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Using Surfer to Investigate Algebraic Surfaces

Children love sculpting clay or building sand castles, creating objects in three dimensions before they have the motor skills to draw in two dimensions. Play-doh® before drawing—this progression seems relevant to development of spatial sense (van de Walle 2006). Similar arguments applied to the study of curves and graphs in high school mathematics would suggest that students' work and calculation with shapes should move sequentially from concrete to more abstract thought (Carson and Rowlands 2007). We live in a three-dimensional world, and two-dimensional shapes are either a mathematical abstraction of reality or the projection of three-dimensional objects onto flat surfaces.

Edited by Heather Lynn Johnson

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To introduce three-dimensional algebraic surfaces to our ninth-grade students, we used Surfer®, the free software behind an installation called *Formula Morph* at the National Museum of Mathematics (MoMath) in New York City. (*Formula Morph* is one of MoMath's permanent exhibits and can be found on floor 0 of the museum [momath.org/about/exhibit-guide].) The immediate visual appeal of algebraic surfaces is apparent, and the developments in technology have made them easily accessible.

WHAT IS AN ALGEBRAIC SURFACE?

A circle in a coordinate plane gives us an example of a one-dimensional object in a two-dimensional space as the set of all points satisfying a polynomial equation in two variables (in this case, $x^2 + y^2 = 1$). Likewise, the sphere $x^2 + y^2 + z^2 = 1$ represents the set of all points satisfying a polynomial equation in three variables as a two-dimensional object located in three-dimensional space—that is, an algebraic surface. Each point on an algebraic surface corresponds to a unique solution (x, y, z) to the polynomial equation. For example, (1, 0, 0) is a point on the sphere $x^2 + y^2 + z^2 = 1$, whereas (2, 1, 0) is not, because the latter does not satisfy the equation.

A more interesting example, resulting from the equation $x^3 + x^2z^2 = y^2$, is an object that might be called the "hummingbird" (see **fig. 1**). After we find some coordinate points by trial and error and plot them in a three-dimensional coordinate system, it becomes apparent that actually picturing the algebraic surface can be daunting.

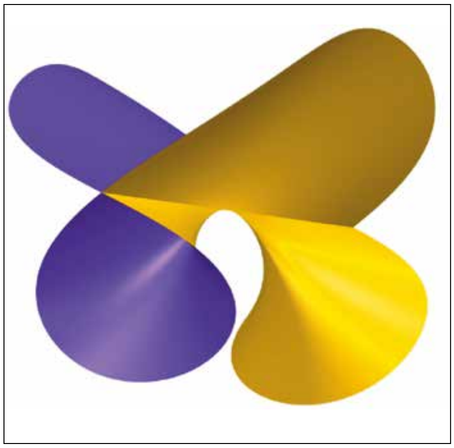


Fig. 1 Algebraic surfaces represent the solution set to polynomial equations with three variables—here, the hummingbird, $x^3 + x^2z^2 = y^2$.

To explore algebraic surfaces in the classroom, we used Surfer, an open-source software available freely on Imaginary (www.imaginary.org). Surfer has been specifically designed as a platform to communicate modern mathematics in public exhibitions. As a result, using the software requires almost no training, making it particularly accessible to mathematics classrooms (see the sidebar **Surfer FAQs**). Surfer is no typical graphing software; it plots algebraic surfaces using ray tracing, rather than triangulation, which cannot deal well with spikes (*singularities*), folds, or intersections. (Interested readers might compare renditions of the cubic $x^2 + y^2 + z^2 + 2xyz - 1 = 0$, using various

graphers to see the limitations.) It still allows a user to enter or edit equations, rotate surfaces, zoom, add parameters, change colors, and manipulate graphs in expected ways. We also find that the aesthetic quality of the images is unmatched, which is important for student motivation—they are making works of art!

The hummingbird equation can serve as a springboard for exploration and experimentation. For example, what happens if we change the power of the first x from 3 to 4? **Figure 2** shows the resulting graph. What happens if we add a constant? Such an exploration would be well suited for students even with little experience of sketching graphs representing solutions to polynomial equations in two variables. We encouraged our ninth-grade students to explore playfully, talking of windmills and other objects, before moving to a more systematic investigation.

START WITH THE FAMILIAR: STRAIGHT-LINE GRAPHS

Our students had studied linear graphs and had seen how higher exponents of x and y produce plane curves. So we started our investigation by comparing the graph of $y = x$ in two-dimensional and three-dimensional graphing environments.

In Surfer, it is necessary to rewrite $y = x$ as $y - x = 0$ because the software accepts only equations set equal to zero. When graphed in a three-dimensional environment, $y = x$ is actually a plane (see **fig. 3**). It appears as a disc (see **fig. 3c**) because the software cuts a sphere around objects that are bigger than the actual viewing space, portraying only what is inside the sphere.

Next, we asked students to consider graphing $y = \pm x$ (i.e., the union of $y = x$ with $y = -x$) in two and three dimensions. To simultaneously graph $y = x$ and $y = -x$ in Surfer, students entered $y^2 - x^2 = 0$ or $(y + x)(y - x) = 0$.

The results are compared in both two and three dimensions (see **fig. 4**).

This result led us to produce a visual reference for the xy-coordinate system using Surfer. In two dimensions, $y = 0$ and $x = 0$ are the x-axis and y-axis, respectively, so the product $y \cdot x = 0$ should, when viewed from the appropriate angle in three dimensions, look like the x- and y-coordinate axes (**fig. 5**). However, from a different perspective in three dimensions, it is clear that $x = 0$ and $y = 0$ are in fact planes.

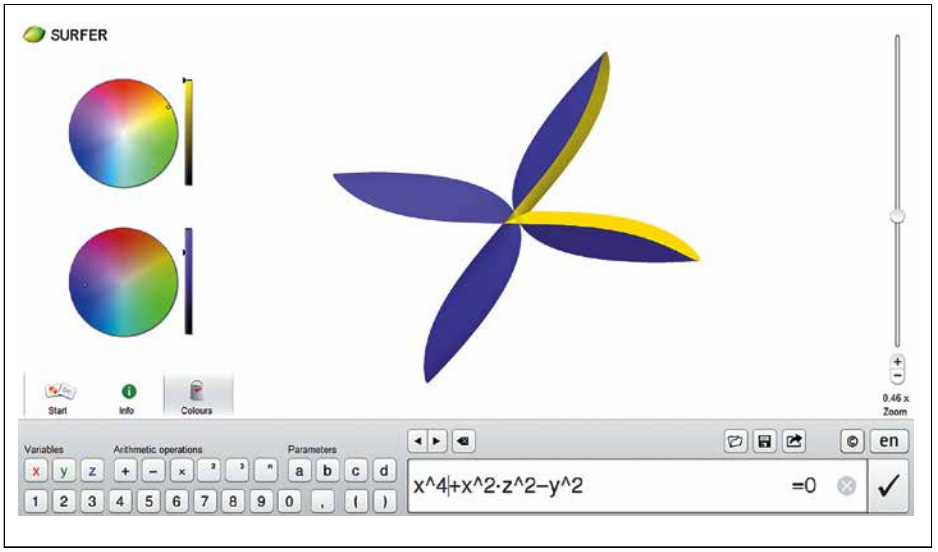


Fig. 2 Changing the exponent on the first x from 3 to 4 in the hummingbird equation gives a butterfly.

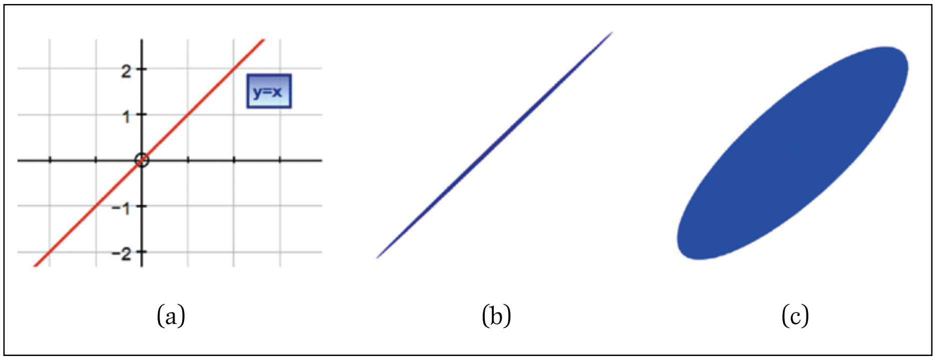


Fig. 3 The graph of $y = x$ is a line in two dimensions (**a**) and a plane in three dimensions, seen here sideways (**b**) and at an angle (**c**).

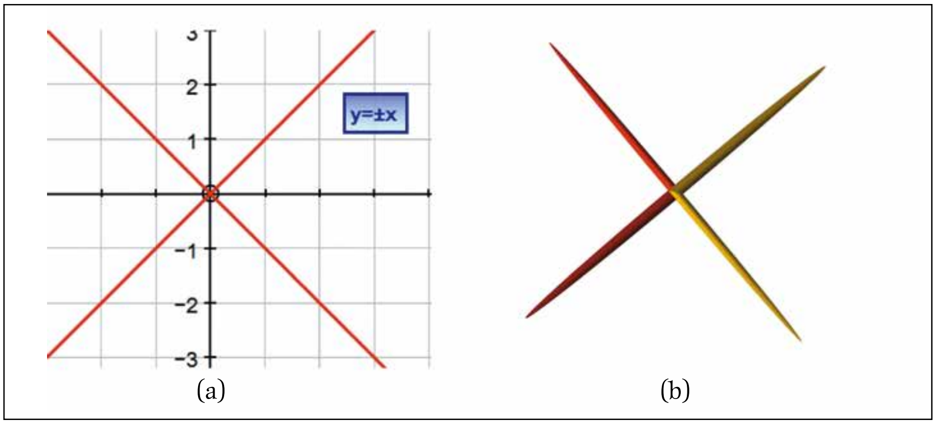


Fig. 4 Superimposing two objects requires multiplying: $(y + x)(y - x) = 0$.

During the course of the next few lessons, we started experimenting. A description of several suggested experiments, meant to build on one another, follow.

EXPERIMENT 1: "SUPERIMPOSING" SURFACES

The purpose of experiment 1 is to investigate how surfaces can be the result of a union of solution sets—that is,

Fig. 5 From this view, with the z-axis almost perpendicular to the page, the two planes $x = 0$ and $y = 0$ appear as coordinate axes.

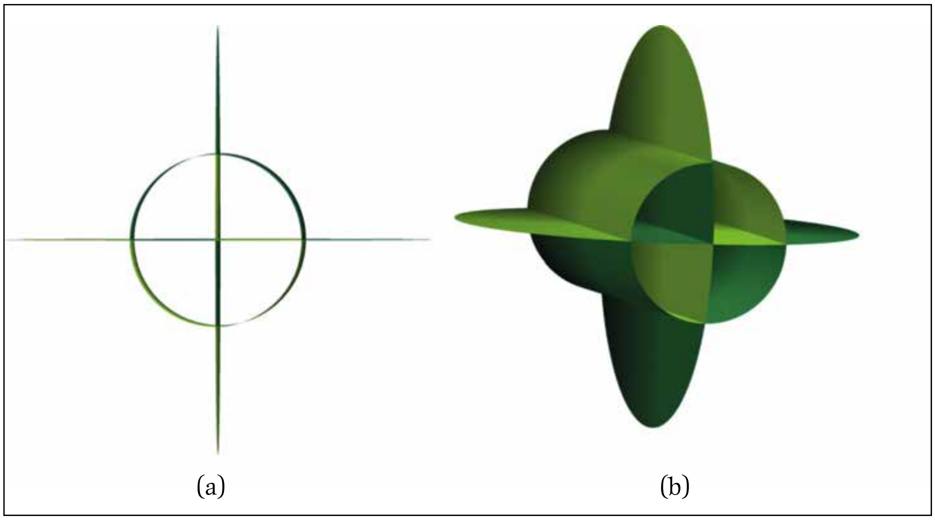


Fig. 7 A circle in two dimensions (**a**) extends to a cylinder in three dimensions (**b**).

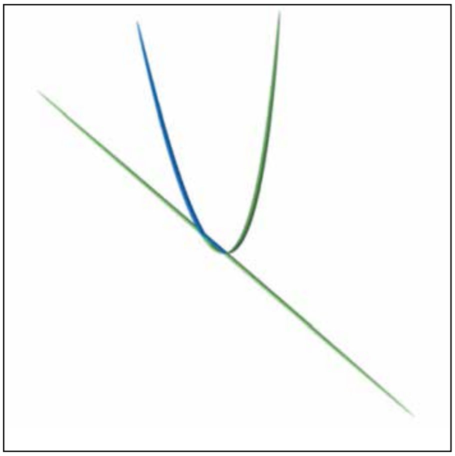


Fig. 6 Changing a power superimposes a line on a parabola: $(y + x)(y - x^2) = 0$.

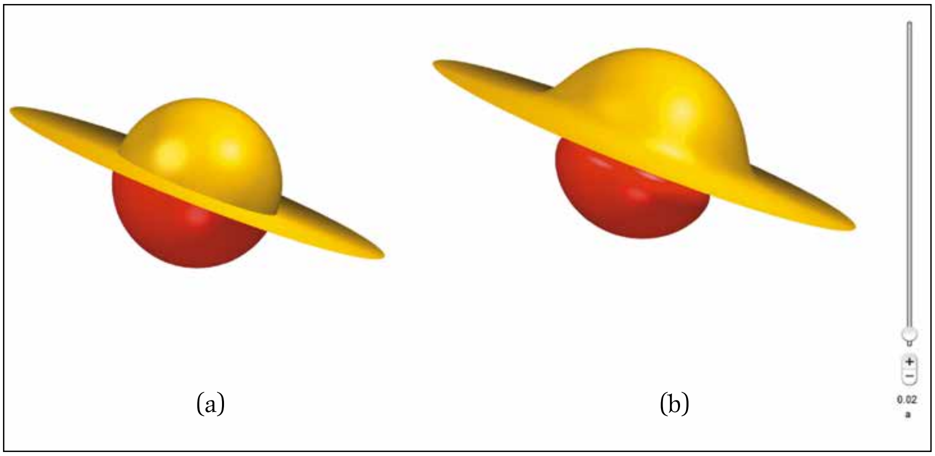


Fig. 8 Superimposing a plane and a sphere creates an object that resembles Saturn. The graph of $x(x^2 + y^2 + z^2 - 1) - a = 0$ transitions from $a = 0$ to $a = 0.02$ with the use of a slider.

"superimposed" objects represented by a single equation.

Part 1: A Parabola and a Line

Having seen the dramatic effect of changing exponents in the hummingbird example (see **figs. 1** and **2**), students will be eager to start editing the equation $(y + x)(y - x) = 0$ from the previous discussion. They may wish to change it to, say, $(y + x)(y - x^2) = 0$, thus superimposing the line $y = -x$ on the parabola $y = x^2$ (see **fig. 6**).

Part 2: Coordinate Axes and a Circle—Bull's-Eye

Extending ideas from part 1, have students superimpose the two-dimensional unit circle onto the axes as

$$(xy) \cdot (x^2 + y^2 - 1) = 0$$

(see **fig. 7a**). Rotating this object in Surfer will, in fact, show a cylinder. The teacher can explain how, with the z-variable free, the circle extends to create the cylinder. For some students, understanding this object may help clarify the intersecting planes represented in **figure 5** and help explain a three-dimensional version of the object graphed in **figure 6**.

Part 3: A Sphere and a Plane—Saturn

For another visual example of combined surfaces, substitute an expression corresponding to the equation of a unit sphere for y to give $x \cdot (x^2 + y^2 + z^2 - 1) = 0$ (see **fig. 8a**). Note that the intersection of the sphere and the plane, the set of points for which both $x = 0$ and $x^2 + y^2 + z^2 - 1 = 0$, is a circle—in this case, a circle in the yz-plane with radius 1 and equation $y^2 + z^2 - 1 = 0$.

At this stage, the teacher could instruct students to consider the sphere and explain the result when they substitute various values for z between -1 and 1. A variety of circles represents cuts through the sphere (see **fig. 9**). The largest red circle is a slice through the "equator" at $z = 0$. The smallest red circle is a slice near the "north pole" at $z = 0.9$.

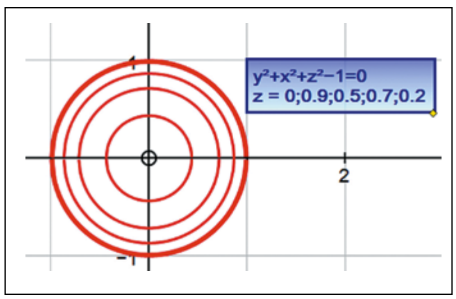


Fig. 9 Circles are level sets of the sphere at different values of z.

EXPERIMENT 2: MORPHING SURFACES—POTATO WITH HAT

In this experiment, we go beyond superimposing surfaces. Students investigate how to "smudge" the intersection so that one surface morphs into the other instead.

Compare the equations

$$x(x^2 + y^2 + z^2 - 1) = 0$$

and

$$x(x^2 + y^2 + z^2 - 1) + a = 0.$$

Introducing a parameter such as a in Surfer will automatically generate a corresponding slider enabling a to take values between 0 and 1. In this example, students will notice that increasing a seems to "morph" the plane and the sphere together (see **fig. 8**). When the added constant, a, is small, the coordinate values that satisfy $p \cdot q + a = 0$ can be approximated by the solutions of $p = 0$ and $q = 0$. Points of the original combined object are near, but no longer on, the surface. The intersecting surfaces of the plane and the sphere are "smoothed" as the two separate objects "melt" together. The larger the parameter a, the bigger the effect will be.

EXPERIMENT 3: TRANSLATIONS AND STRETCHES—TWO SPHERES

This experiment is designed to formalize notions of translations and stretches across representations, algebraic and visual.

Using the now familiar sphere and the results from the previous experiments, we suggested to students that $(x^2 + y^2 + z^2 - 1) \cdot (x^2 + y^2 + z^2 - 1) = 0$ must give two copies of an identical sphere superimposed on each other (see **fig. 10a**). Next we asked students how they might separate the spheres. If students have difficulty, the teacher can remind them that the equation $(x - 3)^2 + y^2 + z^2 - 1 = 0$ will translate the unit sphere centered at the origin by three units in the positive x-direction. (In Surfer, dragging with the mouse will rotate the surface in three-dimensional space, emphasizing, in this example, which is the x-direction on the graph.) And finally, multiplying the x-variable by a constant will result in a stretch or compression. For example, graphing

SURFER FAQs

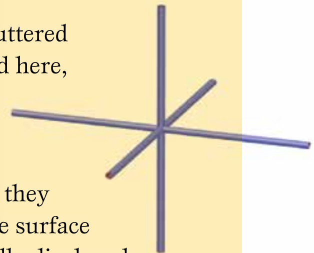
What is ray tracing?

Ray tracing is a three-dimensional graphing technology that simulates the way in which light rays are reflected on an object (and on objects among each other) before meeting the eye of the observer. This enables more photo-realistic rendering, as the reader has probably experienced in digital imagery or visual effects in movies.

Why does a plane not look like a plane?

What does an infinitely big plane—for example, the xy-plane ($z = 0$)—look like on the screen or on the paper? Only a portion of it can be seen. Typically, coordinate systems in rectangular form are truncated with an invisible rectangle or cube. Surfer cuts all surfaces with an invisible sphere. Thus, the plane becomes a disc!

Why are the coordinate axes and scale missing?



The emphasis is on the aesthetics of the rendering, with an uncluttered user interface. In the experiments on algebraic surfaces presented here, scale plays a secondary role; the features of the surface matter more than the precise location of points. Surfer allows free rotation in all directions. To identify the direction of each axis, students can perform a translation on one variable (for example, they could write $(x + 1)$ replacing each x) to see in which direction the surface moves. Alternatively, a thin cylinder along any axis can be usefully displayed.

Will I need an account on the IMAGINARY website?

No. The program is downloadable for free. However, you can upload your own or student creations to the website by registering at no cost.

Does the software come with a manual?

The software includes helpful hints on standard surfaces available in a "tutorial" section, with ideas for small experiments (available in nine languages). The download page also provides access to a manual (pdf) and a "SURFER tips and tricks" document (pdf) with further ideas.

Can I print my work?

Surfer does not have a print button. However, there is an **Export** button to save surfaces as a .png (picture) file. This file can then be embedded in other documents; in addition, the formula can be easily copied and pasted into a text box. (There is a **Print** button, but it is intended for exhibitions and museum installations and requires LaTex.) Surfer produces surface points to be graphed, whereas three-dimensional printing requires data prepared as a mesh of triangles. The translation from points to triangles is mathematically complex, and it is still an open question whether this can be done automatically.

For more information, go to imaginary.org/program/surfer.

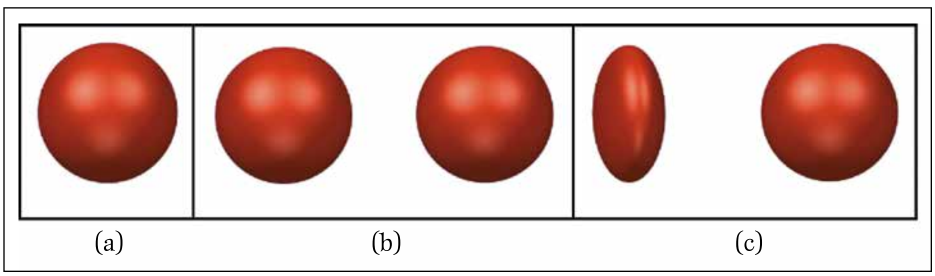


Fig. 10 Two superimposed spheres are used to demonstrate translation and dilation.

Step	Equation	Commentary	Result
1	$x^2 + y^2 + z^2 - 1 = 0$	Start with the equation of the sphere.	
2	$x^2 + y^2 + z^3 - 1 = 0$	Experiment with changing the power of z. Does this look like a hat? Can you explain the change? (Which is the z direction? Why is the sphere "exploded" in the z direction?)	
3	$(x^2 + y^2 + z^3 - 1) \cdot (x^2 + y^2 + z^2 - 1) = 0$	Experiment with intersecting this object with a sphere. How do we get the "head" out of the "hat"?	
4	$(x^2 + y^2 + z^3 - 1) \cdot ((x + 3)^2 + y^2 + z^2 - 1) = 0$	Try translating in the x-direction. Oops—wrong direction! Try translating in the z-direction. That works!	
	$(x^2 + y^2 + (z - 1)^3 - 1) \cdot (x^2 + y^2 + (z + 1)^2 - 1) = 0$	The head is out of the hat, but the hat is a little small.	
5	$(x^2 + y^2 + (z - 1.5)^3 - 1) \cdot (x^2 + y^2 + (z + 1)^2 - 3) = 0$	Try increasing the radius of the sphere. This looks better!	
6	$(x^2 + y^2 + (z - 1.5)^3 - 1) \cdot (x^2 + y^2 + (z + 1)^2 - 3) \cdot ((x + 0.5)^2 + 50y^2 + 50(z + 1.5)^2 - 4)^2 = 0$	Add a "nose" by multiplying with another expression representing a sphere stretched (in the y-direction) and translated (for placement).	

Fig. 11 Separate visual components are combined to create a three-dimensional figure.

$$((2x)^2 + y^2 + z^2 - 1) \cdot ((x - 3)^2 + y^2 + z^2 - 1) = 0$$

shows the translation alongside a compression (see **fig. 10c**).

EXPERIMENT 4: BRINGING IT ALL TOGETHER

Having been introduced to basic ideas of manipulating expressions, students can create more complex objects, such as a head or a die. Consider, for example, the sequence that we discussed with students (see **fig. 11**). (As an extension project, a parameter and slider might be used to cause Pinocchio's nose to grow.)

CULMINATING TASKS

After further play and experimentation, the unit culminated with the creation of two pieces by each student, the first using an imposed theme ("Create the most sophisticated snowman you can") and the second as a free creation ("What is the most interesting composition you have come across?") For each task, students were encouraged to name their artwork, and the surfaces were displayed on posters, with the equation and the attributed name. A few are shown here (see **figs. 12** and **13**).

The prospect of having to work with three variables and higher powers might at first seem too complex for ninth graders. In fact, Surfer has been used in exhibitions all around the world, allowing even younger students to experiment with algebraic surfaces. We have found that students' attempts to gain control of the images created in playful exploration drive them to want to understand the underlying mathematics. Opportunities for learning (e.g., the study of function transformations, inequalities used to represent regions, and implicitly defined functions) that are usually met in a more formal setting arise in creative experimentation as students are motivated to complement surfaces with visually appealing design elements and to move or distort them. In this unit, which required about six to seven hours, students discovered how changes in a polynomial equation can affect a related surface. Many further investigations, such as rotating a surface through the equation or showing just the intersection of two surfaces, might motivate students to continue to make mathematical sense of our three-dimensional world.

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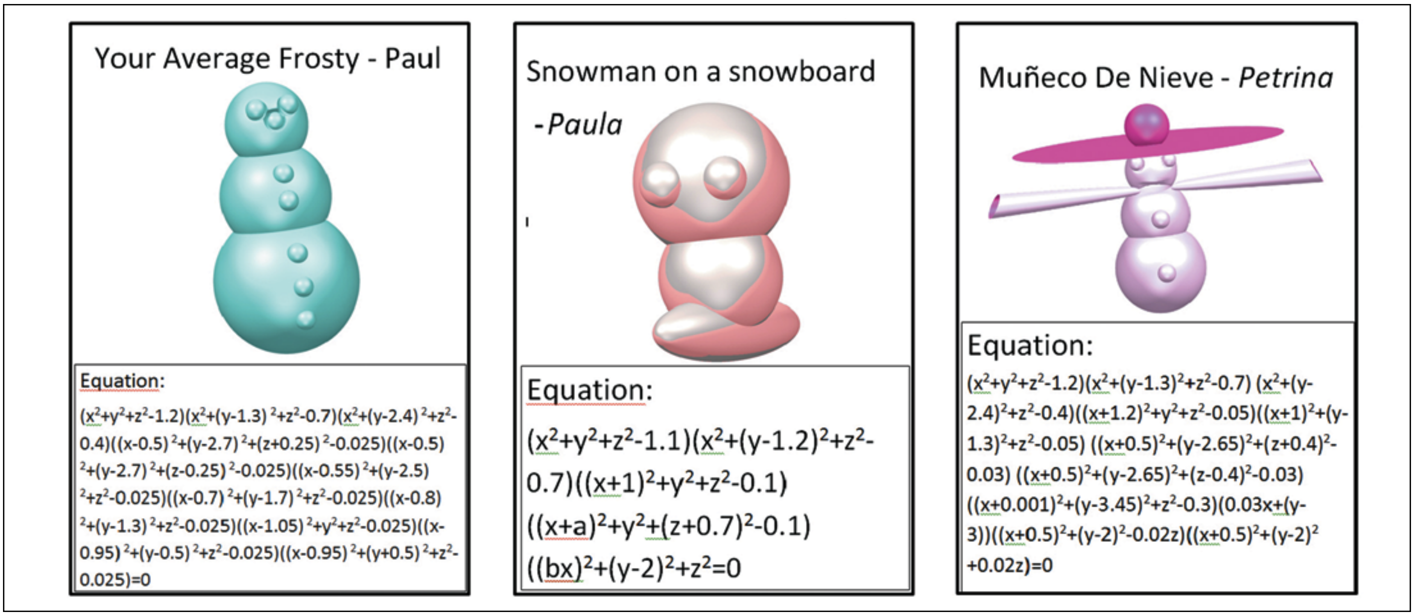

Fig. 12 For one part of their final project, students had an assigned theme—create a snowman.

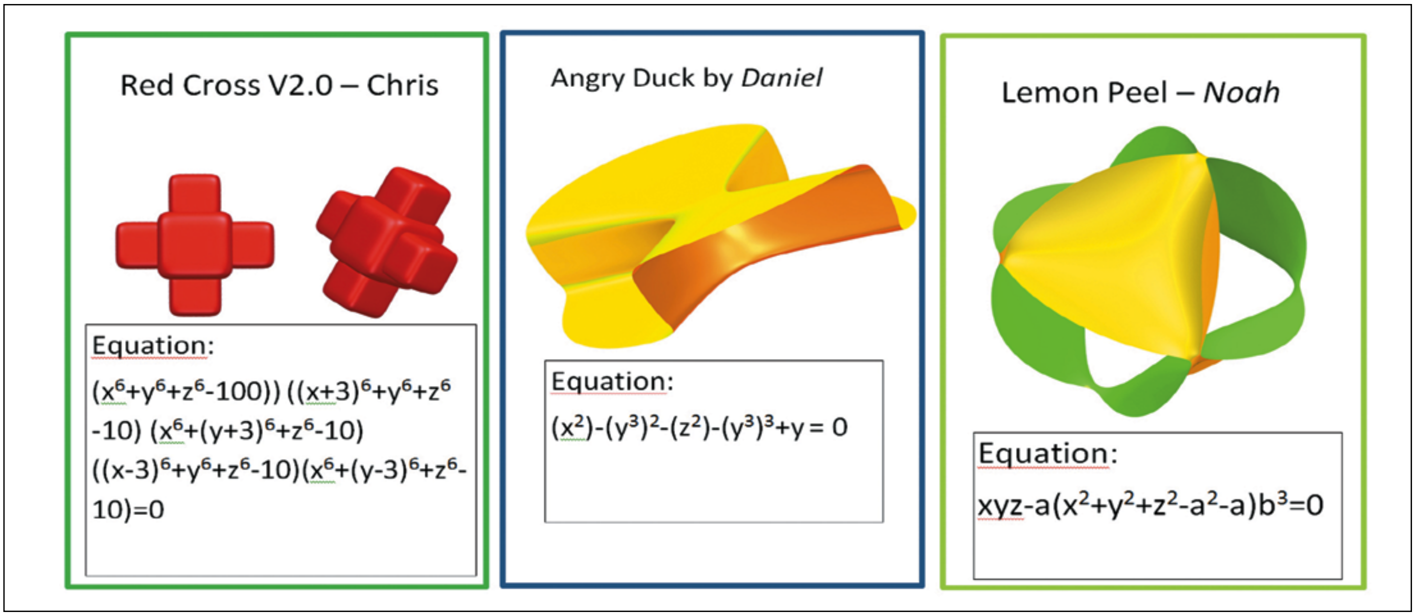

Fig. 13 Students presented their own creations.

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