometry are available. The ideas from Krause's classic book (1986) have been picked up in recent NCTM publications (Dreiling 2012) and (Smith 2013; this includes an extensive pedagogy discussion); House (2005) provides useful activity sheets.

To begin, we need a definition of distance. In a real-world context, locations on a city grid would be associated with points having at least one integer coordinate. However, the following definition applies to all points in the plane.

> *Definition*: Let A and B be points with coordinates (x_1, y_1) and (x_2, y_2), respectively. The taxicab distance from A to B is defined as $dist(A, B) = |x_1 - x_2| + |y_1 - y_2|$.

To compute the distance between two points in taxicab geometry, we need to add the horizontal and vertical distances that separate them. For example, the distance between A $(-2, 1)$ and B $(3, 5)$ is $|-2 - 3| + |1 - 5| = 5 + 4 = 9$. To get from A to B, we move a total of 5 units right and 4 up. Interestingly, the shortest path is not unique; many sequences of 9 steps up and right create a path from A to B (see **fig. 1**).

The way that distance is measured has huge implications for the shapes of well-known geometric figures. Working in taxicab geometry requires careful attention to definitions. Assumptions and intuition grounded in experience with Euclidean geometry are sometimes misleading.

SHAPES IN TAXICAB GEOMETRY

What does a circle look like in this geometry? The set of all points of distance 5 from the origin is shown in **figure 2**. By definition, this circle consists of all points (x, y) with $|x| + |y| = 5$. From this, we see that four lines contain the straight-line segments making up the circle (concisely represented as $y = \pm x \pm 5$). All circles, regardless of their center point or radius, result in a diamond shape.

> *Fact 1*: In taxicab geometry, a circle consists of four congruent segments of slope ± 1.

To check your students' understanding, hold a pen of length 5 inches,

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Activities for Students appears five times each year in *Mathematics Teacher*, often providing reproducible activity sheets that teachers can adapt for use in their own classroom. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more information, visit http://www.nctm.org/mtcalls.

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aligning it vertically on a grid so that it extends from (0, 0) to (0, 5). Now tilt it so that the tip is at (3, 4). The taxicab distance from base to tip is 3 + 4 = 7; the pen became longer! This can be confirmed by holding the pen over the circle (that is, the taxicab circle) of radius 5 and noting that the tilted pen sticks out beyond the circle. Keep rotating. It will be longest at 45 degrees and then shrink back to length 5 in the horizontal position.

This activity shows that an object's length changes when it is tilted, so be sure never to use a ruler to measure distances in taxicab geometry. Always count the number of horizontal and vertical moves to get from A to B. A more formal observation is that the distance is invariant under translation but not rotation.

To facilitate investigations, we use the following notation:

- Two points on the same horizontal or vertical line will be called *h/v-aligned*.
- The rectangle with diagonal vertex points A and B will be referred to as the AB-box. If A and B are h/v-aligned, the AB-box collapses to segment AB.
- We will designate the shorter side length of an AB-box as s and the longer as t.
- We will often label points Q that are on an AB-box.

We have already observed that in taxicab geometry there can be many paths of minimal length between points A and B. This observation can be used to justify several useful facts:

Fact 2:
For any point X in or on an AB-box, $dist(A, X) + dist(X, B) = dist(A, B)$.

Fact 3:
(a) Let P be at perpendicular distance r from any point Q on an AB box (see **fig. 3**). Then $dist(P, A) = r + dist(Q, A)$ and $dist(P, B) = r + dist(Q, B)$.

(b) The same holds true for points P on a quarter-circle segment of radius r centered at vertex Q.

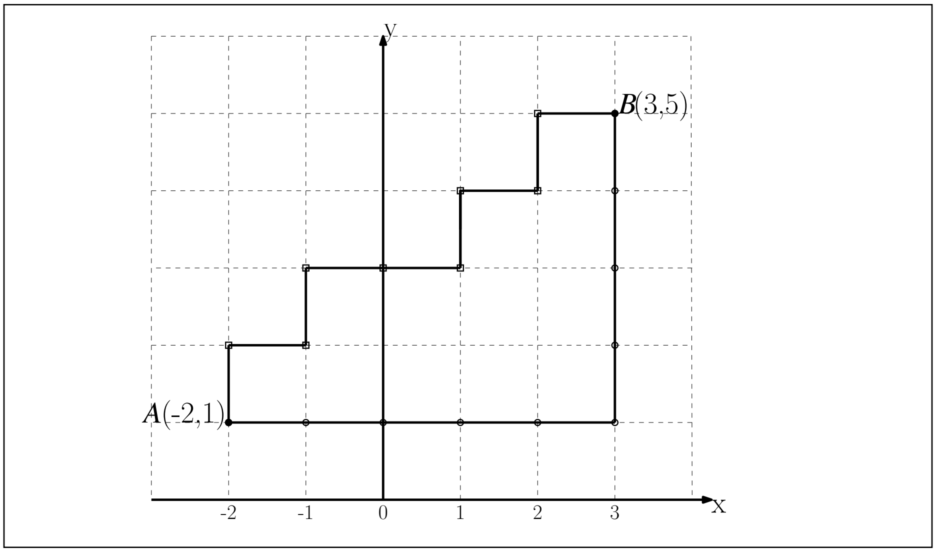

Fig. 1 The shortest distance from A to B is 9 units, but the path is not unique.

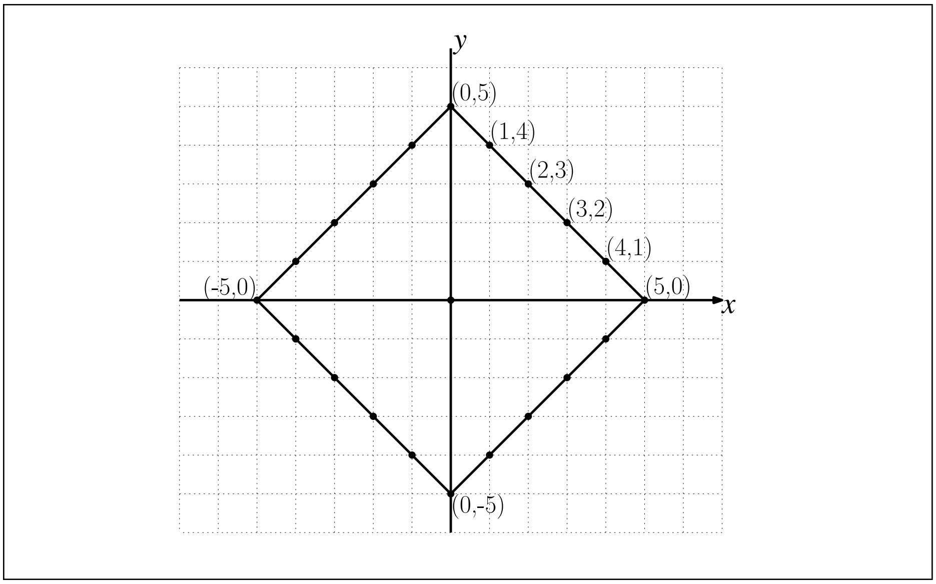

Fig. 2 The set of points 5 units from the origin defines a circle.

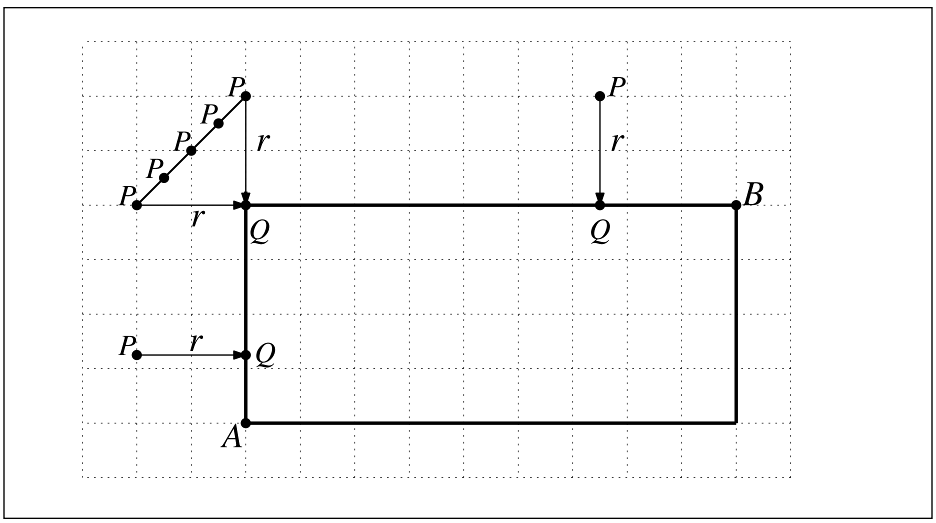

Fig. 3 The distance from P (which is r units from the AB-box) to A depends on the distance from Q to A.

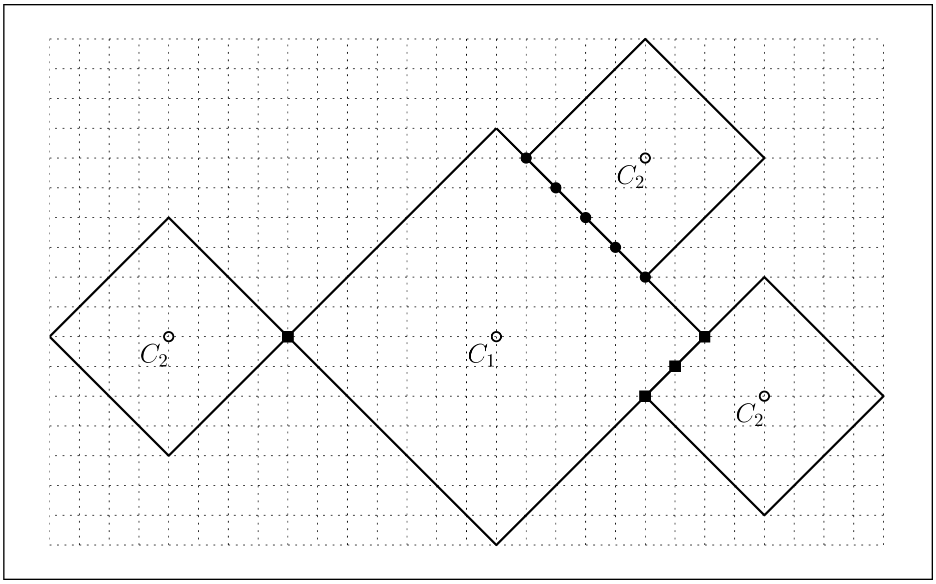

Fig. 4 If $r_1 + r_2 = D$, the intersection of two circles can be a point or a segment, depending on the relative position of the circles' centers.

The questions on the included Grid City activity sheet lead to the taxicab analogs of some well-known figures in Euclidean geometry. In the activity, students will encounter circles, hyperbolas, and ellipses.

SOLUTIONS TO GRID CITY PROBLEMS

Solution A: Circles

Adam lives on a circle of radius 5 around the high school (*HS*) (see **fig. 2**).

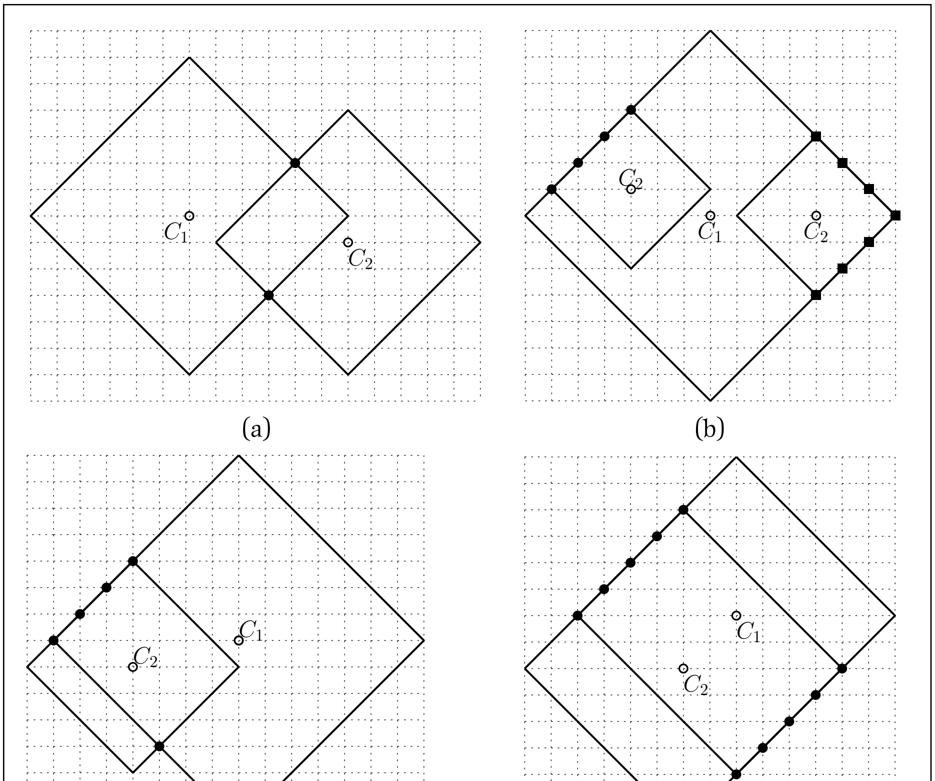

Fig. 5 If $r_1 + r_2 > D$, then several cases represent the intersection of the two circles.

Solution B: Tangent and Intersecting Circles

Brenna lives at the intersection of circles of radius 3 and 7 around *HS* and the middle school (*MS*), respectively. The sum of the radii, 10, happens to be the distance between the centers, *HS* and *MS*, so experience with our usual distance measure would suggest two circles that are tangent at a single point. However, in taxicab geometry, two circles can be "tangent" in other ways—namely, by intersecting in segments of slope $m = \pm 1$ (see **fig. 4**).

Consider two circles of radius r_1 and r_2, with centers C_1 and C_2, respectively. Let $D = dist(C_1, C_2)$ be the distance between their centers. If $r_1 + r_2 < D$, the circles do not intersect. However, if $r_1 + r_2 = D$, we could say that the circles are tangent. The intersection is a single point if the centers are h/v-aligned. If the centers are not h/v-aligned, then the intersection is a segment of slope $m = \pm 1$ inside the C_1C_2-box. (See **fig. 4**.)

Another situation is possible when circles intersect (see **fig. 5**). If $r_1 + r_2 > D$, the circles intersect in two points (see **fig. 5a**) except in the following cases:

- If $|r_1 - r_2| = D$, the intersection is one segment or two adjacent segments if the centers are h/v-aligned (see **fig. 5b**).
- If $|r_1 - r_2| = t - s$, where $0 < s < t$ are the sides of the C_1C_2-box, the circles intersect in a segment and a point (see **fig. 5c**).
- If the C_1C_2-box is a square (i.e., $s = t$) and $r_1 = r_2$, the circles intersect in two segments (see **fig. 5d**).

Solution C: Equidistance

For Carl's apartment, we need to find the set of all points P that are equidistant from two points A and B—that is, $dist(P, A) = dist(P, B)$.

We could begin by drawing pairs of congruent circles of varying radii around A and B and finding their intersections. If A and B are h/v-aligned, the solution is a familiar figure—the points on the perpendicular bisector of segment AB. For other arrangements of A and B, this set of intersection points has surprising shapes (see **fig. 6**). The smallest congruent circles that intersect have radius $r = dist(A, B)/2$; they

intersect in a segment of slope $m = \pm 1$ in the interior of the AB-box. From the endpoints (labeled Q in **fig. 6**) on the longer sides of the box, the set of equidistant points P extends in rays perpendicular to the sides of the box.

If the AB-box is square, the interior segments end at vertex points Q, so by fact 3(b) the set of points P equidistant to A and B contains not only vertical and horizontal components but also all points in between.

Solution D: Hyperbolas

Definition: If we have two points, foci F_1 and F_2, separated by distance D and a positive real number $k < D$, then the set of all points P such that $|dist(P, F_1) - dist(P, F_2)| = k$ is a hyperbola.

Dana and Denise live on different branches of a hyperbola with foci HS and MS and constant difference $k = 4$. Dana's house is located on the branch closer to focus point MS.

To construct the hyperbola, look for intersections of circles of varying radii r and $r + k$, respectively, around the foci. For $k = 0$, we obtain the set of points equidistant to the foci, so the hyperbola has similar features (see **fig. 7**).

If the foci are h/v-aligned, the hyperbola consists of two vertical or horizontal lines a distance of k apart (see **fig. 7a**). For other arrangements, we see segments of slope $m = \pm 1$ inside the F_1F_2-box. These segments represent the intersections of circles of radius $(D - k)/2$ and $(D + k)/2$ around the foci. Once the sides of the box are reached at Q, the hyperbola continues with points P on rays perpendicular to the side. We justify the graph using fact 3(a):

$$\begin{aligned} &dist(P, F_1) - dist(P, F_2) \\ &\quad = [r + dist(Q, F_1)] - [r + dist(Q, F_2)] \\ &\quad = dist(Q, F_1) - dist(Q, F_2) \\ &\quad = (D + k)/2 - (D - k)/2 = k \end{aligned}$$

The locations of segment endpoints Q depend on the F_1F_2-box. Let s be the length of the smaller side of the box. If the shorter side is small enough, $s < (D - k)/2$, both segments will end at points Q on the two longer sides. Otherwise, there will be one endpoint

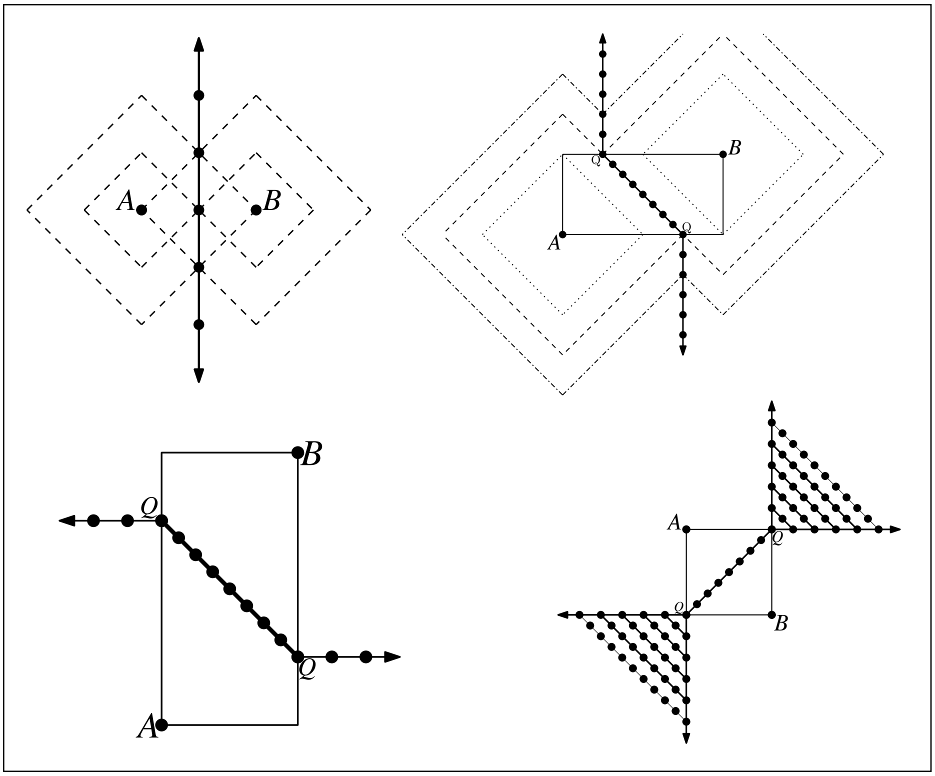

Fig. 6 The set of points equidistant from *A* and *B* depends on the relative position of *A* and *B*.

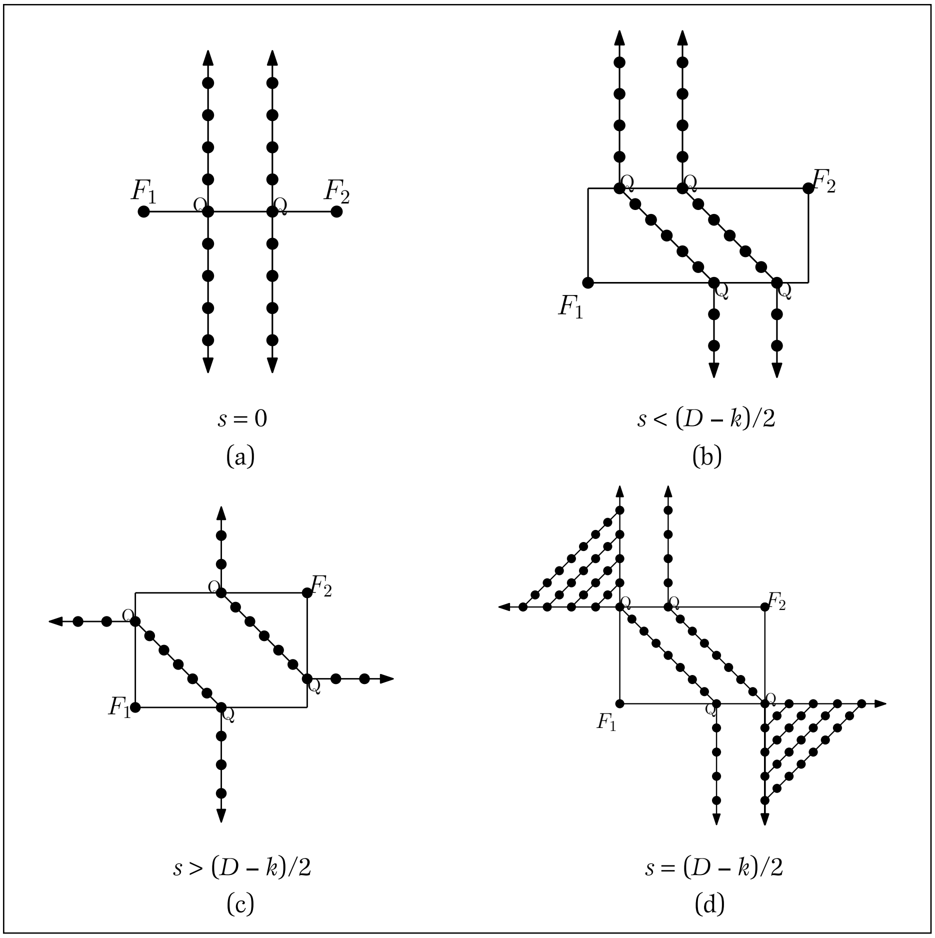

Fig. 7 Different types of hyperbolas result, depending on the orientation of the foci and the length of the short side.

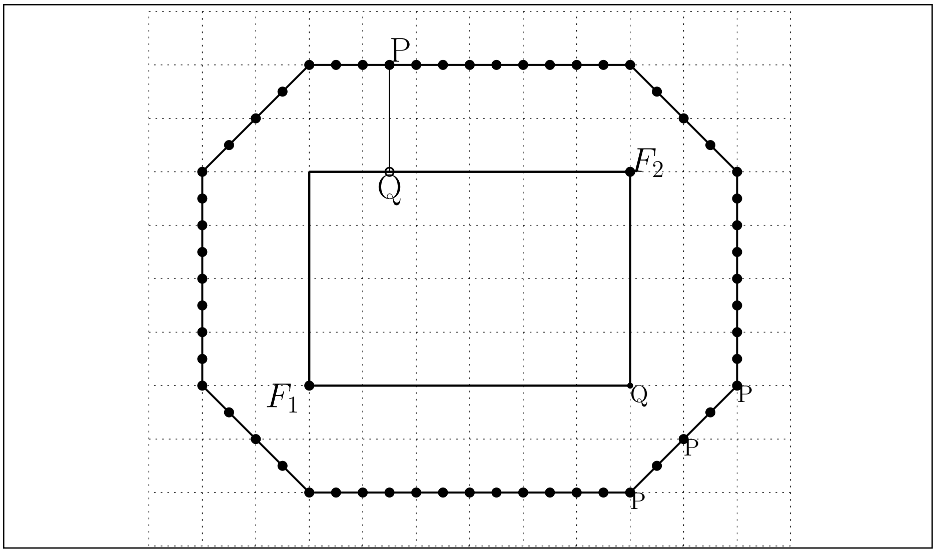

Fig. 8 An ellipse results for nonaligned foci separated by a distance of 10 and constant sum 14.

on each side of the box. If the shorter side is of a precise length with respect to the separation of the foci and the constant difference k, so that $s = (D - k)/2$, then two of the endpoints Q are vertices of the box. In that case, the hyperbola extends to contain not only vertical and horizontal rays but also all points in between!

For street grid problems, the hyperbola should contain many grid points. Choosing D and k as both even or both odd makes $(D \pm k)/2$ an integer.

Solution E: Ellipses

Definition: Given two points, foci F_1 and F_2, separated by distance D, and a real number $k > D$, then the set of all points P such that $dist(P, F_1) + dist(P, F_2) = k$ is an ellipse.

Erik's house is located on an ellipse with foci *HS* and *MS*. Construct the ellipse around the F_1F_2-box: Consider any point Q on the box, and let P be a point of perpendicular distance $r = (k - D)/2$ from Q (see **fig. 8**). Since $dist(P, F_i) = r + dist(Q, F_i)$ for both $i = 1$ and $i = 2$ and $dist(Q, F_1) + dist(Q, F_2) = D$, we have $dist(P, F_1) + dist(P, F_2) = D + 2 \cdot (k - D)/2 = k$. If Q is a vertex of the F_1F_2-box, all points P on a quarter-circle of radius $r = (k - D)/2$ around Q also satisfy this distance requirement, by fact 3(b).

An ellipse consists of eight segments (six if the foci are h/v-aligned). The four slanted segments have slope $m = \pm 1$. The others are horizontal and vertical segments of the same length as the sides of the F_1F_2-box (see **fig. 9**). To ensure that the ellipse contains grid points, choose D and k as both even or both odd, so $(k - D)/2$ is an integer.

EXTENSIONS

It is noteworthy that Euclidean geometry uses two equivalent definitions for ellipses and hyperbolas, respectively, but they lead to very different figures in taxicab geometry. Laatsch (1982) shows that if the definition of ellipse and hyperbola based on a focus point and directrix is used, the resulting figures can be obtained as sections of a square pyramid. But if we use the more popular high school definition based on two focus points, we obtain other ellipse and hyperbola shapes, which are the ones found more commonly in the taxicab geometry literature.

Finally, teachers could provide an extra challenge by writing additional problems introducing a subway line that does not follow the street grid. These problems lead students to draw the taxicab analogs of parabolas and require a discussion of the distance of a point to a line. Interested readers may find supplemental material related to parabolas in taxicab geometry and answers to two additional scenarios with the online article at **www.nctm.org/mt**.

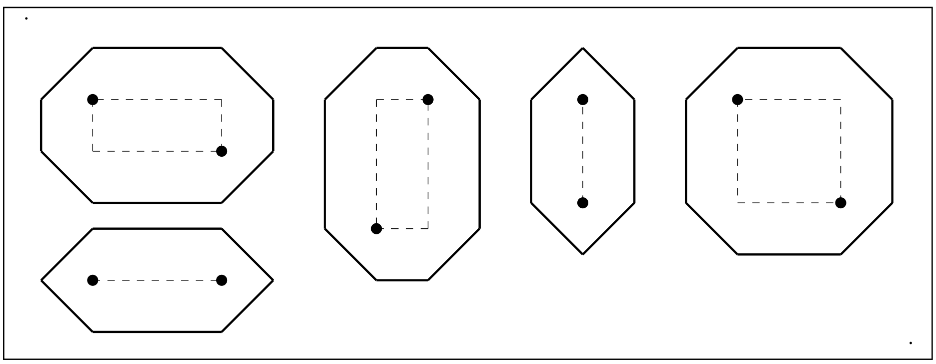

Fig. 9 Ellipses may comprise six or eight sides, depending on the alignment of the foci.

REFERENCES

Dreiling, Keith M. 2012. "Delving Deeper: Triangle Construction in Taxicab Geometry." *Mathematics Teacher* 105 (6): 474–78.

House, Peggy. 2005. *Hey, Taxi.* NCTM Student Math Notes (Student Explorations in Mathematics). http://www.nctm.org/publications/article.aspx?id=22334

Krause, Eugene F. 1986. *Taxicab Geometry: An Adventure in Non-Euclidean Geometry.* New York: Dover Publications.

Laatsch, Richard. 1982. "Pyramidal Sections in Taxicab Geometry." *Mathematics Magazine* 55: 205–12.

Smith, Christopher E. 2013. "Is That Square Really a Circle?" *Mathematics Teacher* 106 (8): 615–19.

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Additional problems related to parabolas in taxicab geometry can be found with this article online at **www.nctm.org/mt**.

GRID CITY ACTIVITY SHEET

Adam, Brenna, Carl, Dana, Denise, and Erik live in Grid City, where each city block is exactly 300 feet wide. They were in middle school (*MS*) last year, but now they attend Grid City High School (*HS*). If *MS* is located 6 blocks east and 4 blocks north of *HS*, use the information below to plot all possible locations for everyone's homes on separate sheets. Assume that students must stay on the street grid, taking no shortcuts through yards or buildings.

Warm-up question: Find the shortest path from *HS* to *MS*. How long is it? Is it unique?

A. Adam lives 5 blocks (1500 ft.) from *HS*.

B. Last year Brenna walked 7 blocks (2100 ft.) to *MS*. This year she has to walk only 3 blocks (900 ft.) to *HS*.

C. Carl uses his skateboard to get to school. He says that it is exactly the same distance from his apartment to *HS* as it was from his apartment to *MS* last year.

D. Dana rides her bike to school. She complains that this year she has to ride 4 blocks (1200 ft.) farther every morning than when she biked to *MS* last year. The exact opposite is true for her best friend Denise, whose bike ride to school is now 1200 ft. shorter.

E. Last year Erik's best friend, Earl, moved to the suburbs but finished the school year in the city. After school he walked from *MS* to Erik's house, and later he walked to *HS* to catch a ride home with his sister after her basketball practice. Earl's total walking distance was 14 blocks (4200 ft.).

Additional Questions to Consider

How do the solutions to the problems above change if *MS* is located 5 blocks east and 5 blocks north of *HS*? If *MS* is 8 blocks due east of *HS*?