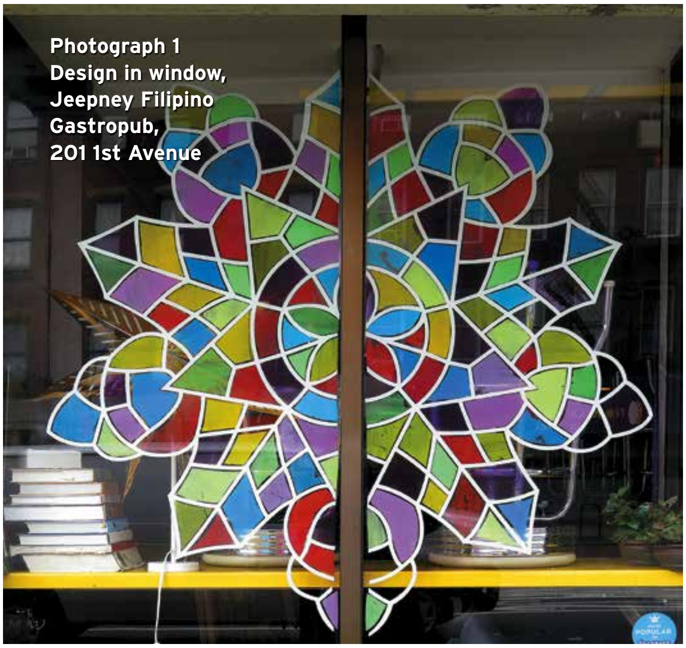
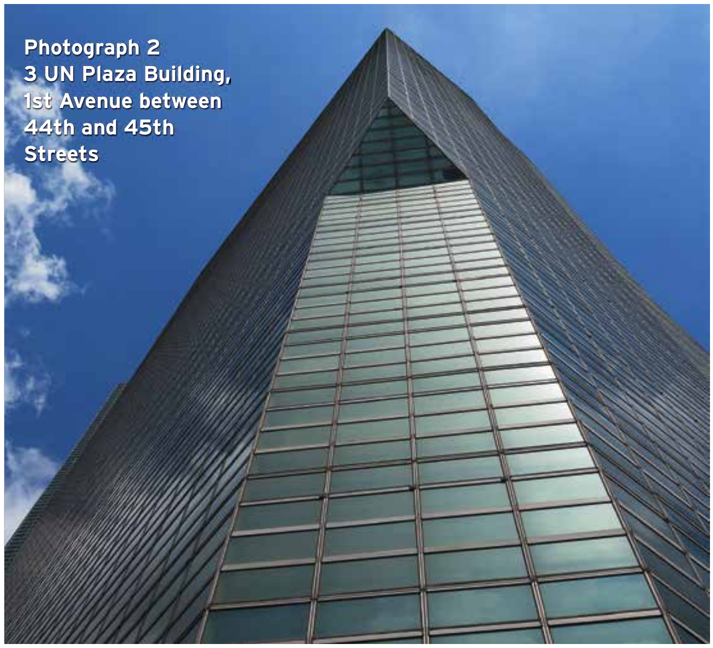
The 100-Block Walk for Math (Part 2)



Photograph 1
Design in window,
Jeepney Filipino
Gastropub,
201 1st Avenue



Photograph 2
3 UN Plaza Building,
1st Avenue between
44th and 45th
Streets

RON LANCASTER

Ron Lancaster and Brigitte Bentele continue their 100-block math walk up First Avenue in New York City.

1. (*a*) Jeepney Filipino Gastropub on 1st Avenue in New York City displays a stained-glass star in the window (see **photograph 1**); **figure 1** shows the inner pentagonal star of this design, centered at O. Suppose that triangle DEF is equilateral, and let G be its centroid. The point G along with the vertices of the triangle DEF and the midpoints of the sides form three congruent quadrilaterals. Arc DE and segment DE form a circular segment of circle O that reduces the areas of two of these quadrilaterals. What is the ratio of the area of the reduced region of one quadrilateral to the area of the (nonreduced) quadrilateral?

(*b*) The point H' on the inner circle is located in the following way: the point H at the intersection of the outer circle and the segment GO is translated toward the center along segment GO by vector GH. What is the ratio of the radius OH' of the inner circle to the radius OH of the outer circle?

(*c*) Each of the five petals inside the smaller circle is formed by two arcs. Each arc is formed in the following way: Ten points divide the inner circle into congruent arcs. Two adjacent points, such as E' and H' define radii (one drawn and one not drawn) whose perpendicular bisectors intersect at a point. An arc is then formed through three points: the center of the circle, the point of intersection of the perpendicular bisectors, and the endpoint, such as E', of a radius that is not drawn. This construction forms half a petal. Use this information to find the area ratio of the region inside the petals

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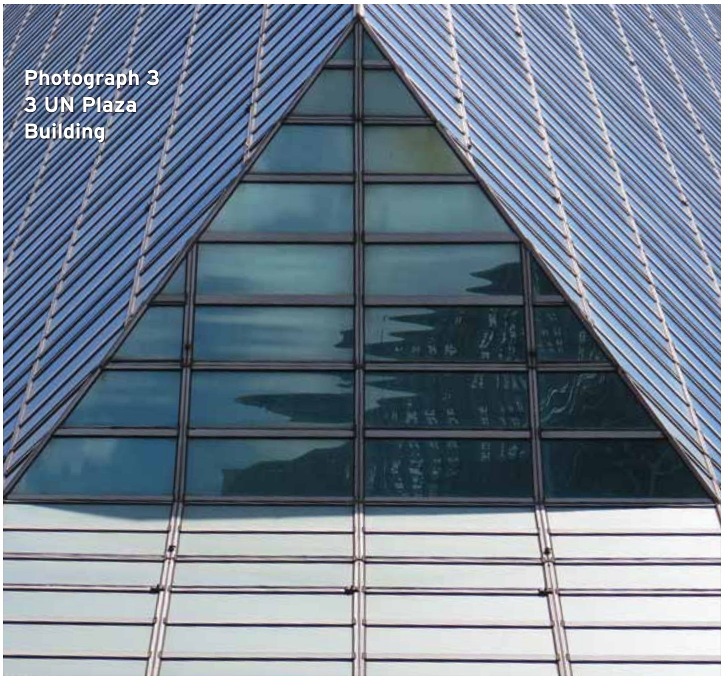

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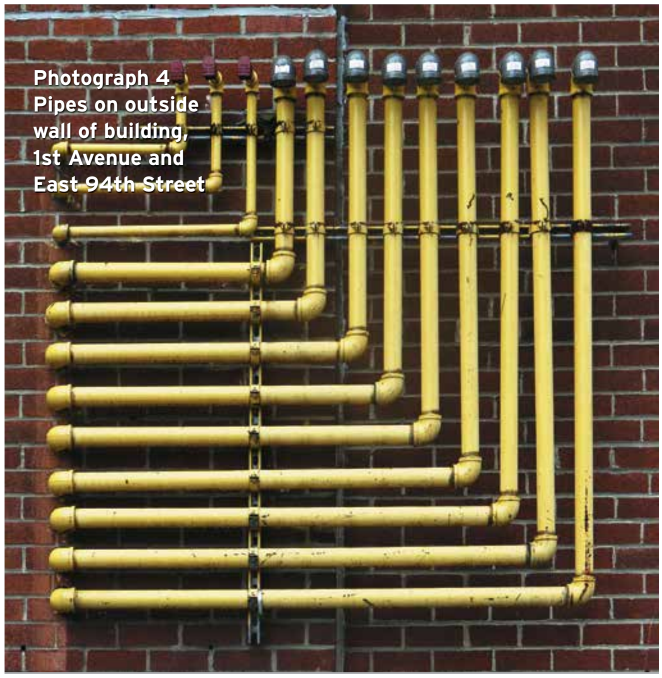

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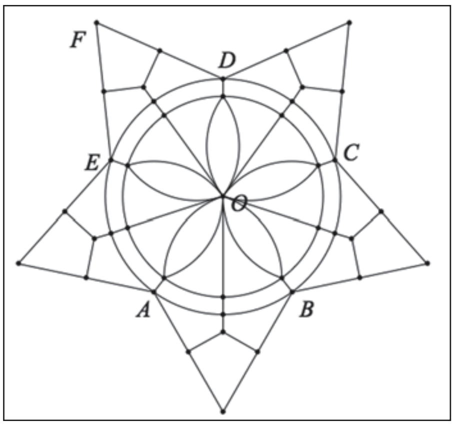

Fig. 1 The central pentagram includes five petals within an inner circle.

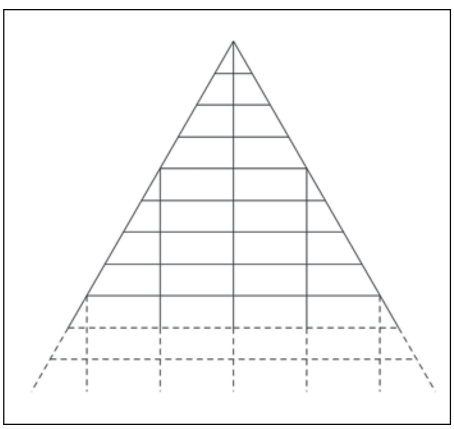

Fig. 2 Panes of different shapes comprise the rows in an extended pattern.

to the region outside the petals. You may first want to estimate.

2. The triangular overhang of 3 UN Plaza Building is seen from below in **photographs 2** and **3**. Panes of glass in the first eight rows form an interesting pattern of nonoverlapping regions—triangles, rectangles, and trapezoids (see **fig. 2**).

   (*a*) State the number of nonoverlapping regions in each of the first 8 rows in this pattern.

   (*b*) If the pattern continues, as shown in **figure 2**, to n rows, find a formula for the number of regions in the nth row.

   (*c*) In the first 8 rows, only rows 1 and 5 have triangles. How could you determine whether row n has triangles?

   (*d*) Let the ratio of the sides of the rectangular panes of glass be $1:r$, where $r > 1$. In terms of r, find the angles of the triangle formed by n rows.

3. Pipes on the outside of a building make an interesting design (see **photograph 4**). The shortest pipe makes an L with horizontal length twice its width (see **fig. 3**). Suppose that the spacing between the pipes is a constant, k, and that the total length of the shortest pipe is 3.

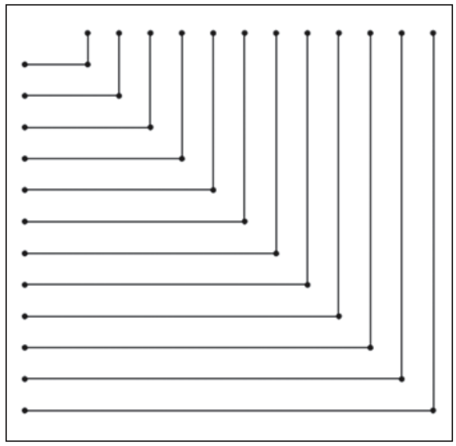

Fig. 3 The central axes of L-shaped pipes are represented by horizontal and vertical segments.

   (*a*) In terms of k and n, express the ratio of the horizontal length to the vertical width of the L-shaped pipe in row n.

   (*b*) What is the ratio of the length to the width of the pipe in row n as n approaches infinity?

1. (*a*) Let the side of the equilateral triangle be 1. Then its area is $\sqrt{3}/4$, and the area of each quadrilateral is $\sqrt{3}/12$. The radius OE can be found using trigonometry. In radians, the central angle of arc DE is $2\pi/5$, and length of arc DE is $(2\pi/5)R$, where R is the radius OE. Therefore, $\sin(\pi/5) = 0.5/R$, so $R = 0.5/\sin(\pi/5) \approx 0.85065$.

Using the formula $(1/2)sR$, where s is the arc length and R is the radius, the area of the sector is

$$\frac{1}{2}\left(\frac{2\pi}{5}R\right)R = \frac{\pi}{5}R^2$$
$$= \frac{\pi}{5}\left(\frac{0.5}{\sin\left(\pi/5\right)}\right)^2 \approx 0.4547.$$

Subtracting the area of triangle ODE, we find that the area of the segment is

$$\frac{\pi}{5}\left(\frac{0.5}{\sin\left(\pi/5\right)}\right)^2 - \frac{1}{2}\left(1\right)\cos\left(\pi/5\right)$$
$$\approx 0.1106.$$

The area of the quadrilateral that has been cut off by half the segment is $(\sqrt{3}/12) - (1/2)(0.1106) \approx 0.890$, and the ratio of the areas is 0.6169.

(*b*) $DO = HO = R$, where R is the radius of the outer circle. $OH' = r$ is the radius of the small circle:

$$FO = \frac{\sqrt{3}}{2} + R\cos\frac{\pi}{5}$$
$$FO = FG + GH + HO$$
$$\therefore \frac{\sqrt{3}}{2} + R\cos\frac{\pi}{5} = \frac{2}{3}\cdot\frac{\sqrt{3}}{2} + GH + R$$
$$GH = \frac{1}{3}\cdot\frac{\sqrt{3}}{2} + R\cos\frac{\pi}{5} - R$$
$$OH' = R - \left(\frac{1}{3}\cdot\frac{\sqrt{3}}{2} + R\cos\frac{\pi}{5} - R\right)$$
$$= 2R - \frac{\sqrt{3}}{6} - R\cos\frac{\pi}{5} = r \approx 0.7244$$

Therefore, the ratio of the radii of the two circles is $r/R \approx 0.8516$.

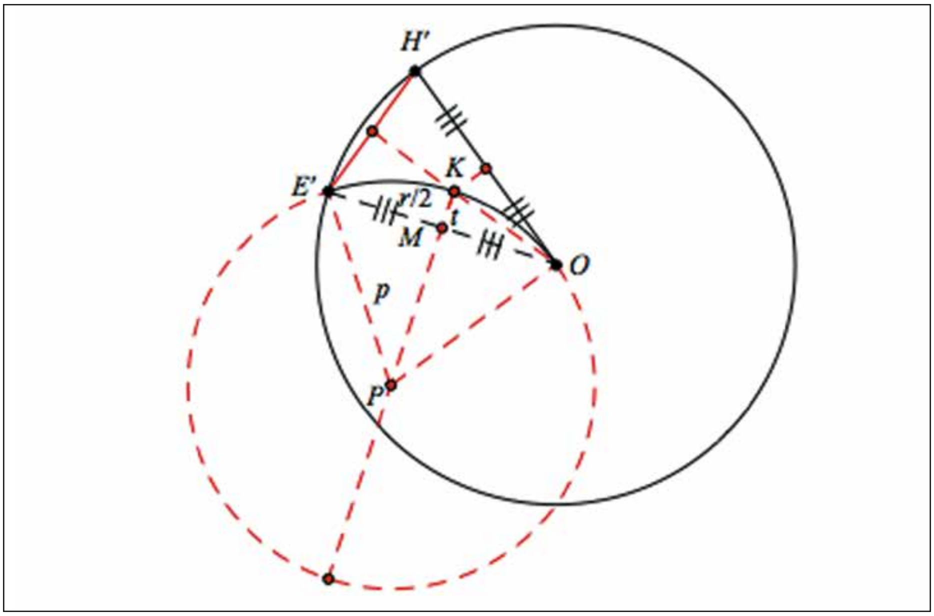

Fig. 4 Point K is located to draw arc OKE′ as one edge of the petal.

(*c*) Circle O with radius r is the small circle of **figure 1**, and K is the intersection of the perpendicular bisectors of radii OE' and OH'. **Figure 4** shows one of the arcs, OKE', formed by a circle with radius p and central angle OPE'.

We use trigonometry to find the distance, t, between K and OE': $\tan(\pi/10) = t/(r/2)$ leads to $t = (r/2) \cdot \tan(\pi/10) \approx 0.7244/2 \cdot \tan(\pi/10) \approx 0.1177$. Then, the two-chord power theorem (applied to the chord OE' and the diameter of circle P through endpoint K) can be used to find the radius, p: $t(2p - t) = (r/2)^2$ leads to

$$p = \frac{\frac{(r/2)^2}{t} + t}{2} \approx \frac{\frac{(0.7244/2)^2}{0.1177} + 0.1177}{2}$$
$$\approx 0.6162.$$

Now we find the central angle: $m(OPE') = 2\sin^{-1}((r/2)/p)$. The area of the circular segment OKE' is found by subtracting the area of the triangle from that of the circular sector:

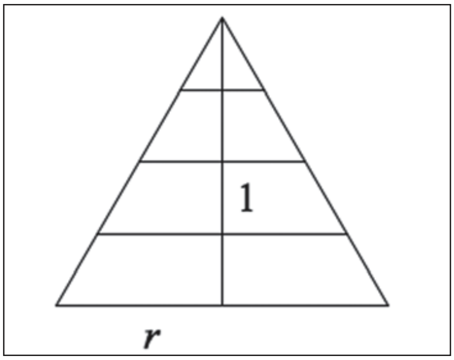

Fig. 5 Eight panes in four rows combine to create an isosceles triangle.

$$\text{area of sector } OPE' = \frac{m\angle OPE'}{2\pi}\pi p^2$$
$$= \sin^{-1}\left(\frac{r/2}{p}\right) \bullet p^2$$
$$\approx \sin^{-1}\left(\frac{(0.7244/2)}{0.6162}\right) 0.6162^2$$
$$\approx 0.2386$$

$$\text{area of triangle } OPE' = \frac{1}{2} r \sqrt{p^2 - (r/2)^2}$$
$$\approx \frac{1}{2} 0.7244 \sqrt{0.6162^2 - (.7244/2)^2}$$
$$\approx 0.1806$$

Thus, the area of segment OKE' is approximately $0.2386 - 0.1806 = 0.580$ units². Ten segments form the five petals with total area 0.580 units². The area of the circle is $\pi r^2 \approx \pi \cdot 0.7244^2 \approx 1.6486$, and the ratio of the areas inside to outside the petals is $0.580/(1.6486 - 0.580) \approx 0.543$. Students who estimated that the area outside the petals was about twice as much as the area inside the petals would find that the calculations support their estimate.

2\. (*a*) The number of panes of glass are 2, 2, 2, 2, 4, 4, 4, 4.

(*b*) In the nth row, there would be

$$2\left(1 + \left\lfloor \frac{n-1}{4} \right\rfloor\right)$$

regions, where $\lfloor x \rfloor$ is the greatest integer function.

(*c*) Row n has triangles, rather than trapezoids, on the outside whenever $n \equiv 1 \pmod 4$. That is, $n - 1$ is divisible by 4.

(*d*) **Figure 5** shows the first four rows of the triangular overhang creating a large isosceles triangle with height 4 and base $2r$. Thus, the two base angles are each $\tan^{-1}(4/r)$, and the vertex angle is $180° - 2\tan^{-1}(4/r)$. The number of rows does not affect the base angle of the isosceles triangles.

3\. (*a*) Since the length of the shortest pipe is 3 and the ratio of the length to width of the L is 2 : 1, the dimensions of the shortest L are 2×1. Each row adds k to each dimension. Therefore, the ratio of the length to the width of the L in the nth row is

$$\frac{2+(n-1)k}{1+(n-1)k}.$$

(*b*) For large n, we see that

$$\lim_{n\to\infty}\left(\frac{2+(n-1)k}{1+(n-1)k}\right) = \lim_{n\to\infty}\left(\frac{2/k+n-1}{1/k+n-1}\right) = 1.$$

In other words, repeatedly adding the same amount to the length and width of a rectangle creates new rectangles that approach the shape of a square.