Celebrating Diversity by Sharing Multiple Solution Methods

A classroom vignette presents teaching practices that develop an appreciation of diverse thinking.

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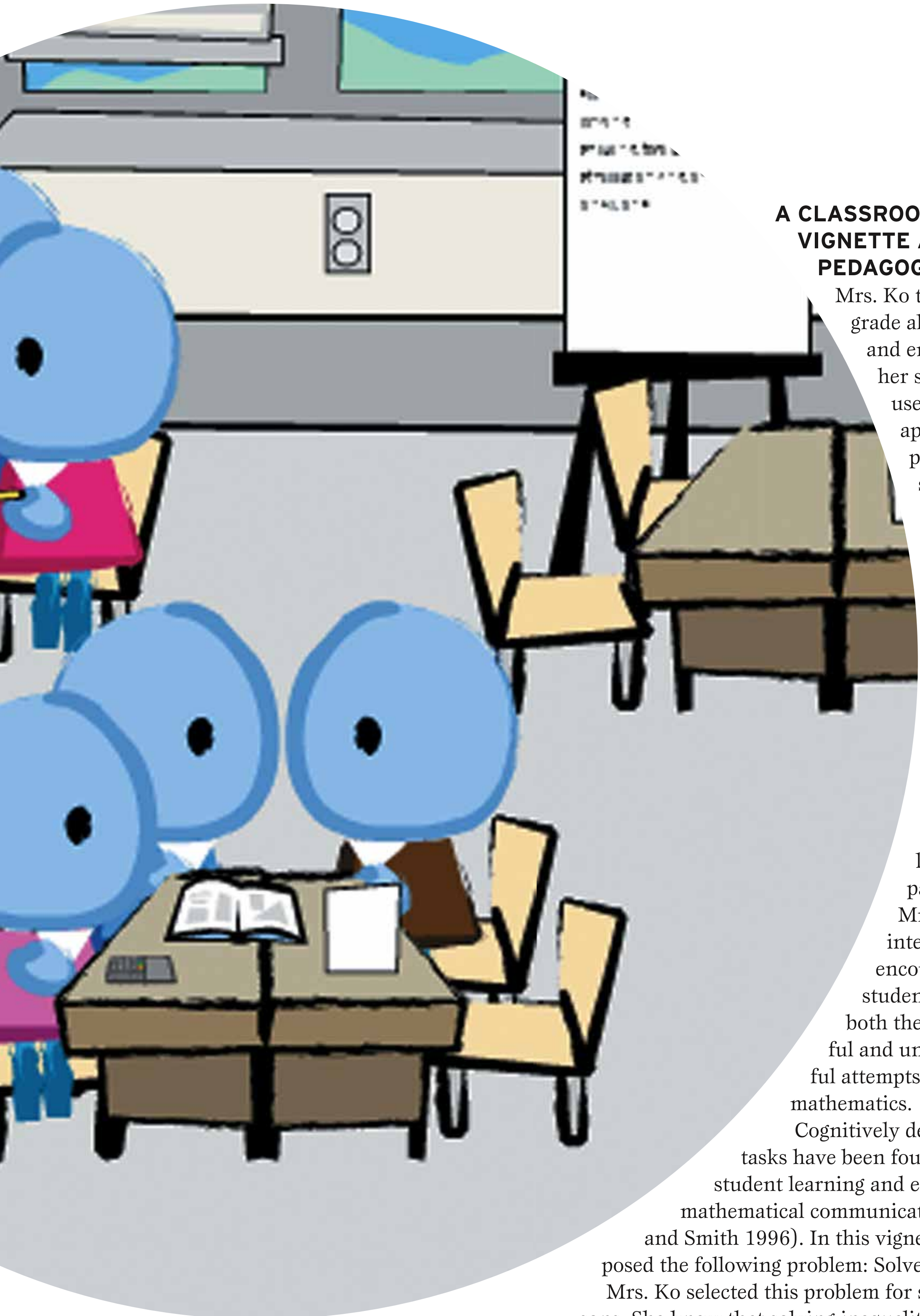
Are your students sometimes reluctant to express their mathematical thinking? Are you concerned that your students do mathematics in isolation with few opportunities to communicate?

Research has found that students, particularly those in middle school and high school, believe that their participation in mathematics classrooms includes listening to teachers explaining, working in textbooks, and solving exercise problems quickly (Franke, Kazemi, and Battey 2007). In this article, we describe a classroom vignette and illustrate how the teacher implements a teaching strategy of sharing multiple solution methods. This pedagogical approach fosters an environment in which students learn how to communicate and critique mathematical ideas and grow to be meaningful participants in their mathematics communities.

Key characteristics of the two-lesson sequence of the vignette include selecting an appropriate task and modeling productive student interactions. This



lesson sequence requires two forty-five-minute class periods, depending on the number of students and the variety of student solutions. Such a community-building lesson must be carefully planned and structured because students are learning the norms of listening to, critiquing, and building on the mathematical thinking of their peers (Walshaw and Anthony 2008).

A CLASSROOM VIGNETTE AND ITS PEDAGOGY

Mrs. Ko teaches ninth-grade algebra students and encourages her students to use multiple approaches to problems and share their mathematical thinking. **Table 1** illustrates a series of seven teaching procedures and describes specific teaching moves to facilitate learning. In particular, Mrs. Ko was interested in encouraging her students to share both their successful and unsuccessful attempts to learn mathematics.

Cognitively demanding tasks have been found to enrich student learning and encourage mathematical communication (Silver and Smith 1996). In this vignette, Mrs. Ko posed the following problem: Solve $|2x - 12| > 8$.

Mrs. Ko selected this problem for several reasons. She knew that solving inequalities with absolute values was typically confusing for algebra students (Almog and Ilany 2012). Although the problem could be solved procedurally with little mathematical reasoning, Mrs. Ko had noticed that her students attempted similar problems in a variety of ways on their homework and quizzes. Perhaps students used strategies learned from tutors, the Internet, or other textbooks. For example, she found that some students solved the problem

Table 1 Mrs. Ko's Teaching Notes on the Practice of Classroom Community Building

Key Procedure	Pedagogical Moves
Day 1 (45 min.)	
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1. The teacher poses a rich mathematical problem (1–3 min.).	• Select a problem that encourages multiple solution methods and the use of various representations. • Select a problem for which students already know the prerequisite concepts so that they can apply and make connections between concepts. • Choose a problem tied to the unit's central objectives.
2. Students solve the problem (15 min.).	• Allow students enough time to develop solutions. • Encourage students to work together. (Some students may respond positively to having the option to work solo.) • Ask students to document their work, including missteps and all traces of their strategies and thinking. • As students work, walk around to survey the patterns of solutions and field questions. Use your observations to group students for the presentations.
3. The teacher provides the correct answer (1–3 min.).	• Provide the answer so that students will not hesitate to present their own approaches to the problem. • Avoid asking the class who is right and who is wrong. • Focus on using the opportunity to validate student approaches and build a classroom culture in which students value process as much as product.
4.The teacher groups students by type of solution method and has them prepare presentations of their solution methods, whether correct or incorrect (at least 25 min.). (The teacher collects individual work and keeps it for the next day.)	• Group students for presentations on the basis of types of representation used, theorems used, or other key elements of the solution process. • Guide students in selecting the representations and organizing the visual portion of their presentations so that they communicate clearly. • Communicate the value of this activity for all students by modeling a conscientious effort to understand each student's solution process, whether correct or incorrect. • Allow students enough time to practice the oral part of their presentations. • Continue to make notes on what topics need clarification, emphasis, or further discussion.
After class, the teacher prepares for the next day.	• Review student work and your notes to determine key topics for emphasis in the closing discussion. • Construct an organizer or notes to help shape the classroom discourse in the closing discussion.
Day 2 (45 min.)	
Key Procedure	**Pedagogical Moves**
5. Students review and edit their work (5 min.).	• Display students' solutions and mathematical struggles on classroom walls.
6. Students tour the gallery of presentations (20–25 min.): (a) Half of each presenter group takes the tour. The other half stands by their work and interacts with visitors. (b) The presenter group and the visitor group switch roles.	• Require students to make note of which students they visited and what they learned from each presentation. This requirement keeps students engaged in mathematical discussions. • Model the appropriate behaviors of peer interactions, including useful questioning prompts; provide a list of appropriate questions and responses, if needed.
7. The teacher facilitates the closing discussion (15–20 min.).	• Focus the closing discussion on sharing insights and connections or generating new knowledge. • Ask for volunteers or call on students to share their conversations with peers; ask students to use notes when speaking. • Ask students to credit other groups or students whose ideas influenced their learning to continue to build community. • Build on unique student approaches that help students make connections. Save copies of the approaches that might be relevant as later mathematical topics are learned. • Make an effort to balance sharing mathematical ideas and mathematical struggles, new learning and meta-cognition. • Model and encourage students to appreciate the diversity of approaches and recognize the value of peer discussion of mathematical ideas. • Emphasize efficient and useful approaches to problem solving. Make connections among solution methods.

graphically. She thought that the variety of solutions made this problem a rich one for exploration.

The students' prior knowledge included arithmetic skills with signed numbers, solving linear equations, and validating the solutions by testing the numbers in the solution set to see whether the equation holds true. A few students knew how to graph linear functions by using graphing utilities such as GeoGebra (go to http://www.geogebra.org), an interactive geometry software freely available for teachers and students. Students in Mrs. Ko's classroom had already been introduced to the mathematical concept of absolute value as distance and its algebraic implications. For example, students were expected to understand that $|x - 5|$ can be represented in two ways: as $x - 5$ if $x \ge 5$ or as $-x + 5$ if $x < 5$.

At the beginning of the lesson, Mrs. Ko provided students with the problem. As they worked on it, she walked around to observe the solution methods that they attempted. She found that about half the class was solving the problem algebraically by setting up the two cases (see **fig. 1a**). Some students attempted to use GeoGebra as a problem-solving tool by graphing two functions, $y = |2x - 12|$ and $y = 8$. Observing their work (see **fig. 1b**), Mrs. Ko noticed a pattern of student misunderstanding about the graphs of the inequalities. Students were mistaking the line segment of the graph for the solution set instead of identifying intervals within the domain of the function.

$|2x - 12| > 8$

$2|x - 6| > 8$

$|x - 6| > 4$

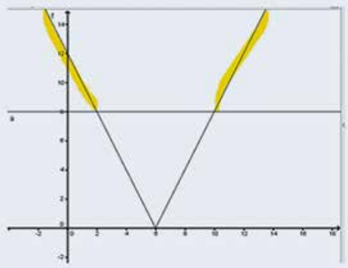
If $|M| > 4$, then $M > 4$ or $M < -4$

It follows that $x - 6 > 4$, $x - 6 < -4$

$x > 10$ or $x < 2$

(a)

We drew two graphs. I showed the part of $|2x - 12|$ that is greater than 8. This is the best way to solve a difficult inequality problem.



(b)

Fig. 1 About half the class solved the problem by setting up two cases (**a**). The graphical solution using GeoGebra demonstrated a misconception about the solution set (**b**).

As she walked around the room, Mrs. Ko observed various solution methods and responded to students' questions. She was cautious about her language while interacting with students. For example, she blended colloquial language (e.g., *plug in*, *check*, *these numbers*, *match*) and formal language (e.g., *substitute*, *validate*, *intervals*, *correspond*) to scaffold students' ability to use mathematical language in communicating their ideas. Mrs. Ko was also careful in crafting her response to student questions. Typically, she *revoiced* (O'Connor and Michaels 1993) students' comments or questions in more refined language and then subtly offered new ideas or suggestions to facilitate student thinking.

When most of the students were ready, Mrs. Ko announced that the correct answer was $x < 2$ or $x > 10$. This announcement was a deliberate move to shift students' focus from the correct answers to analyzing mistakes or explaining the solution process. Mrs. Ko wanted the class to participate in sharing the various solution methods, including failed attempts or struggles. She gave students various options to choose from for presenting their solutions: providing symbolic texts in a logical sequence; describing their solution methods in words (see **fig. 2a**); using graphs and diagrams with brief notes; or demonstrating their solutions with objects. Students who did not have the correct answer were also asked to show their work and explain their solution process in detail (see **fig. 2b**).

Mrs. Ko knew that one challenge of this lesson would be persuading students to value the process of explaining mathematical misunderstandings and struggles. It is important for everyone to understand that even students who could not solve the problem contributed to learning, that all solutions help demonstrate various paths of mathematical thinking and reasoning and build connections in students' mathematical knowledge base. Mrs. Ko explained one primary benefit of sharing common

Students submitted a list of names of peers they had visited and noted how they had helped or received help from these classmates.

Absolute value inequality always has two solutions. It is more complicated if you see ">" in the problem, because the pattern is to solve as it is and solve another by switching both the sign and the direction. You will have two inequalities. One is $2(x - 6) > 8$. The other is $2(x - 6) < -8$. It is easy from now on. You remove 2 by dividing and move –6 to the other side. Finally, you have $x > 10$ and $x < 2$. You draw this on the number line. You have a line extending both sides except 2 and 10 and the line between 2 and 10. That is the solution.

(a)

We are always confused when the problem is like $|M|$ > a number. But we can solve $|M|$ < a number easily. We show you here. $|x| < 5$ can be solved by setting up like this: $-5 < x < 5$. We are done. But we don't know what to do with $|2x - 12| > 8$. Can we do $-8 > |2x - 12| > 8$?

(b)

Fig. 2 Some students wrote a narrative solution (**a**). Some students wrote about their struggles (**b**).

mistakes and struggle: Collaborating to find ways to resolve peers' difficulties in understanding can improve learning. For example, when students articulate their misunderstandings, teacher and students can work together to provide support. One common unsuccessful solution (see **fig. 2b**) showed that some students had attempted to solve the inequalities in the same way that they would solve a simple linear equation. As a result, Mrs. Ko devised strategies for helping students learn how to solve both equations and inequalities.

For their presentations, Mrs. Ko grouped the students by their methods of solution. She regrouped the students during the first lesson but was willing to assign a few new groups for the subsequent lesson, if necessary. Students were grouped according to those who applied the algorithm taught by the teacher; those who solved graphically; those who calculated critical numbers and tested intervals (see **fig. 3a**); and those who came up with uncommon solutions. One student used the concept

$|2x - 12| > 8$
$2x - 12 = +8$ or $2x - 12 = -8$
$x = 10$, $x = 2$
Important numbers: 2, 10
Test numbers: 1, 3, 12

pass fail pass
←--------------2----------------10--------------→

For 1, $|2(1) - 12| = |-10| > 8$ Yes!
For 3, $|2(3) - 12| = |-6| > 8$ No!
For 12, $|2(12) - 12| = |12| > 8$ Yes!
Solution: $x < 2$, $x > 10$

(a)

$|2x - 12| > 8$ has many ways to look at. I look at $|2x - 12|$. It is the distance between $2x$ and 12. " > 8" means "larger than 8." So all together it means the distance between $2x$ and 12 is larger than 8. I need to think of a number (actually two numbers) that is distant from 12 by 8. It is 20. Also 4. But I have $2x$ as the unknown. So I take half of it. So x is 10 or 2. If you think hard about it, the problem is not an equation. So the possible numbers are 11, 12, 13, 14 . . . and 1, 0, –1, . . . So I decide the solution is $x > 10$ and $x < 2$.

(b)

I learned this trick from my sister who is a university math student. When the question pattern is: $|M| > a$. First solve the equality assuming it is an equation: $2x - 12 = 8$; $2x = 20$; $x = 10$. Now solve the equation in the $| \, |$: $2x - 12 = 0$; $x = 6$. 6 is always in the middle of 10 and the other number like this:

??------------6--------------10

Easily, the other number must be 2. We call these two numbers, 2 and 10, the boundary numbers. So the answer must be $x > 10$, $x < 2$.

(c)

Fig. 3 Some students tested intervals (**a**). Some students used the concept of distance on the number line (**b**). Some students used combined strategies (**c**).

of distance on the number line (see **fig. 3b**); other students used a strategy combining the interval test method and graphic analysis (see **fig. 3c**).

Among the students who attempted similar methods, Mrs. Ko encouraged those who had produced incorrect work to collaborate with those who had produced correct solutions, in anticipation that students would begin to understand one another's solution processes. Mrs. Ko commented, "I was hoping students could discuss the similar approaches so that they could learn from someone who thought about the problem in the same way. It was interesting to see [that] some math terms were very contagious and how quickly some students fixed their mistakes and mingled." To increase participation, Mrs. Ko also asked students to submit a list of names of peers they had visited and to note how they had helped or received help from these classmates. To resolve conflicts or challenging questions, a poster (or "parking lot") was placed on the wall so that students could post questions raised during the interactions.

As she toured the student work gallery, Mrs. Ko modeled ways in which students could initiate discussion of mathematical ideas (see **fig. 4**). For example, she encouraged the students who used algebraic procedures to talk with those who attempted the problem graphically and to make connections between the symbolic and graphic representations. Also, she provided a list of appropriate questions and responses. Her questioning prompts included these:

- "What did you do after this step? And why?"
- "Where does this number come from?"
- "I am not sure about this; can you help me with this?"
- "Which step are you having a hard time with?"
- "I think you can do this to move on to the next step."
- "I think you can use my work to help you figure out the next step."
- "How did you draw this graph?"
- "Let's ask him [or her]."
- "I can help you with this step. Can you help me with that step?"

Mrs. Ko had walked through the gallery, made note of common errors and misunderstandings, and arranged for a few volunteers to share their work with the class later in the closing discussion. As students visited different groups, Mrs. Ko told them that they must be aware that students outside the initial group might not be familiar with the group's common reasoning and strategies and encouraged students to think about how to explain better to those who were not familiar with the strategies.

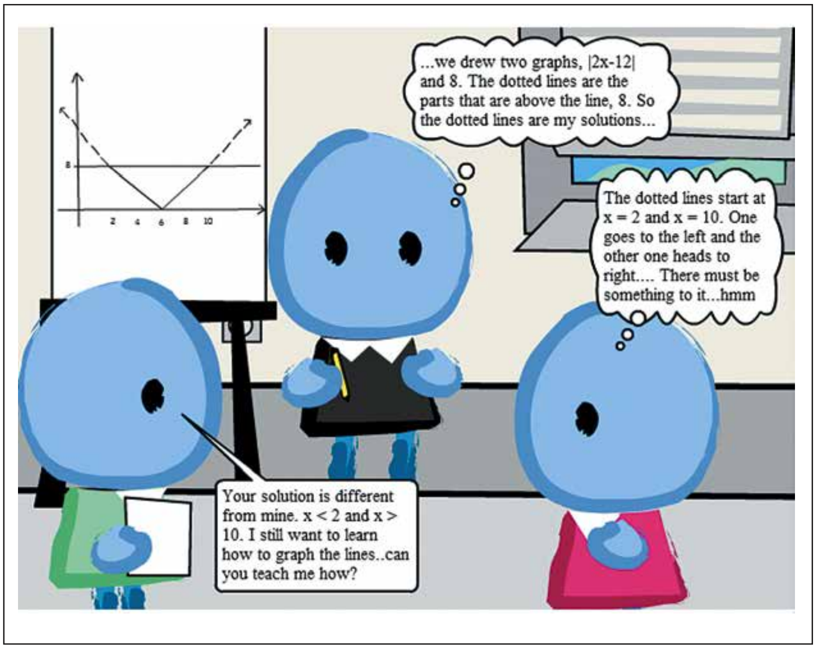


Fig. 4 Students share their strategies.

To start the closing discussion, Mrs. Ko used the questions that students had written on the poster:

- "How do we decide which numbers to write down as solutions?"
- "What are the advantages or disadvantages of this method over that method?"
- "How do we calculate the [critical] numbers?"
- "Do we have to know the graph method?"
- "Hung-soo's method seems very simple and correct. Can we use it?"

Mrs. Ko focused on giving students opportunities to speak and used the questions to focus the classroom conversation. For example, because some students (see **fig. 3b**) were testing the solution set by substituting integers, she used the graphic solutions to demonstrate that the solution set includes the intervals of real numbers rather than only the integers. Also, she showed interest in Mi-ran's work (see **fig. 3c**) by asking the class to help Mi-ran make sense of the strategy to find the middle number between 2 and 10 by using graphs developed by other students. Mrs. Ko

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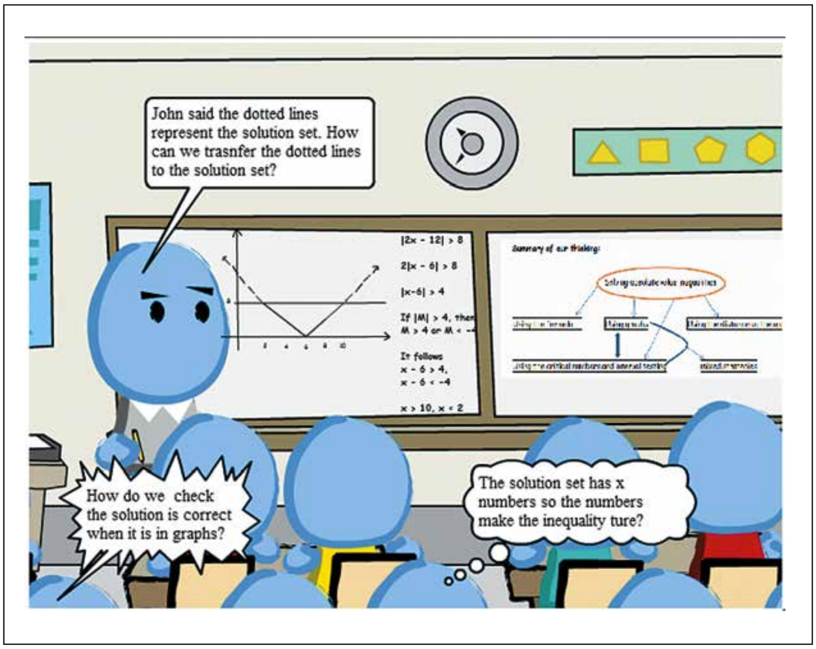

Fig. 5 The teacher closes the lesson with a summary.

used Mi-ran's strategy to illustrate a way to find the first critical number, 10, and find the other number, 2, by using the reflective attribute of the absolute value graph with $x = 6$ as the line of reflection. Toward the end of the closing discussion, Mrs. Ko provided a summary of the discussion based on the types of student solutions (see **fig. 5**).

FINAL WORDS

Research supports pedagogical strategies that develop the social and mathematical classroom norms—the sociomathematical norms (Yackel and Cobb 1996)—for building a classroom community in which students communicate and reason about mathematics. Mrs. Ko's lesson sequence and pedagogical strategies provided a structure that established classroom norms related to communicating mathematically and valuing various approaches and thus developed a classroom community.

Mrs. Ko's lesson aimed to improve student communication and engage students in productive struggle (NCTM 2014), and her teaching practice aligns well with the standards-based classroom practice (NCTM 2000). Her careful development of classroom norms builds a community in which students recognize divergent thinking, develop a sense of belonging, and skillfully communicate mathematical ideas. As one student commented, "It can be confusing because there are so many ways to learn. But Mrs. Ko says I [should] use my method and try to learn one or two additional methods. That is doable, and I actually want to try as many different ways as possible." Another student added, "Speaking math is fun, but it is really hard and can be embarrassing when you don't know what you're talking about. That's why I listen to my classmates and think about how I would talk about my idea carefully. I really want to talk like Mrs. Ko and be able to draw pictures and do the work on the chalk board."

When a rich classroom discourse becomes a regular part of classroom instruction (NCTM 2014), teachers are better positioned to create the classroom community in which students learn to share their diverse thinking, remain open about the struggles in doing mathematics, and gradually develop a sense of belonging in the classroom community.

Development of classroom norms builds a community in which students recognize divergent thinking, develop a sense of belonging, and skillfully communicate mathematical ideas.

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REFERENCES

Almog, Nava, and Bat-Sheva Ilany. 2012. "Absolute Value Inequalities: High School Students' Solutions and Misconceptions." *Educational Studies in Mathematics* 81 (3): 347–64.

Franke, Megan L., Elham Kazemi, and Dan Battey. 2007. "Mathematics Teaching and Classroom Practice." In *Second Handbook of Research on Mathematics Teaching and Learning,* edited by Frank K. Lester, pp. 225–56. Charlotte, NC: Information Age.

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics*. Reston, VA: NCTM.

———. 2014. *Principles to Actions: Ensuring Mathematical Success for All*. Reston, VA: NCTM, 2014.

O'Connor, Mary Catherine, and Sarah Michaels. 1993. "Aligning Academic Task and Participation Status through Revoicing: Analysis of a Classroom Discourse Strategy." *Anthropology and Education Quarterly* 24: 318–35.

Silver, Edward A., and Margaret Schwan Smith.

1996. “Building Discourse Communities in Mathematics Classrooms: A Worthwhile but Challenging Journey.” In *Communication in Mathematics, K–12 and Beyond*, edited by Portia C. Elliott, pp. 20–28. Reston, VA: National Council of Teachers of Mathematics.

Walshaw, Margaret, and Glenda Anthony. 2008. “The Teacher’s Role in Classroom Discourse: A Review of Recent Research into Mathematics Classrooms.” *Review of Educational Research* 78 (3): 516–51.

Yackel, Erna, and Paul Cobb. 1996. “Sociomathematical Norms, Argumentation, and Autonomy in Mathematics.” *Journal for Research in Mathematics Education* 27: 458–77.

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