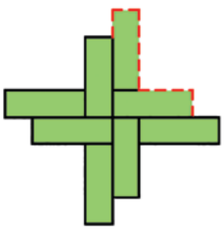
SOLUTIONS to calendar

1. 10301. The formula for the nth triangular number is $t_n = 1 + 2 + \cdots + n = n(n + 1)/2$. Notice that $s_n = 2t_{n+1} - 1$. So $s_{100} = 2t_{101} - 1 = 2(101)(102)/2 - 1 = 10301$.

2. 32 in. The part of the boundary of the shape that is marked with dashed lines is the same as the perimeter of one of the 8 congruent rectangles. The perimeter of the entire shape must be 4 times the length of the part of the boundary marked with dashed lines. Hence, the perimeter of the entire shape (consisting of 8 rectangles) must be 32 in.



3. $3\sqrt{15}/8$ ft. The altitude of length h splits the side of length 4 ft. into segments of lengths x and $4 - x$. By the Pythagorean theorem, we have $h^2 + x^2 = 4$ and $h^2 + (4 - x)^2 = 9$. Subtracting the first equation from the second gives $16 - 8x = 5$, or $x = 11/8$. Thus, $h^2 = 4 - x^2 = 4 - 121/64 = 135/64$, and $h = 3\sqrt{15}/8$.

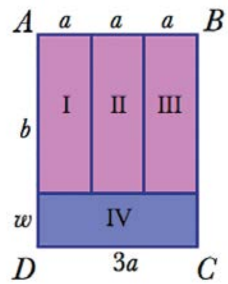
4. 148 cm^2. The figure suggests that the perimeter of the dodecagon created by just the four congruent rectangles must also be 56 cm. Since the side length of each of the four squares is 3 cm, the sum of the length and the width of each of the four rectangles must be $56/4 - 3 = 11$ cm. The positive difference between the length and the width of the rectangle must be 3 cm. Therefore, the length and the width of each rectangle must be 7 cm and 4 cm, respectively. The area of each of the four rectangles is 28 cm^2, and the area of each of the four squares is 9 cm^2, so the area of the dodecagon must be $4(28 + 9) = 148$ cm^2.

Alternative solution: Let the short side of the rectangle be x. Then, 1/4 of the perimeter of the shape without the squares is $x + 3 + (3 + x) = 56/4 = 14$. Thus, $x = 4$; the area of 1 rectangle is $4(4 + 3) = 28$; and the total area is $4(28 + 9) = 148$ cm^2.

5. 18 apples. Suppose that Mr. Rozario bought $2x$ apples and $3x$ oranges. After Mr. Rozario gives away some of his apples and oranges, the ratio of apples to oranges is $2x/3 : (3x - 23) = 3:2$. Solve for x to obtain $x = 9$. Therefore, Mr. Rozario bought 18 apples.

6. 18 pelicans. Noting that none of the pelicans flew away, we let the number of pelicans on the beach be $6x$. Then the original number of herons and the original number of flamingos on the beach must be $12x$ and $18x$, respectively. Moreover, the number of herons and the number of flamingos that were left on the beach must be $3x$ and $2x$, respectively. Therefore, $12x - 3x = 9x$ must the number of herons that flew away, and $18x - 2x = 16x$ must the number of flamingos that flew away. We are told that $16x - 9x = 21$, implying that $x = 3$. Therefore, there must have been 18 pelicans.

7. 12 cm. We can label the segments' lengths as shown because rectangles I, II, and III are congruent. All four rectangles have the same area, so $ab = 3aw$, implying that $b = 3w$. We know that the perimeter of rectangle I is 12, so $a + 3w = 6$, and we know that $ABCD$ has perimeter $21 = 6a + 8w$. Solve the system by substitution to find that $w = 1.5$ and $a = 1.5$. Thus, all four rectangles are congruent, so the perimeter of rectangle IV is 12 cm.



8. (21, 4), (12, 5), and (9, 6). Factor the left side by grouping to obtain $(a - 3) \cdot (b - 3) = 18$. Since a and b are positive integers, $a - 3$ and $b - 3$ are integers, both greater than -3. Integer factorizations of 18, such that $a \geq b$, result in $a - 3 = 18$ and $b - 3 = 1$, or $a - 3 = 9$ and $b - 3 = 2$, or $a - 3 = 6$ and $b - 3 = 3$. These factorizations give us three pairs (a, b): (21, 4), (12, 5), and (9, 6).

Alternative solution: Solve for a: $a = (9 + 3b)/(b - 3) = (3b - 9 + 18)/(b - 3) = 3 + 18/(b - 3)$. Therefore, $b - 3$ must divide into 18. The only positive values for b that result in $a > b$ are 4, 5, and 6. These result in the same three ordered pairs listed above.

9. The areas are equal. With an assist from the Pythagorean theorem, we can show the equality through a proof without words.

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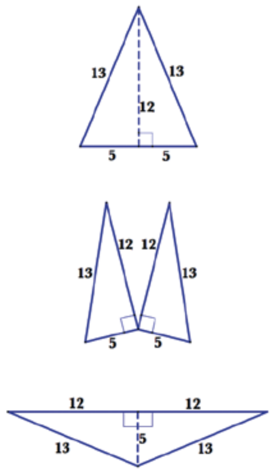


10. 15 cm. The additional areas of the butter stick that are exposed as a result of the seven cuts must be equal to $14 \times (5 \times 5) = 350\ \text{cm}^2$. We are told that the total surface area of the eight slices is twice that of the original stick, so 350 cm^2 must be equal to the original surface area of the butter stick. Hence, we can write the equation $50 + 20b = 350$. Solve for b to find that $b = 15$ cm.

11. 2.5 hours. Altogether Alice worked for 3 hours, so she completed 3/5 of the entire job herself. Alice's friend must have completed the remaining 2/5 of the job; thus, it took her 1 hour to complete 2/5 of the job. Therefore, working alone, Alice's friend will take half an hour to complete 1/5 of the job, and she will need five times that amount of time, 2.5 hours, to complete the entire job on her own.

12. 8/5. The distance from a point to a line is the length of the perpendicular segment from the point to the line. The point P, therefore, must be the intersection of the given line $4x + 2y = 8$ and the line through the origin perpendicular to it. Since the given line has slope -2, the perpendicular through the origin is $y = x/2$. Substituting into the first equation gives $4x + 2(x/2) = 8$, so $x = 8/5$.

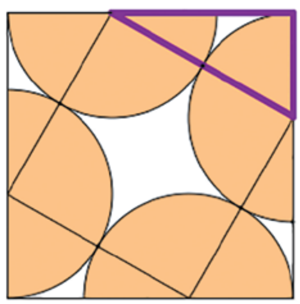
13. 9 sides. We make use of the fact that the interior angle sum is $(n - 2)180°$ for both convex and nonconvex polygons. The part of the n-gon that is visible has a heptagonal shape with one reflex angle. This reflex angle measures $360° - 108° = 252°$. Since $a + b + 252° = 88° + 252° = 340°$, the remaining angles of the heptagon have a sum of $180°(7 - 2) - 340° = 560°$. As angles of the regular n-gon, these four angles must have equal measure: $560°/4 = 140°$. Finally, if a single interior angle has measure 140°, the adjacent exterior angle has measure 40°, and $360°/40° = 9$ tells us that the n-gon has 9 sides.

14. 15 pieces. Suppose that Justin cut the pizza into n equal slices. According to the question, the following must be true:

$$\frac{n-3}{n} = \frac{3}{4} + \frac{3}{4}\left(\frac{1}{n}\right)$$

Multiply both sides by $4n$ to write $4n - 12 = 3n + 3$ and find that $n = 15$.

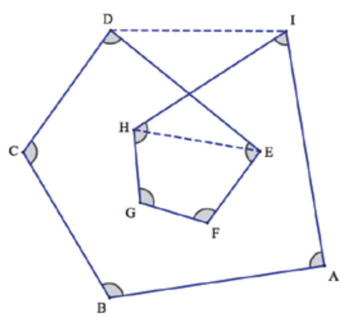
15. $2\pi\left(\sqrt{3}-1\right)^2$. If we join the centers of the four semicircles, we obtain a smaller square, as shown in the figure. The sides of this smaller square pass through the points of tangency of the semicircles, and each side of this square forms a right triangle within the larger square. If we let r be the radius of one of the four semicircles, then each right triangle has hypotenuse $2r$ and shorter leg r. Recognize that these triangles must be 30-60-90° right triangles, or use the Pythagorean relationship to find the length of the longer leg: $(2r)^2 - r^2 = (2 - r)^2$ so that $r = -1 \pm\sqrt{3}$ Since $r > 0$, we have $r = \sqrt{3} - 1$. Therefore, the sum of the areas of the four semicircles must be equal to $2\pi(\sqrt{3} - 1)^2$. A decimal approximation gives us 3.367, which is approximately 84.2% of the area of the square with side length 2 inches.



16. 1,000,000. Computing directly, we find that $f(1) + f(2) + f(3) + \cdots + f(100) = (1^3 - 0^3) + (2^3 - 1^3) + (3^3 - 2^3) + \cdots + (100^3 - 99^3)$. On cancellation, this sum collapses like a spyglass telescope, leaving $100^3 - 0^3$.

17. 2 units^2. First, since $\angle ACB$ is a right angle, the circular sector on the arc from A to B is a quarter circle with radius 2; hence, the sector has area $A = \pi(2)^2/4 = \pi$. Next, chord AB has length $2\sqrt{2}$ (it is the hypotenuse of an isosceles right triangle with leg 2) so the radius of the semicircle on $\overline{AB}$ is $\sqrt{2}$. The area of this semicircle is $\pi(\sqrt{2})^2/2 = \pi$, and the area of isosceles right triangle ABC is 2. The desired area is found by subtraction: area of semicircle on $\overline{AB}$ – (area circular sector on $\overset{\frown}{AB}$ – area $\triangle ABC$) $= \pi - (\pi - 2) = 2$. It may seem surprising, but this area, bounded by two circular arcs, is a rational number, equal to the area of $\triangle ABC$.

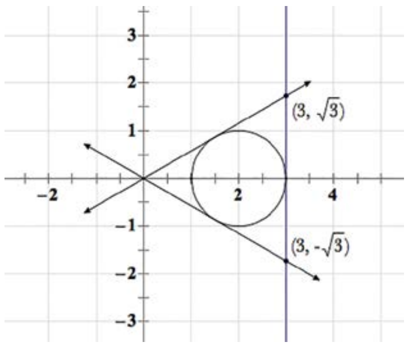
18. 900°. Since $m\angle IHE + m\angle DEH = m\angle IDE + m\angle DIH$, the necessary sum must be the sum of the sum of the angles of pentagon $ABCDI$ and the sum of the angles of quadrilateral $EFGH$. That sum is $540° + 360° = 900°$.



***Alternative solution*:** Draw $\overline{DI}$. Then the sum of the angles is twice the sum of the angles in the two pentagons minus the sum of the angles in $\triangle DIK$, where K is the intersection of $\overline{DE}$ and $\overline{HI}$: $2(540°) - 180° = 900°$.

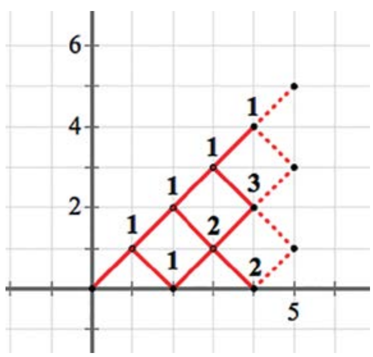
19. $3\sqrt{3}$ units^2. Note that the circle has center (2, 0) and radius 1. Consider the intersection of $y = mx$ and the circle. Substitution leads to the equation $(x - 2)^2 + m^2x^2 = 1$ or $(m^2 + 1)x^2 - 4x + 3 = 0$. If the line $y = mx$ is a tangent line, then this quadratic equation must have exactly one solution, which occurs when the discriminant $16 - 12(m^2 + 1)$ is 0. Solve for m to find $m = \pm\sqrt{3}/3$; the positive slope corresponds to the line that is tangent to the circle in quadrant I, and the negative slope corresponds to the line that is tangent to the circle in quadrant IV. The third tangent has equation

$x = 3$. The tangent lines $y = \pm\sqrt{3}x / 3$ intersect the vertical tangent at $(3, \pm\sqrt{3})$, so the triangle has height 3 and base $2\sqrt{3}$. Its area is $3\sqrt{3}$. (Using the vertical tangent line with equation $x = 1$ would result in a smaller triangle.)



***Alternative solution*:** Note, as above, that the circle has center (2, 0) and radius 1. Draw the radius to one of the diagonal tangents to form a right triangle with hypotenuse 2 and a leg 1. Thus, the angle between the tangent and the x-axis is 30°, meaning that the triangle formed by the tangents is equilateral with altitude 3. Half the base of this triangle is $3 / \sqrt{3}$, and its area is $9 / \sqrt{3} = 3\sqrt{3}$.

20. 10. If $k \geq 1$, each path through (n, k) leads to two paths: one ending at $(n + 1, k + 1)$ and one ending at $(n + 1, k - 1)$. But if $k = 0$, the path leads to $(n + 1, k + 1)$ only. With this in mind, we proceed left to right, to find that there are 5, 4, and 1 paths ending at (5, 1), (5, 3), and (5, 5), respectively, for a total of 10 paths.



21. 5/8. Each switch may be closed, in state C, which passes current, or open, in state O. The probability of passing current is $P[(s_2$ is closed and s_3 is closed) or s_1 is closed]. The sample space can be represented by eight equally likely triples, each with probability $p^3 = 1/8$: (C, C, C), (C, C, O), (C, O, C), (C, O, O), (O, C, C), (O, C, O), (O, O, C), and (O, O, O), with the first, second, and third entries corresponding to the states of switches 1, 2, and 3, respectively. The first five triples correspond to the event of passing current.

***Alternative solution*:** Let A be the event that the upper path passes current, and let B be the event that the lower path passes current. $P(A)$ is the probability that s_2 and s_3 are closed; this probability is $.5^2$ by the assumption of independence. $P(B) = .5$ is the probability that s_1 is closed. $P(A$ and $B)$ is the probability that s_1, s_2, and s_3 are closed; this probability is $.5^3$ by independence. Therefore, $P(A$ or $B) = P(A) + P(B) - P(A$ and $B) = .5^2 + .5 - .5^3 = .625$, or 5/8.

22. 4/5. By the given condition, the sample space in the previous solution has been reduced to the five equally likely triples: (C, C, C), (C, C, O), (C, O, C), (C, O, O), and (O, C, C). The first four triples are included in the event that the first switch is closed.

***Alternative solution*:** Let S be the event that the circuit passes current. Then

$$P\left(s_1 \text{ is closed} \middle| S\right) = \frac{P\left(s_1 \text{ is closed and } S\right)}{P\left(S\right)}.$$

From problem 21 we know that $P(S) = .625$. Also, $P(s_1$ is closed and $S) = P(s_1$ is closed$) = .5$. So $P(s_1$ is closed$|S) = .5/.625 = .8 = 4/5$.

23. 125/216. Let X be the number of tosses by Paul until a match first occurs. Then, $P[X = n]$ is the product of the probabilities that a match did not occur on tosses 1, 2, $\cdots$, $n - 1$, with the probability of a match on the nth toss. Thus,

$$P[X = n] = \left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{5}{6}\right)\left(\frac{1}{6}\right) = \left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right).$$

The probability of needing at most 3 tosses to obtain a match is

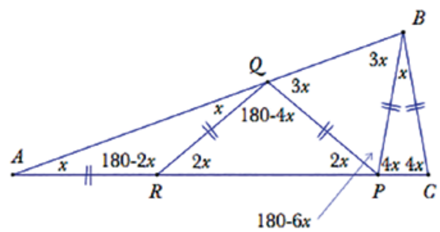
$$\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right) = \frac{91}{216}.$$

The probability of needing at least 4 tosses to obtain a match is the complement of the calculated probability: $1 - 91/216 = 125/216$.

***Alternative solution*:** The probability of needing at least 4 tosses until success is the probability of three consecutive failures; that is, $(5/6)^3 = 125/216$.

24. 125. Suppose the original cake was n inches × n inches × n inches. Then the number of unit cubes that have powdered sugar on three faces (at the corners) is 8, and the number of unit cubes that have no powdered sugar on any of the faces must be $(n - 2)^3$. According to the problem, $(n - 2)^3 = 27$. That is, $n = 5$ inches. Therefore, the cake was cut into 5^3, or 125, cubes.

25. 20°. Suppose that $m\angle QAR = x$. Then, using the properties of isosceles triangles, we can show that $m\angle PCB = 4x$. Since $\triangle ABC$ is isosceles as well, $m\angle ABC = 4x$. Therefore, $4x + 4x + x = 180°$, or $x = 20°$.



26. 113400. Assume that for every couple, the two people are of different heights. Let $C(n, k)$ denote the number of combinations of n distinct objects taken k at a time; there are 10 spots for 10 individuals. The first couple can be assigned two spots in $C(10, 2)$ different ways. Once we choose two spots for a couple, the couple can be assigned to those two spots in only one way because the taller person is to be behind the shorter person. The second couple can now be assigned two spots in $C(8, 2)$ different ways, and so on. Therefore, $C(10, 2) \cdot C(8, 2) \cdot C(6, 2) \cdot C(4, 2) \cdot C(2, 2) = 113400$, on the basis of the fundamental counting principle. By using $C(n, k) = n!/(k!(n - k)!)$, the previous product "telescopes" to $10!/2^5$. In general, for n couples the number of arrangements is $(2n)!/2^n$.

***Alternative solution*:** For $n \geq 1$, let $A(n)$ be the number of arrangements of n

couples according to the rule. We count directly that $A(1) = 1$ and $A(2) = 6$. Consider $n \geq 3$. One of the n shorter partners must come first; for the second position there are two disjoint cases: (1) the taller partner corresponding to the first person comes second; or (2) one of the remaining $n - 1$ shorter partners comes second. In option 1, there are $A(n - 1)$ ways to arrange the remaining $n - 1$ couples. In option 2, the two taller partners corresponding to the first two shorter partners may be placed in any of the $2n - 2$ remaining slots; there are then $A(n - 2)$ ways to arrange the remaining $n - 2$ couples. In summary, for $n \geq 3$, we have

$$A(n) = n \cdot A(n-1) + n(n-1)(2n-2)(2n-3) \cdot A(n-2).$$

(Actually, this result holds for $n \geq 2$, where $A(0) = 1$, because there is one way to arrange zero couples—namely, the "empty" line.) Starting at $n = 0$, this formula generates the number of arrangements as 1, 1, 6, 90, 2520, and 113400. This interpretation of this sequence and others may be found in The On-Line Encyclopedia of Integer Sequences® (oeis.org).

27. 213 jugs. If Tristan sold 3/4 of what he had and ended with x jugs then he began with $4x$ jugs. We will use this reasoning and work backward. After selling jugs to Beth and before selling any jugs to Cindy, Tristan must have had $4 \cdot (3\ 1/4) = 13$ jugs. After selling jugs to Ann and before selling any jugs to Beth, he must have had $4 \cdot (13\ 1/4) = 53$ jugs. Similarly, before selling any jugs to Ann, he must have had $4 \cdot (53\ 1/4) = 213$ jugs. Therefore, Tristan originally had 213 jugs of milk.

Alternative solution: Let the function $f(x) = x/4 - 1/4 = (x - 1)/4$. This function models one iteration of selling; x represents the number of jugs that Tristan has to sell, and $(x - 1)/4$ represents the number of jugs Tristan has left after a transaction. We need the composition $f(f(f(x)))$, which must equal 3. If $f(x) = (x - 1)/4$, then $f((x - 1)/4) = (x - 5)/16$, and also $f((x - 5)/16) = (x - 21)/64$. Set this expression equal to 3 to find $x = 213$.

28. $f(x) = x$ or $f(x) = -x + b$. To find the inverse, set $y = mx + b$ and solve for x. So $x = (y - b)/m$ and $f^{-1}(x) = (x - b)/m = (1/m)x + (-b/m)$. For $f(x)$ and $f^{-1}(x)$ to be the same polynomial functions, the corresponding coefficients must be equal, so $m = 1/m$ and $b = -b/m$. If $m = 1/m$, we know that $m = 1$ or $m = -1$. The case $m = 1$ implies that $b = -b$, so $b = 0$. In the second case, $m = -1$, we see that $b = -b/m$ places no restrictions on b. We could also reason out the solution graphically because the problem is equivalent to finding all lines that are symmetric with respect to $y = x$.

29. 1/16. This family of functions has an interesting effect on inputs of the form $x = 1/n$:

$$f_a(1/n) = \frac{1/n}{1 + a/n} = \frac{1}{n + a}$$

Beginning with $x = 1 = 1/1$, we have $f_5(1/1) = 1/(1 + 5) = 1/6$. Then

$$(f_4 \circ f_5)(1) = f_4(f_5(1)) = f_4(1/6) = 1/(6 + 4) = 1/10,$$

and so on. As the functions are applied successively, we have, schematically, $1 = 1/1 \to 1/6 \to 1/10 \to 1/13 \to 1/15 \to 1/16$.

Alternative solution: This family of functions has an interesting property:

$(f_a \circ f_b)(x) = f_{a+b}(x)$, verified by

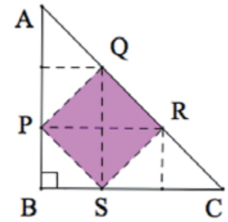
$$(f_a \circ f_b)(x) = \frac{x/(1 + bx)}{1 + a(x/(1 + bx))} = \frac{x}{1 + (a + b)x}.$$

So

$$(f_1 \circ f_2 \circ f_3 \circ f_4 \circ f_5)(x) = f_{15}(x) = x/(1 + 15x)$$

and $f_{15}(1) = 1/16$.

30. 4/9. The figure provides a proof without words.



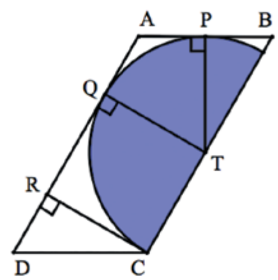
Alternative solution: Both the proof without words above and the computational solution that follows require us to recognize that $RC = QR$, because both equal RS, and $AQ = QR$, because both equal QP. Let $AB = BC = x$. Then $AC = \sqrt{2}x$, and $QR = \sqrt{2}x/3$. We have the area of square $PQRS = 2x^2/9$ and the area of $\triangle ABC = x^2/2$. Therefore,

$$\frac{\text{area of square } PQRS}{\text{area of } \triangle ABC} = \frac{2x^2/9}{x^2/2} = \frac{4}{9}.$$

31. $36\sqrt{3}/(2 + \sqrt{3})$ or $72\sqrt{3} - 108$. Let the radius of the semicircle be r inches, let T denote the center of the semicircle, and let the circle be tangent to segment AB at point P. The following must be true from the properties of $\triangle BPT$:

$$\frac{r}{6 - r} = \sin 60° = \frac{\sqrt{3}}{2},$$

implying that $r = 6\sqrt{3}/(2 + \sqrt{3})$. Because TQ is the height of the parallelogram using "base" AD, the area is $AD \cdot TQ = 6 \cdot r = 36\sqrt{3}/(2 + \sqrt{3})$ or $72\sqrt{3} - 108$.





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