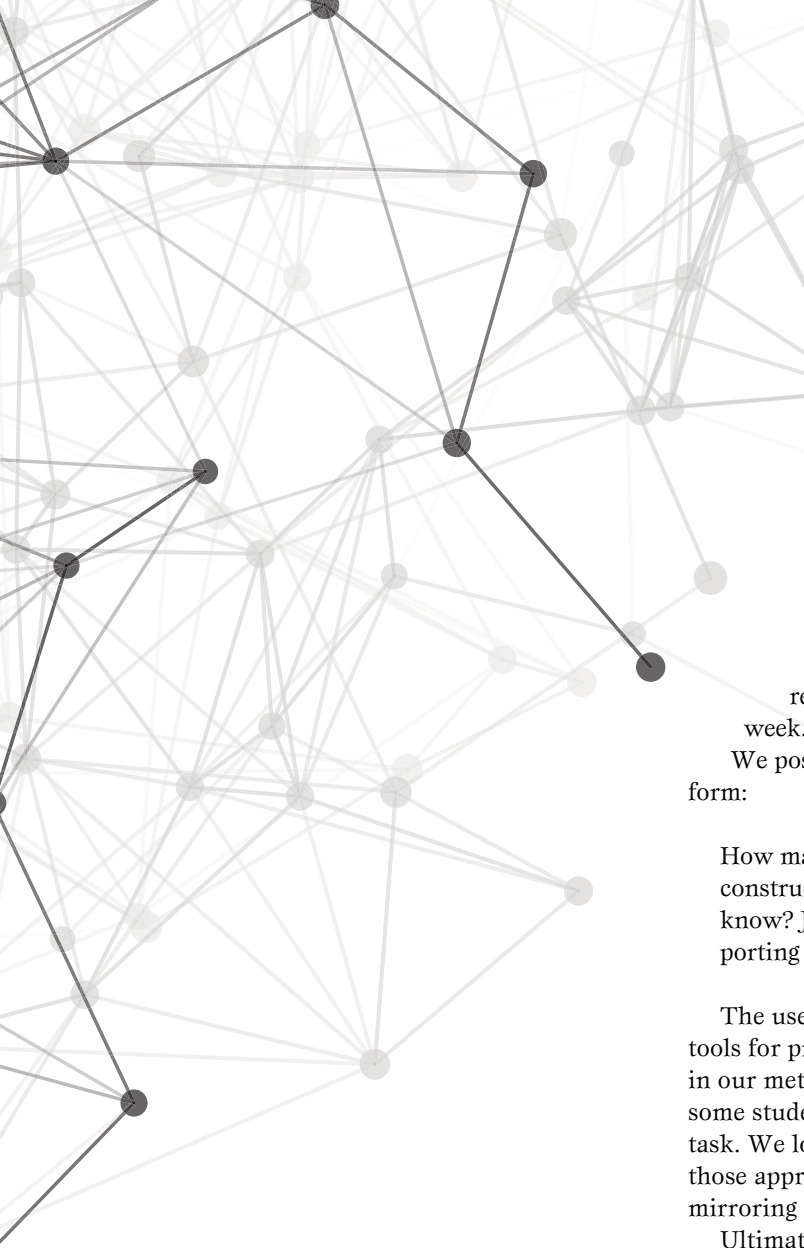


What's Your Answer? Searching for Triangles

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The Geoboard Triangle Quest yields many results. The challenge for students is to verify which—if any—are correct.

In her *Mathematics Teacher* article “The Geoboard Triangle Quest” (2013), Kasi Allen describes students’ excitement as they explored how many triangles can be constructed on a 4×4 geoboard. Because the task engages students in many Common Core Standards for Mathematical Practice (CCSSI 2010)—for example, “Look for and make use of structure” (SMP 3) and “Model with mathematics” (SMP 4)—and is particularly well suited for multiple solution strategies, we posed the task to our preservice teachers (hereafter referred to as students) as a problem of the week.

We posed the Geoboard Triangle Quest in this form:

How many noncongruent triangles can be constructed on a 4×4 geoboard? How do you know? Justify your answer with significant supporting work.

The use of advanced digital technologies as tools for problem solving receives much attention in our methods classes, and we anticipated that some students would use technology to solve the task. We looked forward to the interplay between those approaches and hands-on approaches mirroring those adopted by Allen’s students.

Ultimately, our students attempted the task in many ways, using both digital and analog techniques. Interestingly, different solution strategies yielded different numbers of triangles. Although Allen’s students tentatively concluded that 72 triangles were possible (2013, p. 118), they did not provide a proof, and none of our students arrived at this answer. In this article, we discuss how we came to what we consider a definitive solution and, on the way, address a challenge that arises in many inquiry-based classrooms: When confronted with multiple, conflicting results, how does one determine which—if any—answers are correct?

INITIAL SOLUTION METHODS

Hands-on with Geoboards

A number of our students used physical geoboards to explore the task, taking approaches that paralleled those discussed by Allen (2013). Students found it easy to construct arbitrary triangles with rubber bands. Soon, however, it became clear that a more systematic approach would be needed.

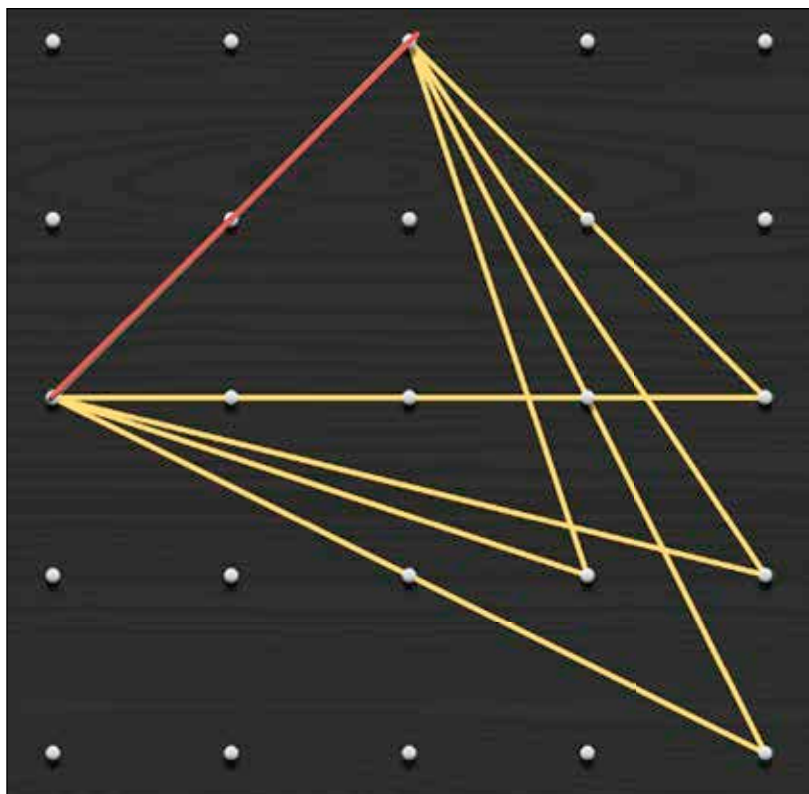


Fig. 1 Geoboard triangles are constructed with $\sqrt{8}$ as the shortest length.

Recognizing that fourteen distinct segment lengths can be constructed—for example,

$$1, \sqrt{2}, 2, \sqrt{5}, \sqrt{8}, 3, \sqrt{10}, \sqrt{13}, 4, \sqrt{17}, \sqrt{18}, \sqrt{20}, 5, \text{ and } \sqrt{32}$$

—several groups began by forming all triangles containing a shortest side of length 1. Next, they constructed all triangles containing a shortest side of length $\sqrt{2}$. They continued, constructing triangles with shortest side lengths 2, then $\sqrt{5}$, then $\sqrt{8}$, and so forth. Constructing an initial segment with desired length, students stretched the band to a third peg to form a triangle with no sides shorter than the first. We illustrate this approach in **figure 1** for initial length $\sqrt{8}$.

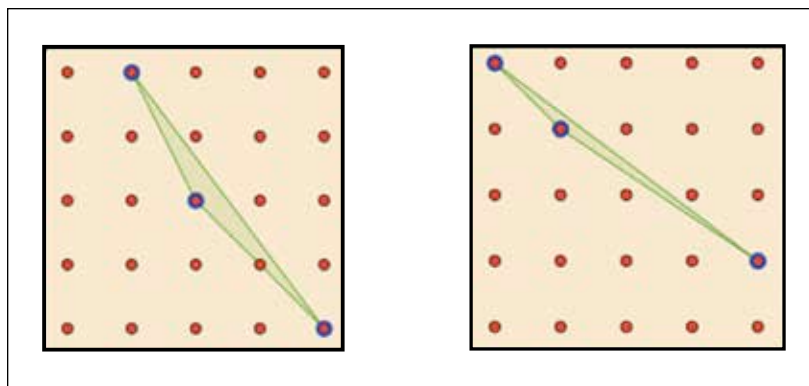


Fig. 2 “Difficult” triangles are uncovered with a virtual geoboard sketch.

Table 1 summarizes the initial work of one group who used this method. Note that the last four rows of the table do not indicate that triangles with side lengths $\sqrt{18}$, $\sqrt{20}$, 5, and $\sqrt{32}$ are not constructible. Rather, those lengths are not possible as *shortest* sides. In all, students uncovered 71 triangles with geoboards.

Virtual Geoboard Applet

A number of students preferred to use GeoGebra to create a virtual geoboard. First, they constructed a digital version of a hands-on board, ultimately modifying the applet to extend its functionality with measurement and data collection features. An online version of the students’ final applet is available at <http://bit.ly/triangle-sketch>.

Interestingly, even the earliest versions of the applet provided students with advantages over a conventional board. For instance, the fixed size of rubber bands made the construction of triangles with small interior angles difficult. Two such triangles, discovered with the applet but initially missed by physical geoboard users, are illustrated in **figure 2**.

Initially, students used the virtual applet as little more than a conventional geoboard—a digital analog that allowed them to construct triangles without the constraints of physical rubber bands. Soon, however, they revised the sketch to make better use of the many capabilities of GeoGebra. An early revision added numeric labels to each side of the triangle. This feature helped students compare side lengths of newly constructed triangles with those already recorded. In a later revision, the students used spreadsheet and scripting features to automate the process of recording triangles. Specifically, a **Record to Spreadsheet** button was added that copied side lengths and perimeter to a spreadsheet. A second button, **Clear Spreadsheet**, was added. These buttons are illustrated in **figure 3**. To help avoid duplicate triangles, the revised applet sorts collected lengths in ascending order before recording the measurements into columns A, B, and C.

A final revision added dynamic text indicating the number of distinct triangles currently recorded in the spreadsheet. Controlled by a short script comparing the perimeter of the most recent triangle with all other values in column D, the script adds one to the **Unique Triangle** count when the perimeter is a new one. This functionality is highlighted in **figure 4**.

Using the final revised sketch, the students systematically constructed and recorded triangles. As **figure 5** suggests, 215 triangles were recorded with the applet, 77 of which had distinct perimeters and thus were considered unique.

Table 1 Organization by Shortest Side Length

Shortest Side	Triangles	<i>n</i>
1	$(1, 1, \sqrt{2}), (1, \sqrt{2}, \sqrt{5}), (1, \sqrt{5}, \sqrt{10}), (1, \sqrt{10}, \sqrt{17}), (1, 2, \sqrt{5}), (1, \sqrt{5}, \sqrt{58}), (1, \sqrt{8}, \sqrt{13}),$ $(1, 3, \sqrt{10}), (1, \sqrt{13}, \sqrt{20}), (1, \sqrt{10}, \sqrt{13}), (1, \sqrt{13}, \sqrt{18}), (1, \sqrt{18}, 5), (1, 4, \sqrt{17}), (1, \sqrt{17}, \sqrt{20}),$ $(1, \sqrt{20}, 5), (1, 5, \sqrt{32}),$	16
$\sqrt{2}$	$(\sqrt{2}, 2, \sqrt{10}), (\sqrt{2}, 3, \sqrt{17}), (\sqrt{2}, \sqrt{2}, 2), (\sqrt{2}, \sqrt{5}, 3), (\sqrt{2}, \sqrt{10}, 4), (\sqrt{2}, \sqrt{5}, \sqrt{5}), (\sqrt{2}, \sqrt{8}, \sqrt{10}),$ $(\sqrt{2}, \sqrt{13}, \sqrt{17}), (\sqrt{2}, \sqrt{13}, \sqrt{13}), (\sqrt{2}, \sqrt{18}, \sqrt{20}), (\sqrt{2}, 5, 5)$	11
2	$(2, \sqrt{5}, \sqrt{17}), (2, 2, \sqrt{8}), (2, \sqrt{5}, \sqrt{5}), (2, \sqrt{5}, \sqrt{13}), (2, \sqrt{8}, \sqrt{20}), (2, 3, \sqrt{13}), (2, \sqrt{10}, \sqrt{10}),$ $(2, \sqrt{10}, \sqrt{18}), (2, \sqrt{13}, 5), (2, 4, \sqrt{20}), (2, \sqrt{17}, \sqrt{17}), (2, \sqrt{17}, 5), (2, \sqrt{20}, \sqrt{32})$	13
$\sqrt{5}$	$(\sqrt{5}, \sqrt{5}, 4), (\sqrt{5}, \sqrt{5}, \sqrt{10}), (\sqrt{5}, \sqrt{8}, \sqrt{17}), (\sqrt{5}, \sqrt{8}, 3), (\sqrt{5}, \sqrt{10}, \sqrt{13}), (\sqrt{5}, \sqrt{13}, \sqrt{20}),$ $(\sqrt{5}, 3, \sqrt{20}), (\sqrt{5}, \sqrt{10}, \sqrt{17}), (\sqrt{5}, \sqrt{13}, 4), (\sqrt{5}, \sqrt{17}, \sqrt{18}), (\sqrt{5}, \sqrt{20}, 5)$	11
$\sqrt{8}$	$(\sqrt{8}, \sqrt{8}, 4), (\sqrt{8}, \sqrt{10}, \sqrt{10}), (\sqrt{8}, \sqrt{13}, \sqrt{17}), (\sqrt{8}, \sqrt{20}, \sqrt{20})$	4
3	$(3, 3, \sqrt{18}), (3, \sqrt{10}, \sqrt{13}), (3, \sqrt{10}, 5), (3, 4, 5), (3, \sqrt{17}, \sqrt{20}), (3, \sqrt{17}, \sqrt{32})$	6
$\sqrt{10}$	$(\sqrt{10}, \sqrt{10}, \sqrt{20}), (\sqrt{10}, \sqrt{13}, \sqrt{17}), (\sqrt{10}, 4, \sqrt{18}), (\sqrt{10}, \sqrt{17}, 5)$	4
$\sqrt{13}$	$(\sqrt{13}, \sqrt{13}, 4), (\sqrt{13}, \sqrt{17}, \sqrt{20})$	2
4	$(4, 4, \sqrt{32}), (4, \sqrt{17}, 5), (4, \sqrt{20}, \sqrt{20})$	3
$\sqrt{17}$	$(\sqrt{17}, \sqrt{17}, \sqrt{18})$	1
$\sqrt{18}$	None	0
$\sqrt{20}$	None	0
$\sqrt{25}$	None	0
$\sqrt{32}$	None	0
Total		71

OVERCOMING THE LIMITATIONS OF VIRTUAL AND PHYSICAL BOARDS

Despite the options afforded by the virtual geoboard applet, a significant limitation is inherent to the use of boards, both physical and virtual: How does one organize the generation of triangles in a systematic manner so that all triangles are accounted for? Alternative approaches, including the writing of computer code, may overcome this limitation. We will explore one such approach.

Scilab Programming

Using either a hands-on geoboard or a virtual applet, students risk omitting triangles. In an alternative approach, we begin with every possible configuration of three pegs on the geoboard

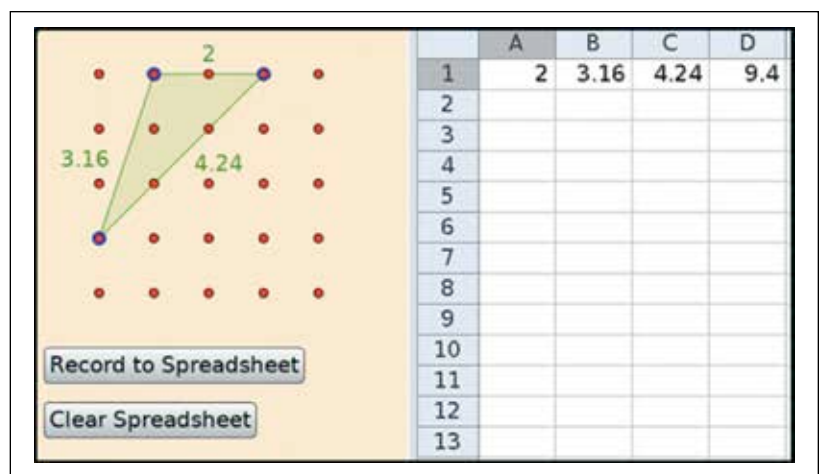


Fig. 3 Data-capturing functionality was included in the revised sketch.

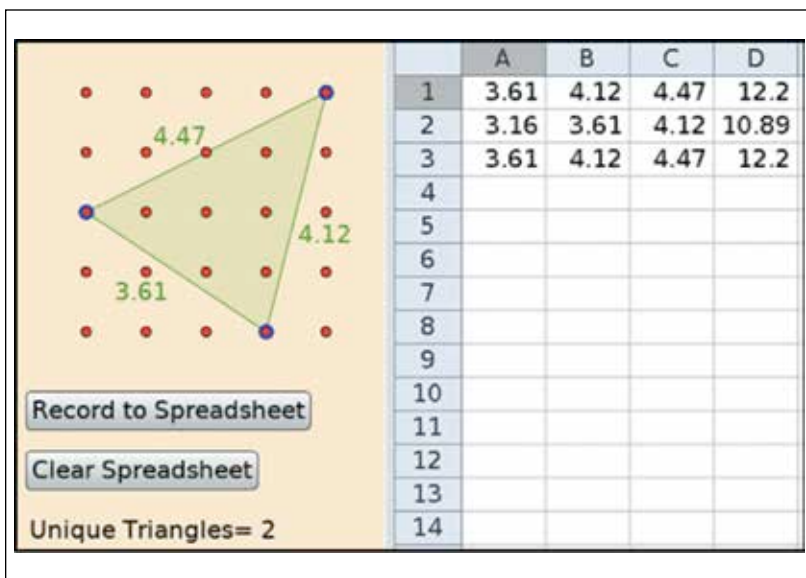


Fig. 4 Although three triangles have been recorded in rows 1-3 of the spreadsheet, the applet indicates that only two are unique (because the third has a perimeter of 12.2 units, as previously recorded).

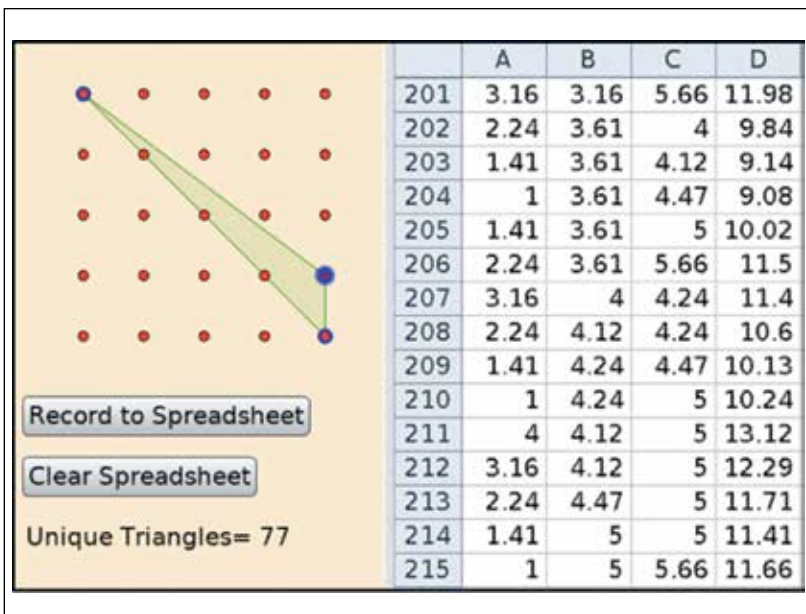


Fig. 5 With the applet, the students found 77 distinct triangles.

and remove those configurations that do not yield triangles. Modeling this approach for our students, we wrote such code in Scilab®. A pseudocode of our program is provided in the **appendix**. A functional Scilab program that implements the methods instructors' pseudocode is also provided.

Our initial Scilab program was short, consisting of eighteen lines of code; however, it makes use of significant secondary school-level mathematics, including the distance formula, floor and modulo functions, and elementary counting (e.g., 25 choose 3). As **figure 6** suggests, an initial run of the code returned a list of 80 triangles.

WHO IS CORRECT? TRIANGULATING RESULTS

At an end-of-week class meeting, the students and their instructor shared initial solutions to the Geoboard Triangle Quest, presenting three different solution strategies. Students who took a hands-on approach with physical geoboards found 71 triangles. Applet users found 77 triangles. Scilab users uncovered 80 triangles. Which, if any, number of triangles was correct? As the class carefully reviewed these findings, sharing lists of triangles across groups, it was apparent that each approach included errors or limitations.

Revising Virtual Geoboard Findings

Our students' virtual geoboard made the faulty assumption that triangles with equal perimeters were congruent. When students analyzed their data more closely, they uncovered two pairs of noncongruent triangles with equal perimeters, thus revising their total from 77 to 79. The pairs shown in **figure 7** led to the inclusion of two additional triangles to the students' list. With these triangles, the virtual geoboard students had 79 triangles in all.

Revising the Scilab Program

A spreadsheet aided the instructors in comparing the results generated by their code with the list of triangles generated by the virtual applet. The results were identical, except for an extra triangle with side lengths 1.414235, 2.8284271, and 4.2426406 (i.e., $\sqrt{2}$, $\sqrt{8}$, and $\sqrt{18}$). Much to our surprise, these side lengths fail the triangle inequality because $\sqrt{2} + \sqrt{8} = \sqrt{18}$. As a result of round-off error, the code as written had failed to eliminate this combination. After it was thrown out, the Scilab-generated list and the virtual applet list were identical—each listing the same 79 triangles.

A Collaborative Solution

When we compared the 71 hands-on geoboard triangles to the 79 generated using GeoGebra and Scilab, we discovered that the following triangles had been overlooked:

$$\begin{aligned} &(\sqrt{2}, \sqrt{5}, \sqrt{13}), (\sqrt{2}, \sqrt{10}, \sqrt{20}), (\sqrt{2}, \sqrt{13}, 5), \\ &(\sqrt{5}, \sqrt{5}, \sqrt{18}), (5, \sqrt{8}, 5), (\sqrt{5}, \sqrt{10}, 5), \\ &(\sqrt{5}, \sqrt{13}, \sqrt{32}), \text{ and } (\sqrt{10}, \sqrt{10}, \sqrt{32}). \end{aligned}$$

After verifying that each combination could be constructed on a conventional geoboard, we were confident that Scilab had provided a proof by exhaustion and that exactly 79 triangles could be constructed on a 4 × 4 geoboard. Through additional research, we found that this conclusion was

supported by a number of *Mathematics Teacher* articles (Moser [1985], p. 603; Kennedy [1993], p. 82; Willcut [1985], p. 614), although none of these articles provided a proof of the result.

CONCLUDING THOUGHTS

The three approaches highlighted in this article demonstrate different aspects of two Standards for Mathematical Practice (CCSSI 2010)—“Look for and make use of structure” and “Model with mathematics,” each highlighting advantages and disadvantages of different solving options. For instance, the hands-on approach with physical geoboards provided students with a kinesthetic model to start visualizing different triangles and begin thinking about noncongruent triangles.

We can look to advanced digital technologies as an amplifier (Sherman 2014) to this initial approach, and we see in the use of the GeoGebra applet that the technology offers many affordances. The examples are easier to generate, and we do not face logistical issues with the physical constraints of the rubber bands. Moreover, the spreadsheet capability of the applet allows for organization of the data and also allows the user to check whether new cases are indeed new. However, this “bottom up” approach has the inherent challenge of any proof by exhaustion approach when there is not a clear and systematic way of working through cases: How do you know when you’re done? As we see above and in Allen’s (2013) article, even with a high level of organization and a well-thought-through system, the students came up short with the constructive approach.

An alternative is a “top down” approach, in which the challenge of knowing when you are done is replaced by the challenge of deciding which of all possible scenarios are not true scenarios for the situation. The Scilab program works as both an amplifier (Sherman 2014) in that the program first “blindly” generates every possible choice of three points on the board, something quite difficult to do by hand. However, the amplification goes further in that the program, like the applet, can be used to test results and organize the data. If done correctly, this approach leads to the correct answer, although, again, in the case of the top-down approach, the limitations of the technology at first led to extraneous answers.

The bottom-up and top-down approach contrast nicely: The latter would appear to work smoothly, but it turned out that having the list of possible triangles generated through the hard work of the bottom-up approach was necessary for our students to understand the limitations of the technology while taking advantage of the affordances.

Using multiple approaches for modeling prob-

A	B	C	A	B	C	A	B	C
1	1	1.4142135	1.4142135	3.6055512	4.1231056	2.2360679	3.6055512	4
1	1.4142135	2.2360679	1.4142135	3.6055512	5	2.2360679	3.6055512	4.4721359
1	2	2.2360679	1.4142135	4.2426406	4.4721359	2.2360679	3.6055512	5.6568542
1	2.2360679	2.8284271	1.4142135	5	5	2.2360679	4.1231056	4.2426406
1	2.2360679	3.1622776	2	2	2.8284271	2.2360679	4.4721359	5
1	2.8284271	3.6055512	2	2.2360679	2.2360679	2.8284271	2.8284271	4
1	3	3.1622776	2	2.2360679	3.6055512	2.8284271	3.1622776	3.1622776
1	3.1622776	3.6055512	2	2.2360679	4.1231056	2.8284271	3.6055512	4.1231056
1	3.1622776	4.1231056	2	2.8284271	4.4721359	2.8284271	4.4721359	4.4721359
1	3.6055512	4.2426406	2	3	3.6055512	3	3	4.2426406
1	3.6055512	4.4721359	2	3.1622776	3.1622776	3	3.1622776	3.6055512
1	4	4.1231056	2	3.1622776	4.2426406	3	3.1622776	5
1	4.1231056	4.4721359	2	3.6055512	5	3	4	5
1	4.2426406	5	2	4	4.4721359	3	4.1231056	4.4721359
1	4.4721359	5	2	4.1231056	4.1231056	3	4.1231056	5.6568542
1	5	5.6568542	2	4.1231056	5	3.1622776	3.1622776	4.4721359
1.4142135	1.4142135	2	2	4.4721359	5.6568542	3.1622776	3.1622776	5.6568542
1.4142135	2	3.1622776	2.2360679	2.2360679	3.1622776	3.1622776	3.6055512	4.1231056
1.4142135	2.2360679	2.2360679	2.2360679	2.2360679	4	3.1622776	4	4.2426406
1.4142135	2.2360679	3	2.2360679	2.2360679	4.2426406	3.1622776	4.1231056	5
1.4142135	2.2360679	3.6055512	2.2360679	2.8284271	3	3.6055512	3.6055512	4
1.4142135	2.8284271	3.1622776	2.2360679	2.8284271	4.1231056	3.6055512	4.1231056	4.4721359
1.4142135	2.8284271	4.2426406	2.2360679	2.8284271	5	4	4	5.6568542
1.4142135	3	4.1231056	2.2360679	3	4.4721359	4	4.1231056	5
1.4142135	3.1622776	4	2.2360679	3.1622776	3.6055512	4	4.4721359	4.4721359
1.4142135	3.1622776	4.4721359	2.2360679	3.1622776	4.1231056	4.1231056	4.1231056	4.2426406
1.4142135	3.6055512	3.6055512	2.2360679	3.1622776	5			

Fig. 6 The Scilab program generates a list of 80 triangles.

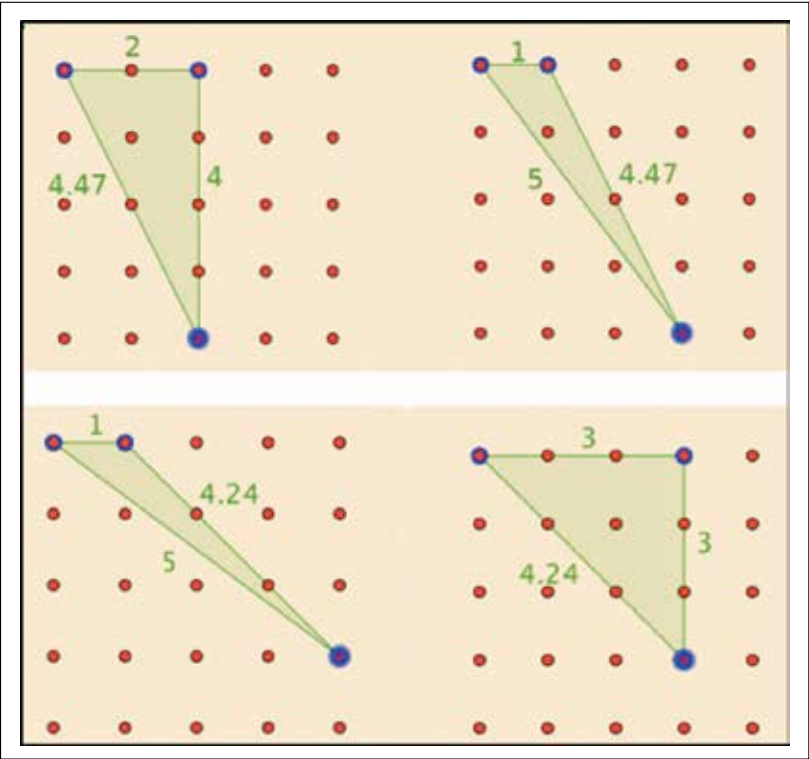


Fig. 7 Pairs of noncongruent triangles have the same perimeters.

lems is useful not only for the insight offered by different perspectives but also, as we see in this case, for the opportunity for each approach to work as a corrective and verifier of the other approach, each honing the other in reaching a correct solution.

Ultimately, in this case, a proof by exhaustion using advanced digital technologies settled the problem. For our students, aside from thinking about proof, the most valuable lesson was seeing how multiple approaches to solving problems can be much more than just different ways to find an answer. Moreover, the commingling and influence of different solutions on each other drives each approach toward a correct answer.

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APPENDIX

Pseudocode (a) and a Scilab program (b) are used to determine all triangles on a 4×4 geoboard.

1. Let the numbers 1 to 25 represent each of the 25 pegs on a 4×4 geoboard.
2. Take all possible combinations of numbers from 1 to 25 three at a time. Place these combinations in a list.
3. Perform the following tasks for each combination of 3 pegs in the list:
 - a. Assign cartesian coordinates (x,y) to each peg.
 - b. Define segments AB, AC, and BC using these (x,y) coordinates.
 - c. Calculate the length of segments AB, AC, and BC to determine if they satisfy the triangle inequality. If they do, record AB, AC, and BC to a table of values. These represent sides of a new triangle in the table.
4. For each triangle in the table, sort the side lengths in ascending order.
5. Remove duplicate (i.e., congruent) triangles from the table.

(a)

```
rows=5; //number of rows on board
triangles=zeros(0,3); //matrix of side lengths of all constructible triangles

//Vector c contains all possible combinations of board pegs taken
//3 at a time. Combin is the number of elements in vector c.
c=specfun_subset(1:rows*rows,3);

for i=1:size(c,1)
    //Assign coordinates to pegs
    A=[modulo(c(i,1),rows),floor((c(i,1)-0.5)/rows)+1];
    B=[modulo(c(i,2),rows),floor((c(i,2)-0.5)/rows)+1];
    C=[modulo(c(i,3),rows),floor((c(i,3)-0.5)/rows)+1];

    //Calculate lengths of segments AB, AC, and BC
    d1=sqrt((A(1)-B(1))^2+(A(2)-B(2))^2);
    d2=sqrt((A(1)-C(1))^2+(A(2)-C(2))^2);
    d3=sqrt((B(1)-C(1))^2+(B(2)-C(2))^2);
    tri_ineq=(d1+d2>d3) & (d1+d3>d2) & (d2+d3>d1);

    //If triangle inequality holds, record side lengths to matrix triangles
    //if tri_ineq then sort lengths so that d1<=d2<=d3
    newrow=gsort([d1,d2,d3], 'g','r');
    triangles=[triangles;newrow];
end
end
```

(b)



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