TRIANG

with Integer

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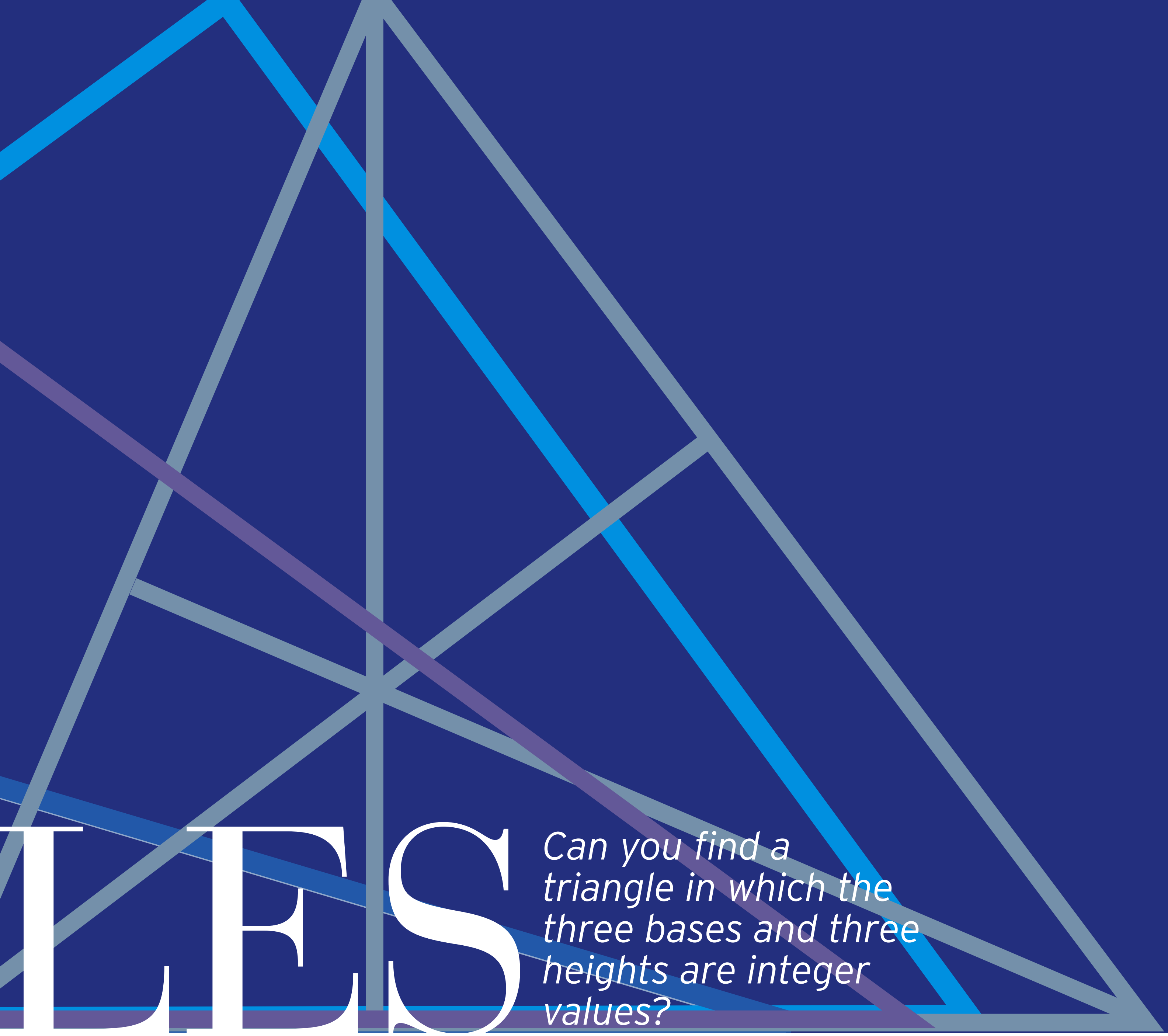
Interesting and engaging mathematics problems can come from anywhere. Sometimes great problems arise from interesting contexts. At other times, interesting problems arise from asking "what if" questions while appreciating the structure and beauty of mathematics. The intriguing problem described in this article resulted from the second context, where trying to create an interesting example to support student thinking produced a rich investigation in its own right.

ONE THING LEADS TO ANOTHER

As students learn about area formulas in the middle grades, they relate irregular or many-sided two-dimensional shapes to areas of more basic shapes (e.g., rectangles, parallelograms, and triangles). Because students can decompose polygons into tri-



LES Dimensions

Can you find a triangle in which the three bases and three heights are integer values?

angles, understanding the area formula of triangles is crucially important to knowing how to find the areas of more challenging shapes. Although students can often see why the area formula of a triangle should be $A = 1/2(b \cdot h)$, thinking more deeply about this formula in relation to the area of a parallelogram raises at least two interesting questions:

1. If two triangles have the same base and height, will they necessarily have the same area?
2. Does it matter which of the three (base, height) pairs are chosen to determine the triangle's area?

Although the first question is interesting, especially in relation to understanding shearing actions, we focus our attention on the second question.

Visually, the area of the triangle should not change simply by choosing one edge over the others

Integers and Triangles

For each type of triangle below, determine whether it is possible to create a triangle where all three side lengths are integer values and all three heights are also integer values. If possible, draw the triangle with the heights and lengths of sides marked. If not possible, describe why it is not possible.

	Equilateral	Isosceles	Scalene
Acute			
Right			
Obtuse			

Fig. 1 A structured approach may help students investigate different triangles with all integer heights and bases.

as the base; less clear numerically is the fact that, for any triangle, the three products $A_1 = (1/2) \cdot (b_1 \cdot h_1)$, $A_2 = (1/2)(b_2 \cdot h_2)$, and $A_3 = (1/2)(b_3 \cdot h_3)$ will be equal. From a pedagogical standpoint, students should be able to articulate why it does not matter which base they choose to calculate the area of a triangle. The relationship $A_1 = A_2 = A_3$ for any triangle may become clearer if students could investigate at least one ideal triangle in which all the bases and heights are easily measurable—that is, they are positive integer values. Such an investigation could serve as an initial step toward generalization. However, it raises the question, Does such an ideal triangle even exist?

This question is the genesis of the activity described here, which is not intended to be a prerequisite activity before investigating areas of triangles. Indeed, the content knowledge needed is more sophisticated than when students first encounter areas of triangles. Rather, this activity was borne out of a pedagogical need to find at least one ideal triangle for students to investigate the relationship of bases and heights. Given the two approaches to implementing this task described below, teachers could use this task as early as grade 8. But because students are able to use ever more sophisticated tools to solve the problem, it could also be used effectively beyond this grade level (even in a professional development setting with practicing teachers).

The main purpose of this activity (see **fig. 1**) is for students to find cases of triangles where the three heights and the lengths of the three sides are all integers. The task is intended to focus primarily on the Common Core State Standard for Mathematical Practice (SMP) 1—"Make sense of problems and persevere in solving them"—while also following SMP 5—"Use appropriate tools strategically" (CCSSI 2010, pp. 6, 7). In the next section, we provide one particular method for solving this problem, but the task itself is sufficiently open to allow for multiple pathways to solve the problem.

STUDENTS EXPLORE TRIANGLE CONSTRUCTIONS

This task is particularly productive because it contains easy entry points as well as allows students to work through some challenges during their solution process. Students could solve this problem in a variety of ways, from guess-and-check to numerical approaches by creating brute-force computer programs on a spreadsheet (Rosenberg, Spillane, and Wulf 2008). Here, we choose a somewhat middle path, focusing on number theory applications and applications of the Pythagorean theorem, with special attention to Pythagorean triples. This activity could be a productive extension for students learning about the Pythagorean theorem and properties. The relationships given in **table 1** assume that a, b, and c are side lengths of a right triangle and that $a < b < c$.

When working toward a solution for the task, we could begin by choosing a particular cell, row, or column (see **fig. 1**) and seeing whether it is possible to construct that type of triangle. Consider, for example, the equilateral triangle column: We should be able to determine that a right equilateral triangle and an obtuse equilateral triangle are not

Table 1 Pythagorean Triple Formulas

Characterization	Triples (a, b, c) with $a < b < c$
One leg length is an odd number.	For any positive integer n, $a = 2n + 1$; $b = (1/2)(a^2 - 1)$; and $c = (1/2)(a^2 + 1)$.
One leg length is a multiple of 4.	For any positive integer m, $a = 4m$; $b = 4m^2 - 1$; and $c = 4m^2 + 1$.
The sum of squares property is preserved under scale factor.	For any positive integer k, (ka, kb, kc) is also a Pythagorean triple.
Note: For proofs of these formulas, see Forman and Rash (2015).	

possible because they cannot even be constructed in Euclidean geometry. When focusing on acute equilateral triangles, therefore, we can deduce that an equilateral triangle with an integer base of length b will have a height of length $(\sqrt{3}/2)b$, which will never be an integer. Similarly, an isosceles right triangle also cannot have integer-length bases and heights because, given the length of the leg of an isosceles right triangle as an integer b, the length of the hypotenuse (one of the other bases) will be $\sqrt{2}b$, which will never be an integer.

Another way to begin working on a solution to this problem is to consider a triangle that is guaranteed to have three integer bases. For example, students might start by using a right triangle consisting of a Pythagorean triple such as a 3-4-5 right triangle. In fact, using right triangles with Pythagorean triples as "building blocks" will be particularly useful because the two legs are also two of the altitudes, meaning that five of the six values (three bases and two of the heights) will be integer lengths. But what about the sixth value, the height where the corresponding base is the hypotenuse?

The area of a 3-4-5 right triangle can be calculated as $A = (1/2) \cdot 3 \cdot 4 = 6$ square units. Now, using the hypotenuse as the base, we can solve for the height: $h = 2A/b = 2(6)/5 = 12/5$ units. This is not an integer, but fortunately it is a rational number. Thus, we can scale the original 3-4-5 triangle by any multiple of 5 to create a triangle with integer values for the lengths of all three heights and all three bases. Specifically, scaling the 3-4-5 triangle by 5, we create a 15-20-25 right triangle with area $A = (1/2) \cdot 15 \cdot 20 = 150$ square units. If we use 25 (the new hypotenuse) as the base, the corresponding height is $h = 2A/b = 2(150)/25 = 12$ units.

Using this strategy of scaling will be useful in building many other types of triangles as well. In **figure 2**, the five triangles represented are composed of two smaller right triangles that share a common side. For example, the acute scalene triangle was built by joining a 5-12-13 right triangle and a 9-12-15 right triangle. The preservation property of Pythagorean triples (see **table 1**) allows for many ways of creating new triangles from two right triangles with all integer sides. Most important, any triangle built this way will have an integer base and height, meaning that the area will be a rational number. This approach allows for scaling to ensure that all heights and side lengths are integers.

CONJECTURES PROVIDE AN ALTERNATIVE APPROACH

A more open approach, without the scaffolding provided for students to consider different types of triangles (see **fig. 1**), is to simply ask students to construct a triangle with three integer bases and

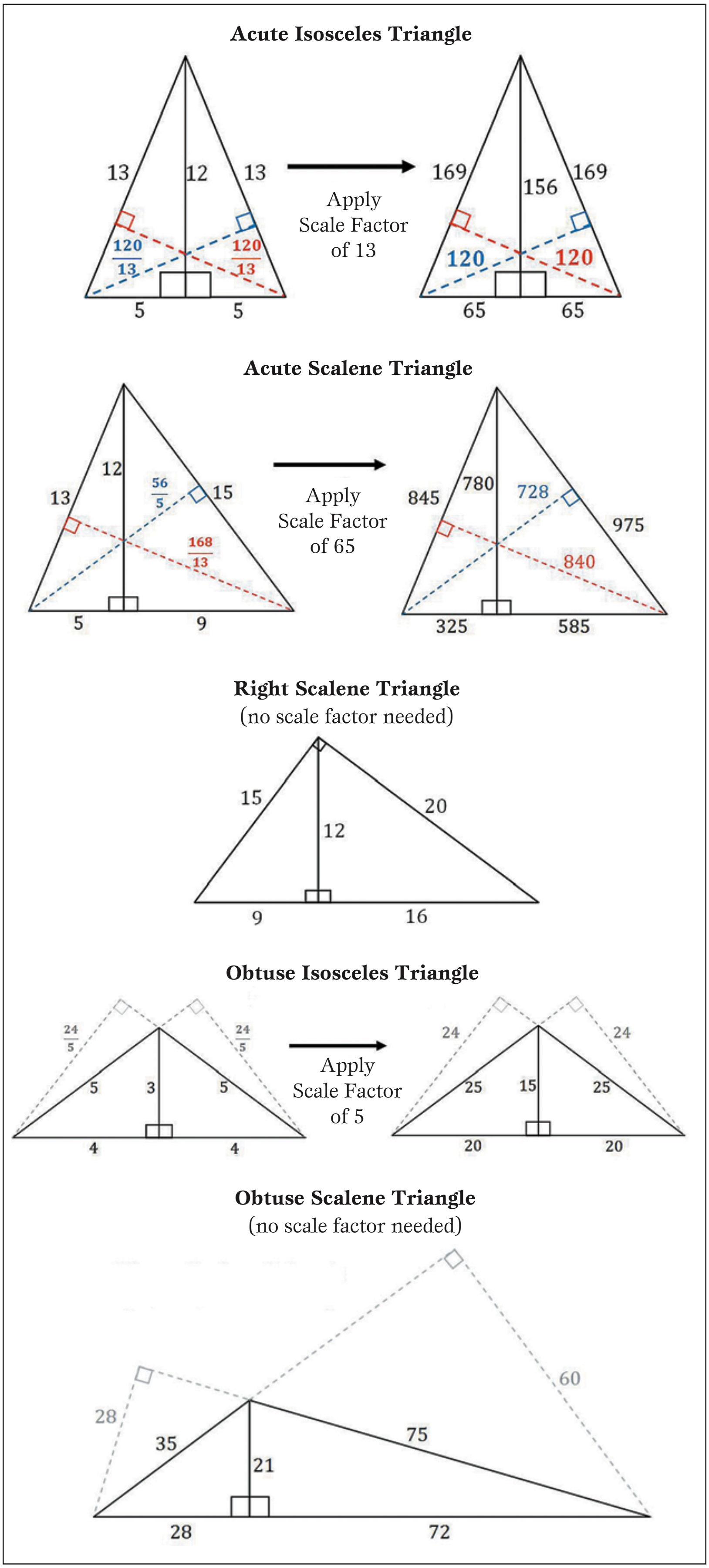


Fig. 2 Different triangles all have heights and bases in integer measurements.

three integer heights. Before letting students work on the problem, ask them whether, intuitively, they believe such a triangle exists. On the one hand, a student could argue that, given the various types of triangles and having infinitely many to choose from, eventually we are bound to find one that works. On the other hand, a student could argue that something about the structure of triangles may make three integer heights and three integer bases impossible.

This second conjecture is, in fact, true for the special cases of equilateral and isosceles right triangles; even though infinitely many of these types of triangles exist, there are no cases where the conditions hold true. In fact, we can construct 87,125 unique triangles with three integer-length sides measuring 100 or less. This count was determined by creating a very large ordered list (i.e., for $\triangle ABC$, where $1 \le a \le b \le c \le 100$) of all possible combinations of side lengths with special attention to using the triangle inequality, where $a + b > c$. Of these 87,125 unique triangles, only 10 satisfy the conditions of having three integer heights (see **table 2**).

Once students have had an opportunity to decide whether they think these triangles exist or not, ask them how they might go about solving the problem. Having students discuss potential approaches may make the activity of choosing appropriate tools strategically (SMP 5, CCSSI 2010, p. 7) more explicit. Students who are looking for an existence case should consider the following questions: What types of triangles should we start with? Are there certain types of triangles that may increase the likelihood that the six values will be integers? Should we start with triangles where the three sides are integers and check the heights, or vice versa? What formulas might be helpful in trying to solve this problem? (Students may not know about Heron's formula, for example, but this would be a good opportunity for them to look into that relationship.)

In contrast, students who claim that no triangle exists also must come up with an argument that convinces the class that this claim is true in general. This is an important idea for students to understand in relation to reasoning and proof: Showing nonexistence must also be established in some deductive way—that is, we start from a neutral position of whether a triangle exists, and both nonexistence and existence have equal burdens of proof. This approach may in fact be more complicated, but students should consider how they would test all possible cases even if the subset is relatively small (such as integer values less than or equal to 100). This discussion may generate the need to develop a computer program to exhaust the sample space, but, at the very least, the teacher could ask students about whether randomly selecting triangles would be enough to convince themselves or whether they should consider selecting examples from different types of triangles (e.g., isosceles, equilateral, right, etc.).

Table 2 Triangles with Integer-Value Side Lengths and Heights of 100 Units or Less

Right Scalene Triangle	Acute Isosceles Triangle	Obtuse Isosceles Triangle	Obtuse Scalene Triangle
15-20-25	25-25-30	25-25-40	35-75-100
30-40-50	50-50-60	50-50-80	
45-60-75	75-75-90		
60-80-100			

Students on both sides of the argument should present their reasoning to the class, likely creating some lively discussion. For a resolution, the class as a whole may need to investigate existence. The 3-4-5 right triangle explained previously is a good starting point because five of the six values are integers. Leaving open the idea of scale factor may be a useful ending point for the discussion without leading students all the way to the solution.

INTUITION AND PERSEVERENCE

Some of the most satisfying problems to solve are those in which our intuition takes us in different directions, compelling us to step back and consider what tools might be useful to solve this problem before digging deeper by, for example, using interactive software, writing a computer program to test cases, or generating paper-and-pencil constructions. Trying to find any of these results in **table 2** through guess-and-check would be daunting. Because they occur so infrequently, we need either to be strategic about finding an existence case or to reason about why no such triangle exists in general. This problem could easily prompt discussion about the inherent limitations of examples-based reasoning (Harel and Sowder 2007)—that finding tens of thousands of cases where no triangles exist is not sufficient in establishing general nonexistence.

In addition to being an interesting mathematical problem in its own right, this problem about finding at least one triangle with integer values for all the lengths of its bases and heights can help students see the need to persevere in looking for an existence case and not simply apply empirical reasoning to support why no such cases exist. Providing opportunities for students to observe that the area of specific triangles remains consistent regardless of the base and height pair chosen is important and can then lead to further discussions about why

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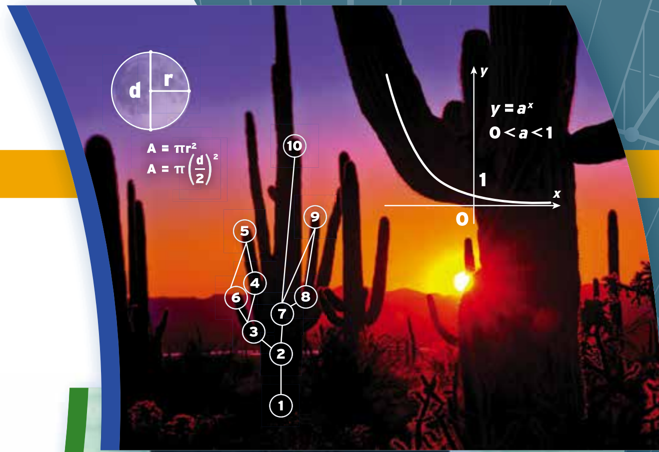

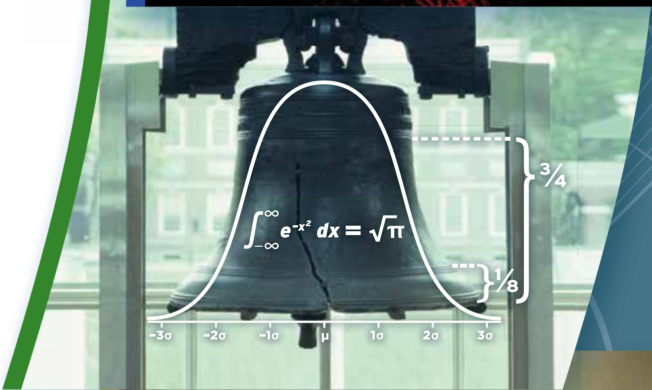

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these calculations are equivalent for any triangle. Using these triangles can be productive to illustrate whether it matters which side is chosen as the base so that students establish the equivalence of the three ways to calculate the area of a triangle.

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