Mediants Make (Number) Sense of Fraction Foibles

Enhance students' number sense and illustrate some surprising properties of this alternative operation.

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By the time they reach middle school, all students have been taught to add fractions. However, not all have *learned* to add fractions. The common mistake, of course, is to report that $a/b + c/d$ is equal to $(a + c)/(b + d)$. Teachers' knee-jerk reaction to this claim is to purge it from our students' minds. Perhaps we might illustrate that 1 + 1 would be equal to 1 if fractions were added this way: 1 + 1 = 1/1 + 1/1 = 2/2 = 1. Or we might indicate that the sum of 1/4 and 3/4 would be a value between these two fractions (i.e., 4/8 or 1/2), whereas the sum of two positive numbers must necessarily be greater than each.

It is certainly necessary to correct this mistake when a student makes it. However, this occasion also presents a valuable opportunity to enhance the student's mathematical confidence while also strengthening her number sense. Rather than placing all the emphasis on why her formula produces wrong answers, we should also acknowledge that she has "invented" an operation that is both useful and interesting. Regardless of her mathematical background, she will be proud and intrigued to learn that she has stumbled upon an operation that has utility in geometry, statistics, calculus, and other areas. In this article, we offer examples and activities that can be used to strengthen weaker

students' basic numerical skills while honing the problem-solving abilities of the best and brightest. All students will be fascinated to see what can be reaped from examining this "mistake" in its proper contexts.

COURSE CONTEXT FOR THE MEDIANT

To distinguish this operation from ordinary addition of fractions, we will use the notation $\oplus$ to mean the operation defined by

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}.$$

Formally, $(a + c)/(b + d)$ is called the *mediant* of a/b and c/d. It has been studied by many mathematicians for many years. Indeed, references to the mediant are found in the writings of Plato as far back as 325 BCE; a thorough investigation of the history and study of the mediant can be found in Guthery (2011). Let's explore some applications of this operation in the settings of several mathematical areas.

Statistics

Suppose that a class consists of 12 boys and 18 girls. If 7 of the boys have brown hair and 8 of the girls have brown hair, then the ratios of brown-haired

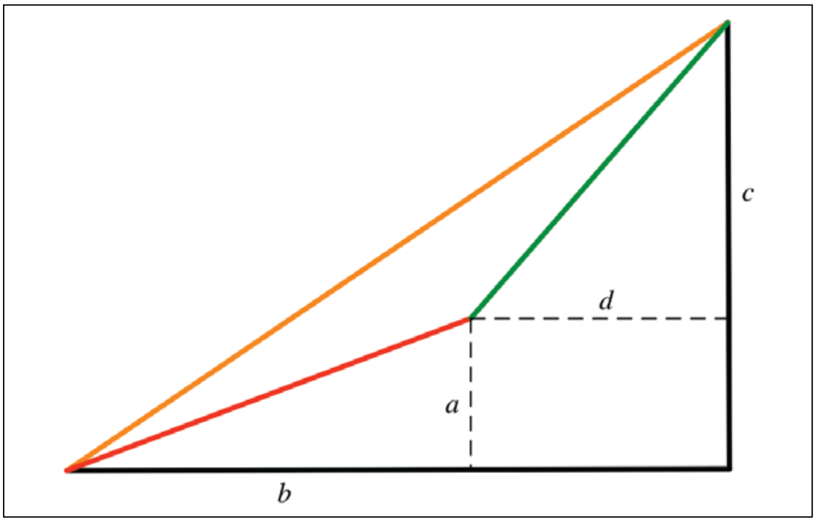

Fig. 1 The mediant is related to slopes. (Figure based on Gibbs 1990.)

boys to all boys and brown-haired girls to all girls are 7/12 and 8/18, respectively. Because there are 15 brown-haired students in this class of 30, the ratio of brown-haired students to all students is 15/30. But 15/30 is exactly $7/12 \oplus 8/18$, which is the mediant of the two ratios.

Algebra

We generally use (a, b) to denote a vector with horizontal component a and vertical component b. The sum of vectors (a, b) and (c, d) is $(a + c, b + d)$. So if we choose to represent vectors as fractions rather than as ordered pairs, then the sum of vectors a/b and c/d is $a/b \oplus c/d$.

Coordinate Geometry

Consider the red line segment in **figure 1** with positive slope a/b and the green segment with steeper positive slope c/d. As the diagram suggests, the orange line segment with slope $a/b \oplus c/d$ is necessarily steeper than the red segment and less steep than the green. As we will explore later, this will always be the case. Specifically, if a, b, c, and d are all positive and $a/b < c/d$, then it will always be true that $a/b < a/b \oplus c/d < c/d$.

Geometry

Suppose that we have two rectangles, one with area A and width w and another with area B and the same width w. (Assume that area and length are measured in square inches and inches, respectively.) Then the heights of these two rectangles are $h_1 = A/w$ and $h_2 = B/w$. Observe that

$$\frac{A}{w} \oplus \frac{B}{w} = \frac{A+B}{w+w} = \frac{h_1 w + h_2 w}{2w} = \frac{(h_1 + h_2)w}{2w} = \frac{h_1 + h_2}{2},$$

which is the ordinary mean of the heights h_1 and h_2. This observation motivates us to define the average height of n rectangles with areas A_i and various widths w_i as

$$\frac{A_1}{w_1} \oplus \frac{A_2}{w_2} \oplus \cdots \oplus \frac{A_n}{w_n}.$$

Geometrically, this is the height of a rectangle with

$$\text{area } \sum_{i=1}^{n} A_i \text{ and width } \sum_{i=1}^{n} w_i.$$

As we will see, this definition for the average height of rectangles is a precursor to the usual notion of the average value of a function that we find in calculus.

Calculus

If f is a real-valued function defined on an interval $[a, b]$, then the average value of f is defined as the (signed) height of the rectangle whose width is $b - a$ and whose area is equal to the (signed) area between the graph of f and the x-axis. If

$$\sum_{i=1}^{n} f(x_i)\Delta x_i$$

is a Riemann sum of f on $[a, b]$ that approximates this area, then the average value of f on $[a, b]$ is approximated as the average (signed) height of n rectangles with (signed) areas $f(x_i)\Delta x_i$ and widths Δx_i for $i = 1, \ldots, n$. As we saw in the geometry section above, this is exactly equal to

$$\frac{f(x_1)\Delta x_1}{\Delta x_1} \oplus \frac{f(x_2)\Delta x_2}{\Delta x_2} \oplus \cdots \oplus \frac{f(x_n)\Delta x_n}{\Delta x_n},$$

which can also be written as

$$\frac{\sum_{i=1}^{n} f(x_i)\Delta x_i}{\sum_{i}^{n} \Delta x_i} = \frac{\sum_{i=1}^{n} f(x_i)\Delta x_i}{b-a}.$$

The limiting value of this expression is

$$\frac{\int_a^b f(x)\,dx}{b-a},$$

which is the standard definition of the average value of f on $[a, b]$.

ACTIVITIES INVOLVING THE MEDIANT

The mediant has a number of interesting properties, some of which are illustrated in the activities that are outlined in this section. Each activity is aligned with one or more of the Common Core

State Standards for Mathematics (CCSSI 2010). The labels, numbers 1–8, will be used to identify the Common Core standards that are addressed by each activity.

1. Develop understanding of fractions as numbers. (Grade 3)
2. Extend understanding of fraction equivalence and ordering. (Grade 4)
3. Understand decimal notation for fractions and compare decimal fractions. (Grade 4)
4. Analyze and solve linear equations and pairs of simultaneous linear equations. (Grade 8)
5. Solve systems of equations. (High School: Algebra)
6. Perform arithmetic with polynomials and rational expressions. (High School: Algebra)
7. See structure in expressions. (High School: Algebra)
8. Reason with equations and inequalities. (High School: Algebra)

All the activities that are described in this section can be enjoyed by any student who has mastered standards 1–4 above. For some of these activities, extensions are suggested that address standards 5–8. Therefore, these activities should be accessible to all students in the upper grades, and the extensions should offer a sufficient degree of challenge to keep even the most talented students engaged.

ACTIVITY 1: FRIENDLY NEIGHBORS

Standards 1, 2, and 3

Ask students to write down all the fractions with values between 0 and 1 (inclusive) that have a denominator of 6 or less. Only fractions in lowest terms are to be included. Have students share their answers with one another until everyone is sure that his or her list is complete. Then have students use calculators to represent the value of each fraction in decimal form. Use these decimal representations to order the fractions from least to greatest. This ordered list (with decimal approximations included) is provided in **table 1**.

Once you have verified that the students' lists are correct, point out that 2/5 is the mediant of its nearest neighbors, 1/3 and 1/2. Ask whether there are any other fractions on the lists with this property. Students should quickly identify 1/6, 3/5, and 5/6 as being the mediants of their neighbors. If no one identifies any others, ask students what $3/4 \oplus 5/6$ is equal to in lowest terms. Once they realize that the value of $3/4 \oplus 5/6$ is equivalent to 4/5, it will not be long until they have verified that every fraction in the list (except for 0/1 and 1/1) is equivalent to the mediant of its neighbors.

Extension 1

Perform this activity exactly as described, but without using a calculator.

Extension 2

We used denominators of 6 or less in this activity only because the list produced seems about the right length for a classroom activity. However, the property of mediants that is illustrated above holds regardless of the greatest denominator used. Ask students to discover whether this property holds for denominators of 7 or less. (They can begin with the list already created and insert 1/7, 2/7, and so on into the appropriate locations.) Challenge students to verify this "neighboring" property for fractions with denominators of 10 or less.

She will be proud and intrigued to learn that she has stumbled upon an operation that has utility in geometry, statistics, calculus, and other areas.

ACTIVITY 2: WHERE'S THE MEDIANT?

Standards 1, 2, 6, 7, and 8

Choose any four positive integers a, b, c, and d and ask your students to calculate $a/b \oplus c/d$. Then have them locate a/b, c/d, and $a/b \oplus c/d$ on a number line and write a number sentence that describes their numerical order using the less-than (<) sign properly. Have them repeat this exercise multiple times with different collections of four integers. Students should eventually recognize that the mediant always lies between a/b and c/d. (This fact was

Table 1 An Ordered List of All Fractions between 0 and 1 with Denominator 6 or Less

0/1	1/6	1/5	1/4	1/3	2/5	1/2	3/5	2/3	3/4	4/5	5/6	1/1
0	0.16	0.20	0.25	0.33	0.40	0.50	0.60	0.66	0.75	0.80	0.83	1

illustrated earlier in the example from coordinate geometry.)

Extension 1

This is actually an extension of activities 1 and 2 combined. Explain that the "betweenness" property of mediants can be used to build lists of rational numbers—in order from least to greatest—that can be represented as fractions with denominator n or less. Have students construct a line segment representing the interval [0, 1] that extends along the full length of a piece of paper and ask them to write down 0/1 above 0 at the far left and 1/1 above 1 at the far right. Calculate the mediant (1/2) and write it in its appropriate location between 0/1 and 1/1. Continue to include mediants of adjacent fractions (always reducing to lowest terms) until all adjacent fractions have mediants (in lowest terms) with denominator greater than n. When students think that they have finished, ask whether they can think of any fraction (in lowest terms) with denominator n or less that does not appear on their list. (If any exist, then there are still adjacent fractions whose mediant can be represented as a fraction with denominator n or less.) Once the lists are complete, have students observe that each fraction that they have written (other than 0/1 and 1/1) is the mediant of its neighbors.

Extension 2

For students with a stronger mathematical background, suggest that they try to prove that if $a/b < c/d$ for positive integers a, b, c, and d, then it is necessarily true that

$$\frac{a}{b} < \frac{a}{b} \oplus \frac{c}{d} < \frac{c}{d}.$$

One approach is to show that $(a/b \oplus c/d) - (a/b)$ and $(c/d) - (a/b \oplus c/d)$ are both positive. Note that

$$\begin{aligned}\left(\frac{a}{b} \oplus \frac{c}{d}\right) - \left(\frac{a}{b}\right) &= \left(\frac{a+c}{b+d}\right) - \left(\frac{a}{b}\right) \\ &= \frac{b(a+c) - a(b+d)}{b(b+d)} \\ &= \frac{bc - ad}{b(b+d)}.\end{aligned}$$

We already know that $b(b + d) > 0$ because the integers were chosen to be positive. To demonstrate that the numerator, $bc - ad$, is positive as well, write

$$bc - ad = \frac{bc - ad}{bd} \bullet bd = \left(\frac{c}{d} - \frac{a}{b}\right) \bullet bd.$$

Because both these factors are positive (recall that a/b is less than c/d by assumption), it follows that $bc - ad > 0$. With both the numerator and denominator positive, we conclude that the equivalent value $(a/b \oplus c/d) - (a/b)$ is positive. Showing that $(c/d) - (a/b \oplus c/d)$ is positive can be done similarly.

ACTIVITY 3: RECKONING RATIOS
Standard 2

Begin this activity by demonstrating how the mediant of two ratios can be used to represent a related third ratio, as we did earlier with the 8 brown-haired girls (out of 18 girls) and the 7 brown-haired boys (out of 12 boys). We found that the ratio of brown-haired students to total students in the classroom was $8/18 \oplus 7/12$, which is equal to 1/2. Point out that 8/18 is numerically equivalent to 4/9 but that the interpretation of the problem would be different if 4/9 were used in place of 8/18 (even though the percentage of brown-haired girls out of all girls would be unchanged). Ask students to consider what this difference would be. (Answer: There are now only 9 girls in the class, and 4 of them have brown hair.) Using the mediant, find the ratio of brown-haired students to total students in this new classroom. When they discover that $4/9 \oplus 7/12 = 11/21$, point out that this value is slightly more than 1/2. Experiment with other fractions that are numerically equivalent to either 8/18 or 7/12 (or both) to see how the resulting ratios change even when the percentages of brown-haired girls and brown-haired boys remain constant.

This activity illustrates that although $\oplus$ is an operation on the set of fractions (with positive denominators), it is not an operation on the set of rational numbers. Recall that an operation is a rule that assigns any pair of elements of a set to a unique member of that set. Because 4/9 and 8/18 are identical as rational numbers, any operation $*$ on the set of rational numbers would have to assign $(4/9) * (7/12)$ and $(8/18) * (7/12)$ to the same thing. However, $4/9 \oplus 7/12$ and $8/18 \oplus 7/12$ are not equal. This idea is illustrated in the next activity as well.

ACTIVITY 4: TARGET PRACTICE
Standards 2, 3, 4, and 5

Choose any four positive integers a, b, c, and d and pick any rational number that lies strictly between a/b and c/d. For example, 0.465 lies between 2/5 and 3/4. Now try to find fractions that are numerically equivalent to a/b and c/d whose mediant is equal to the chosen rational number. Continuing with our example, we see that $2/5 \oplus 3/4$ is equal to 5/9, or 0.555. . . . Too big. And $2/5 \oplus 6/8$ is about 0.615, which is bigger still. However, $4/10 \oplus 3/4$ is 0.5, which is heading in the right direction. And

Although ⊕ is an operation on the set of fractions (with positive denominators), it is not an operation on the set of rational numbers.

8/20 ⊕ 3/4 is just a bit larger than 0.458. Let your students continue to search the (literally) endless combinations for a while before comparing answers. Then ask them to decide whose answer is closest to the target. Identify a winner and celebrate everyone's good work.

Extension 1

Although some students might approach this activity by selecting seemingly random equivalent fractions each time, others might develop a strategy that seems to produce better and better guesses. For example, when a guess is too large, multiplying the numerator and denominator of the lesser fraction by a common integer yields a mediant that is smaller (4/10 ⊕ 3/4 is less than 2/5 ⊕ 3/4). Similarly, multiplying the numerator and denominator of the larger fraction by a common integer yields a greater mediant. Watch for this and other potential strategies as students are working. At the conclusion of the activity, ask them to share their strategies.

Extension 2

It turns out that for any positive rational number p/q that lies strictly between a/b and c/d, there are fractions that are equivalent to a/b and c/d whose mediant is equivalent to p/q. Some stronger students might be able to prove specific cases of this fact if they are adept at solving simultaneous

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equations. We illustrate one possible proof for the example given above.

To show that there are fractions equivalent to 2/5 and 3/4 whose mediant is equivalent to 0.465, find integers x and y for which

$$\frac{2x}{5x} \oplus \frac{3y}{4y} = \frac{465}{1000}.$$

This is equivalent to solving the system

$$2x + 3y = 465$$
$$5x + 4y = 1000$$

for x and y. Doing so yields

$$x = \frac{1140}{7}$$
$$y = \frac{325}{7}.$$

Therefore, we have that

$$2\left(\frac{1140}{7}\right) + 3\left(\frac{325}{7}\right) = 465$$
$$5\left(\frac{1140}{7}\right) + 4\left(\frac{325}{7}\right) = 1000.$$

However, this is not quite what we were looking for: We were seeking *integers* that satisfied the system of equations. By multiplying both sides of the equations above by 7, we arrive at

$$2(1140) + 3(325) = 3255$$
$$5(1140) + 4(325) = 7000.$$

So it follows that

$$\frac{2(1140)}{5(1140)} \oplus \frac{3(325)}{4(325)} = \frac{3255}{7000} = \frac{465}{1000} = 0.465,$$

as desired.

EXPLORE THE "ERROR"

Playing with mediants is fun. Their properties are surprising, interesting, and engaging. Moreover, they offer ample opportunity to enhance the number sense of students at every level of ability. So when your students inevitably confuse $a/b + c/d$ with $a/b \oplus c/d$, introduce them to one or more of these activities. On second thought, why wait? By exploring properties of the mediant, students will have a concrete understanding of what this "error" really produces, and this deeper understanding is likely to encourage them to add fractions properly in the future.

REFERENCES

Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf

Gibbs, Richard A. 1990. "Proof without Words: The Mediant Property." *Mathematics Magazine* 63 (3): 172.

Guthery, Scott B. 2011. *A Motif of Mathematics*. Boston: Docent Press.

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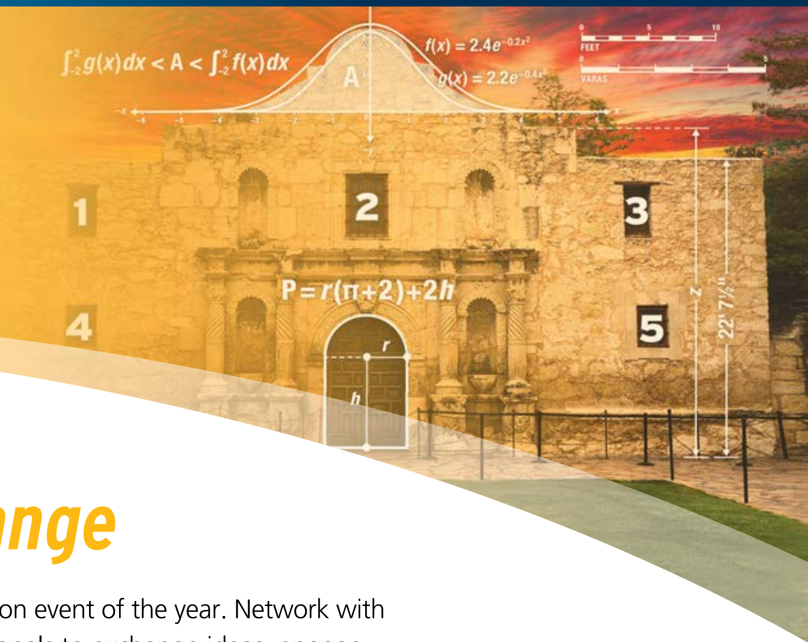

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