SOLUTIONS to calendar

1. 1680. The number of permutations of 8 letters taken 4 at a time, written ${}_8P_4$, is calculated by $8!/(8-4)! = 8!/4! = 8 \bullet 7 \bullet 6 \bullet 5 = 1680$.

Alternative solution. There are 8 choices for the first letter, 7 choices for the second, 6 for the third, and 5 for the last. The fundamental counting principle says these choices should be multiplied: $8 \bullet 7 \bullet 6 \bullet 5 = 1680$.

2. Triangle ABC is isosceles with $AB = BC$, and the measure of its vertex angle is 108°. Therefore, the base angles, $\angle BAC$ and $\angle BCA$, have measure 36°. The same can be said for $\triangle EAB$, which means that $\angle ABP$ also has measure 36°. Two angles of $\triangle APB$, are congruent to two angles of $\triangle ABC$, which is sufficient to show the triangles are similar.

3. $\left(-1+\sqrt{5}\right)/2$. We have established (in the solution for problem 2) that $m\angle BAP = m\angle ABP = 36°$, which implies $m\angle BPQ = 72°$. We can use symmetry (or the fact that $\triangle APB \cong \triangle CQB$) to show $m\angle BQP = 72°$, also. Now $m\angle ABQ = 180° - 36° - 72° = 72°$, so $\triangle ABQ$ is isosceles with $AB = AQ = AP + PQ$. It can be shown that $\triangle APB \sim \triangle ABC$ (see Oct. 2), so $(AP/AB) = (AB/AC)$. The perimeter of $ABCDE$ is 5, so $AB = 1$. Substituting this value into the proportion yields $AP \bullet AC = 1$, or $AP(AP + AP + PQ) = AP(AP + 1) = 1$. Solve the quadratic $AP^2 + AP - 1 = 0$ for AP:

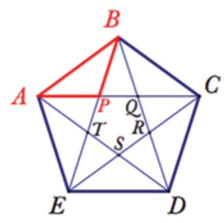
$$\frac{-1 \pm \sqrt{1-4(-1)}}{2}$$

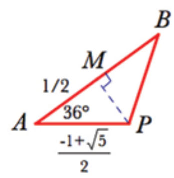
Reject the negative solution.

4. $\left(1+\sqrt{5}\right)/4$. Using the figure from Oct. 2, we have isosceles $\triangle ABP$ with $m\angle A = m\angle B = 36°$, $AB = 1$, and $AP = \left(-1+\sqrt{5}\right)/2$. Construct the altitude from P to $\overline{AB}$, as shown. Triangle AMP is a right triangle with $AM = 1/2$. So

$$\cos 36° = \frac{1}{2} \bullet \frac{2}{-1+\sqrt{5}} = \frac{1}{-1+\sqrt{5}}.$$

Rationalizing the denominator results in $\cos 36° = \left(1+\sqrt{5}\right)/4$.





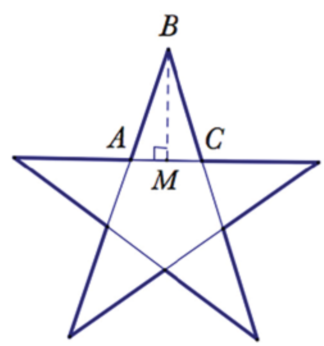
5. $5+5\sqrt{5}$. The star's perimeter is 10 times the length of $\overline{AB}$. This segment is one leg of an isosceles triangle. Each base angle has measure 72° because each is an exterior angle of the original pentagon. Construct the altitude $\overline{BM}$ from B to $\overline{AC}$. We know $AM = 1/2$, because $AC = 1$. To find AB we have $AB = (1/2)/\cos 72°$. We have shown (in the solution to problem 4) that $\cos 36° = \left(1+\sqrt{5}\right)/4°$. Then

$$\cos(2 \bullet 36°) = 2\cos^2 36° - 1$$
$$= 2\left(\frac{1+\sqrt{5}}{4}\right)^2 - 1 = \frac{6+2\sqrt{5}}{8} - 1$$
$$= \frac{-1+\sqrt{5}}{4}.$$

So

$$AB = \frac{1}{2} \bullet \frac{4}{-1+\sqrt{5}},$$

which, after rationalizing the denominator, equals $\left(1+\sqrt{5}\right)/2$. The perimeter of the star is $10 \bullet AB = 5+5\sqrt{5}$.



Problems 8–14 were contributed by Melvin Peralta, a Math for America Fellow who teaches in New York City.

Problems 27–29 were submitted by Charles Kicey and Arsalan Wares, who teach mathematics at Valdosta State University (Georgia), which hosts an annual High School Mathematics Tournament.

The Editorial Panel of Mathematics Teacher is considering sets of problems submitted by individuals, classes of prospective teachers, and mathematics clubs for publication in the monthly Calendar. Send problems to the Calendar editors. Remember to include a complete solution for each problem submitted.

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6. $\left(a/\sqrt{2},\ a/\sqrt{2}\right)$. Since the figure has 8-fold rotational symmetry, the angle of rotation is 360°/8 = 45°. Point B lies the same distance from the origin as point A, namely, a units. So segment OB is the hypotenuse of an isosceles right triangle. The coordinates of B equal the lengths of the legs of that triangle, $\left(a/\sqrt{2},\ a/\sqrt{2}\right)$.

Alternative solution. Multiply the point matrix

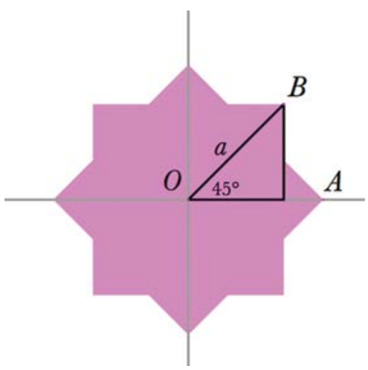
$$\begin{bmatrix} a \\ 0 \end{bmatrix}$$

by the rotation matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

as follows:

$$\begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix}\begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{bmatrix} a\cos 45° + 0 \\ a\sin 45° + 0 \end{bmatrix} = \begin{bmatrix} a/\sqrt{2} \\ a/\sqrt{2} \end{bmatrix}$$



7. $8\sqrt{2}-8 \approx 3.314$. Note the following on the figures: An interior angle of the octagon has measure 135°, so $m\angle XBY = 135°$, and $m\angle XBP = 67.5°$. The sides of the inner octagon adjacent to $\overline{AB}$ intersect at right angles, so $m\angle BXP = 90°$ and $m\angle XPB = 22.5°$. Angle AOB is a central angle of the inner octagon with measure 45°. The remaining angle in $\triangle AOP$ is $\angle OAP$ with measure 112.5°. Before applying the law of sines to $\triangle AOP$, observe that $\sin 112.5° = \sin 67.5° = \cos 22.5°$. Let $r = OA$ be the radius of the inner octagon:

$$\frac{\sin 22.5°}{r} = \frac{\sin 67.5°}{8}$$
$$r = \frac{8\sin 22.5°}{\cos 22.5°}$$

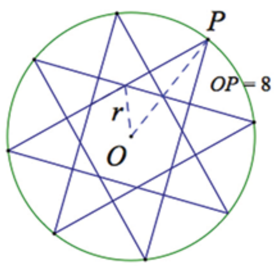
If we wish to approximate now, we obtain $r \approx 3.314$. To obtain an exact value in radical form, use the half-angle formulas:

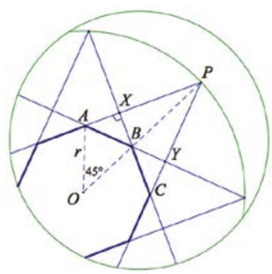
$$r = \frac{8\sqrt{1-\sqrt{2}/2}}{\sqrt{2}} \bullet \frac{\sqrt{2}}{\sqrt{1+\sqrt{2}/2}}$$

Simplifying the expression yields $r = 8\sqrt{3-2\sqrt{2}}$. Note that

$$\left(\sqrt{2}-1\right)^2 = 3-2\sqrt{2},$$

so simplest radical form is $8\sqrt{2}-8$.





8. $\tan 30°$, or $\sqrt{3}/3$. By the angle bisector theorem, $AD/BA = DC/BC$. Since $\overline{BD}$ bisects $\angle ABC$, $\triangle ABD$ is a right triangle with $m\angle ABD = 30°$. Observe that $AD/BA = 1/\sqrt{3} = \sqrt{3}/3$, or $AD/BA = \tan 30°$. Therefore $DC/BC = \tan 30°$.

9. –112,793. The expression contains an arithmetic series, 2 + 5 + 8 + 11 + . . . + 98, and products of pairs of consecutive integers, $-3 \cdot 4 - 6 \cdot 7 - 9 \cdot 10 - \ldots - 99 \cdot 100$. The sum of the arithmetic series is

$$\frac{33(2+98)}{2} = 1650.$$

The products of consecutive integers can be written as

$$\begin{aligned} &-3(3+1)-6(6+1) \\ &\qquad -9(9+1)-\ldots-99(99+1) \\ &= -(3^2+3)-(6^2+6) \\ &\qquad -(9^2+9)-\ldots-(99^2+99) \\ &= -(3^2+6^2+9^2+\ldots+99^2) \\ &\qquad -(3+6+9+\ldots+99) \end{aligned}$$

separating the sum of the squares of multiples of 3 and the sum of multiples of 3. (We actually need the additive inverse of each sum.)

The sum of the multiples of 3 is another arithmetic series:

$$\frac{33(3+99)}{2} = 1683$$

The formula for sums of consecutive squares from 1 to n is

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Replace k with $3k$ to account for multiples of 3:

$$\sum_{k=1}^{33} (3k)^2 = 9\sum_{k=1}^{33} k^2 = \frac{9(33)(34)(67)}{6} = 112761$$

Combining these parts of the sum gives us 1650 – 1683 – 112761 = –112794. We add 1 to account for the initial 1 in the expression to obtain –112793 for the value of the expression.

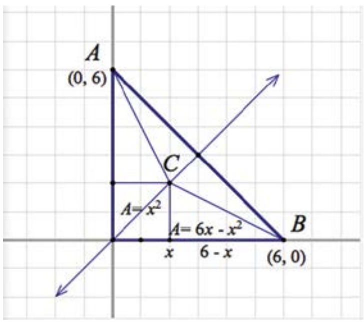
10. 155. Numbers can be written as the product of two consecutive integers if they can be written as $n(n+1)$. Letting $n = 1$, 2, and 3 we find the first three integers that are the product of consecutive natural numbers: $1 \cdot 2 = 2$, $2 \cdot 3 = 6$, and $3 \cdot 4 = 12$. Since there are 3 such numbers less than or equal to 12 (and greater than 1), there must be 12 – 3 – 1 = 8 that are not the product of consecutive positive integers. (In other words, the 8th integer greater than 1 that cannot be written as the product of two consecutive positive integers is $3 \cdot 4 - 1 = 11$.) Generalizing, we need $n(n+1) - n - 1 = n^2 - 1 = 143$. Therefore $n = 12$, and $12 \cdot 13 - 1 = 155$ is

the 143rd integer greater than one that cannot be written as the product of two consecutive positive integers.

11. (2, 2) or (4, 4). Base $\overline{AB}$ has length $6\sqrt{2}$ and slope –1. This is easily seen if one plots the two points and realizes that the diagonal in a unit square has length $\sqrt{2}$. Point C lies on the perpendicular bisector of $\overline{AB}$, the line with slope 1 that contains (3, 3), the midpoint of $\overline{AB}$. The area of $\triangle ABC$ is

$$\begin{aligned} 6 &= (0.5)6\sqrt{2} \bullet h \\ &= 3\sqrt{2} \bullet h. \end{aligned}$$

So $h = \sqrt{2}$, which implies point C is exactly 1 unit square diagonal from (3, 3). Both (2, 2) and (4, 4) satisfy the conditions.



Alternative solution. The vertex, C, must have coordinates (x, x). The area of $\triangle ABO$, where O is the origin, is 18. We must subtract a region of area 12. Assume first that $0 < x < 3$. Remove the square, x^2, and the two triangles that have a combined area of $x(6 - x)$. That is,

$$x^2 + x(6 - x) = 12.$$

Then $6x = 12$, or $x = 2$, so the coordinates of C are (2, 2). The other possible position for C is the reflection in $\overleftrightarrow{AB}$, namely, (4, 4).

12. 9. The number of triangles pointing up in any row is equal to the row number. Since there are nine rows in a composite triangle with 81 triangles, the number of triangles pointing up is $1 + 2 + 3 + \ldots + 9 = 9(9 + 1)/2 = 45$. The number of triangles pointing down in any row is one less than the row number, so the number of triangles pointing down is $1 + 2 + 3 + \ldots + 8 = 8(8 + 1)/2 = 36$. So there are $45 - 36 = 9$ more triangles that point up.

Alternative solution I. Every triangle pointing up shares its base with a triangle pointing down, except for the last row of triangles. The number of triangles in the last row is the square root of the number of interior triangles since the sum of the first n odd numbers is n^2. The square root of 81 is 9.

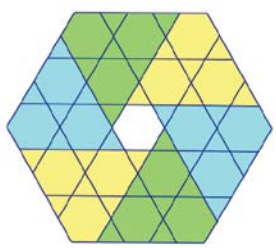
Alternative solution II. To create a large triangle with 81 interior triangles, arrange 9 copies of the triangle with 9 interior triangles so that 6 face up and 3 face down. The "up–down" difference for the first 6 is $6(6 - 3) = 18$, while the "up–down" difference for the remaining 3 is $3(3 - 6) = -9$. The sum of the differences is 9.

13. hexagons: $3k^2/4 + 3k/2 + 1$; triangles: $3k^2/2 + 3k$. Every hexagon can be divided into the six trapezoids as shown in the figure, with one hexagon remaining in the center. Each of these trapezoids contains $1 + 2 + 3 + \ldots + n = n(n + 1)/2$ hexagons, where $n = k/2$. So the number of hexagons is

$$\begin{aligned} 6\left(\frac{n(n+1)}{2}\right)+1 &= \frac{3k}{2}\left(\frac{k}{2}+1\right)+1 \\ &= 3k^2/4+3k/2+1. \end{aligned}$$

Each trapezoid contains $2 + 4 + 6 + \ldots + k$ triangles:

$$6\left(\frac{k(k+2)/2}{2}\right)=\frac{3k^2}{2}+3k$$



14. A + B = 10. The digit sum of any number is congruent to itself modulo 9, and numbers that are congruent modulo 9 have the same remainder when divided by 9. The sum $7 + 1 + 5 + 8 + 3 = 24$ is congruent to 6 mod 9. We need to find A + B such that (A + B) + 6 is congruent to 7 mod 9. Although 6 + 1 is congruent to 7 mod 9, A + B must be at least 2. So A + B must be 10, since 6 + 10 = 16 is congruent to 7 mod 9.

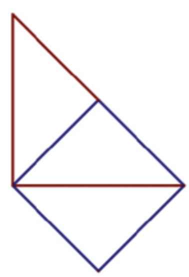
Alternative solution. Since the remainder is 7, if we add 2 to the original number the result is a number divisible by 9. So 7158AB5 is a multiple of 9. Since $7 + 1 + 5 + 8 + 5 = 26$, we must have A + B equal to 1 more than a multiple of 9, that is, 1, 10, 19, etc. But A and B are single-digit positive integers, so the only possible sum is 10.

15. Cut 5 ft. from an end to get a piece for the square. One way to approach this problem begins by choosing an arbitrary area, say $A = 1$ ft^2. Then the square's perimeter is 4 ft. The two legs of the isosceles right triangle must be $\sqrt{2}$ ft. because $0.5\left(\sqrt{2}\right)^2 = 1$. The hypotenuse must be 2, so the triangle's perimeter is $2 + 2\sqrt{2}$, or approximately 4.8 ft. We need a total of 8.8 ft., but we have 11 ft. to use, so apply a scale factor of 11/8.8 = 1.25. Cut (5/4) • (4) = 5 ft. from an end. We confirm that $(5/4)\left(2+2\sqrt{2}\right)$ uses the rest of the wire: $5/2 + 5\sqrt{2}/2$, or approximately 6.03 ft. (The difference is due to rounding.)

Alternative solution. The square (blue) and the large isosceles right triangle (red) shown have the same area. If a side length of the square is x, the hypotenuse of the triangle is $2x$ and each leg is $x\sqrt{2}$. Therefore $6x + 2x\sqrt{2} = 11$. Solve to find

$$\begin{aligned} x &= 11\left(6-2\sqrt{2}\right)/28 \\ &\approx 1.2459751. \end{aligned}$$

Cut off $4x \approx 4.98$ ft. for the square.



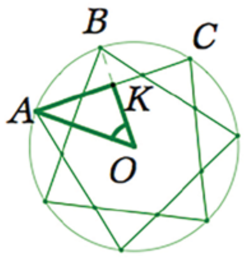
16. We are given that $PA = PO$, so $\triangle APO$ is isosceles with $\angle A$ as one of the two base angles. We also have radii $OA = OB$, which implies $\triangle AOB$ is isosceles with $\angle A$ again as one of the two base angles. Two angles of $\triangle AOB$ are congruent to two angles of $\triangle APO$ so the two triangles are similar.

17. Since $PO = PA$, $\triangle POA$ is isosceles, with $\angle A$ as one of the two base angles. Radii $\overline{OA}$ and $\overline{OB}$ are congruent, which means $\triangle BOA$ is also isosceles with $\angle A$ as one of the two base angles. Since the base angles of $\triangle POA$ are congruent to the base angles of $\triangle BOA$, the two triangles are similar.

18. First method: Begin with a regular heptagon. (While a regular heptagon cannot be constructed using straightedge and compass, use a protractor to obtain a fairly accurate figure.) Then extend all 7 sides in both directions until they intersect each other to form the "star points."

Second method: Begin with 7 points, equally spaced apart on a circle. Suppose the points are numbered consecutively. Then connect them in order: 1–3–5–7–2–4–6–1. (Connect every second point until you reach your starting point.)

19. 10.95. The star points are equally spaced on the circumference, so $AB = BC$. Radii $\overline{OA}$ and $\overline{OC}$ are congruent. Since diagonals of a kite are perpendicular, $\triangle AOK$ is a right triangle, and central angle AOB, with measure $360°/7$, is one of its acute angles. Since we know the measure of $\angle AOB$ and the length of hypotenuse AO, we can find the length of $\overline{AK}$ as $\sin(360°/7) = AK/1$. The length of the path that creates the figure is $14 \cdot AK = 14 \sin(360°/7) \approx 10.95$.



$m\angle AOB = 360°/7$

20. Use $m\angle A = 20°$ to find the measures of all four interior triangles as shown in the figure. Consider $\triangle PBC$. Let $CB = PB = 1$. Half the vertex angle is $10°$, so $PC/2 = \sin 10°$ and the altitude from B is $\cos 10°$. The area of $\triangle PBC$ is therefore $\sin 10° \cdot \cos 10°$. Similarly, the area of $\triangle AQR = \sin 70° \cdot \cos 70°$, and the area of $\triangle PQR = \sin 50° \cdot \cos 50°$. We must show

$$\sin 10° \cdot \cos 10° + \sin 70° \cdot \cos 70° \\ = \sin 50° \cdot \cos 50°.$$

The product-to-sum formula may be used here:

$$\sin a \cdot \cos b = (1/2)[\sin(a+b) + \sin(a-b)]$$

Rewriting both sides of the equation yields

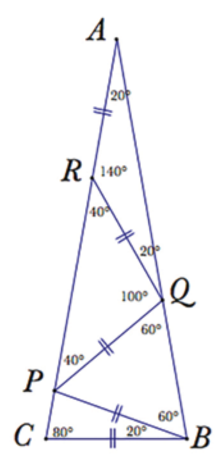
$$(1/2)[\sin 20° + \sin 0] \\ +(1/2)[\sin 140° + \sin 0] \\ = (1/2)[\sin 100° + \sin 0].$$

Using supplementary angles, this equation can be further simplified to

$$(1/2)[\sin 20° + \sin 40°] = (1/2)\sin 80°.$$

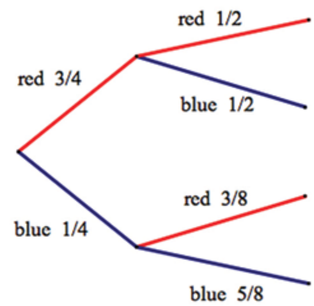
Now use the sum-to-product formula and the co-function theorem to rewrite

$$\frac{1}{2}\left(2\sin\frac{40°+20°}{2} \cdot \cos\frac{40°-20°}{2}\right) \\ = \frac{1}{2}\left(2\sin 30° \cdot \cos 10°\right) \\ = (1/2)\cos 10° \\ = (1/2)\sin 80°.$$



21. 1/5. If a red marble is transferred from the first pouch to the second, the distribution in the second pouch will be 4 red, 4 blue. If a blue marble is transferred, the distribution will be 3 red, 5 blue. We can use a probability tree to shown the color of the marble selected on each draw.

Since the second marble drawn is red, the sequence of draws is either RR or BR. Those outcomes have probability $(3/4)(1/2) = 3/8$ and $(1/4)(3/8) = 3/32$. The conditional probability of blue on first draw given red on second is thus $(3/32)/(3/8 + 3/32) = 1/5$.



22. median from point C. The sum of the perimeters of the two triangles created by a median equals the perimeter of the original triangle plus twice the median. Therefore, we need the shortest median. The shortest median of a triangle extends from a vertex to the midpoint of the longest side. Since $AB = \sqrt{12^2 + 5^2} = 13$, $AC = \sqrt{3^2 + 4^2} = 5$, and $BC = \sqrt{9^2 + 1^2} \approx 9.06$, side AB is the longest. The median from point C will create the triangles with the smallest perimeter sum.

23. Both figures are star polygons with 7 star points and both can be drawn as a path that visits each of 7 points on the circumference of a circle exactly once. Both figures contain a regular heptagon; the polygon featured on October 18 appears in its entirety as part of the figure that accompanies today's problem.

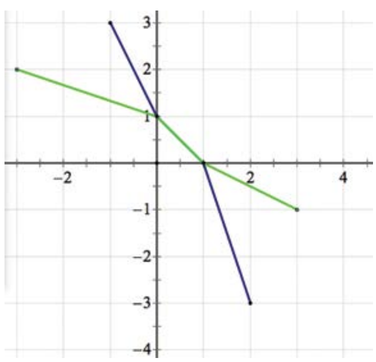
The October 18 star polygon is created by connecting every second point until one returns to the first point, while the October 23 star polygon is created by connecting every third point until one returns to the first point. If the radius of the circle that contains the star points of

today's figure is 1 unit, then the length of the path that forms the polygon is

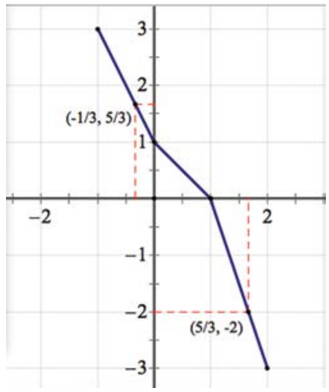
$$7\sqrt{2-2\cos(154\tfrac{2}{7}°)} \approx 7(1.9499)$$
$$= 13.65 \text{ units.}$$

The angle sum of the star points of the October 18 polygon is 540°, while the angle sum of the star points of the October 23 polygon is 180°. The editors encourage readers to find and compare additional characteristics of the polygons.

24. $0 \le x \le 1$. If $f(g(x)) = x$ for all x in the domain of g such that $g(x)$ is in the domain of f, then g and f are inverses. Although f is not its own inverse, we would like to find some subset of the domain of f such that f and f^{-1} coincide. The figure shows the reflection of $f(x)$ (in green) over the line $y = x$. Observe that for $0 \le x \le 1$ we have $f(f(x)) = x$; for example $f(0) = 1$ and $f(1) = 0$, so $f(f(0)) = 0$.



25. –1/3. When the value of the function is –2, we have $x = 5/3$. Now let the value of the function be 5/3; then $x = -1/3$.



26. Total area enclosed increases by about 0.2 ft.^{2} The piece of wire intended for the triangle—a 6-ft. length—was used for a square. Its area is $(6/4)^2 = 9/4$ ft.^{2} The 5-ft. length was bent in the ratio 1:1:1.4. The sum of the parts is 3.4, so the triangle had side lengths 5(1)/3.4, 5(1)/3.4, and 5(1.4)/3.4, or approximately 1.47 ft., 1.47 ft., and 2.06 ft. The area is $(1.47)^2/2 = 1.08$ ft.^{2} The total area enclosed by the two pieces of wire is now 3.33 ft.^{2} If Eve had used the correct piece of wire for the shapes, each would have enclosed the same area, namely $(5/4)^2$; so their combined area would have been 3.125 ft.^{2}, approximately 0.2 ft.^{2} less.

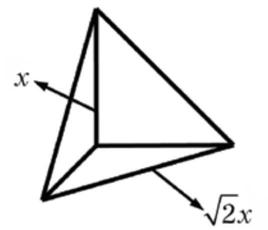
27. decrease of $108 - 36\sqrt{3}$ units^2. Consider one of the eight cut-off pyramids. Let the length of each of three perpendicular edges of the pyramid be x. Then each of the three edges of the pyramid that form the equilateral face of the triangle must be $\sqrt{2}x$. The volume of a removed pyramid is one-third the area of the base times the height, or $x^3/6$. The total volume of these eight removed pyramids must be $4x^3/3$. Therefore, $4x^3/3 = 36$, and hence $x = 3$.

The formula for the area of an equilateral triangle, given a side length s, is $s^2\sqrt{3}/4$; therefore, the sum of the areas of the eight equilateral triangles that became a part of the surface area of the truncated solid must equal

$$8\left(\sqrt{2}x\right)^2 \bullet \sqrt{3}/4 = 2 \cdot 2x^2 \bullet \sqrt{3}$$
$$= 4\sqrt{3}x^2.$$

The total area of the twenty-four isosceles right triangles that were removed from the surface of the original cube must be equal to $12x^2$. Comparing the surface area added to the surface area removed, and using $x = 3$,we see that there is a net decrease of

$$12x^2 - 4\sqrt{3}x^2 = 108 - 36\sqrt{3}.$$



28. 13/27. Imagine a large 3 × 3 × 3 cube made out of 27 unit cubes. Of the 27 unit cubes, 1 is completely hidden and cannot be seen, and each of the remaining 26 cubes has one, two, or three of its faces exposed. Three faces are exposed on 8 of the 27 unit cubes (at the 8 corners of the large cube), 12 of the 27 unit cubes have two faces exposed (on the edges of the large cube), and 6 of the 27 cubes have 1 face exposed (in the middle of each face of the large cube). Each of the 26 unit cubes that have one or more faces exposed has at most one black face exposed. A large 3 × 3 × 3 cube made out of 27 unit cubes has a surface area of 54 unit squares. Therefore, the answer to our question is 26/54 = 13/27.

29. 3750 units^3. When the two black cubes are removed, the four exposed faces are replaced by eight congruent square faces, for a net gain of four square faces. These four faces correspond to the increase of 100 square units. Therefore, each face has area 25 and edge length 5 units. Each of the thirty cubes has volume $5^3 = 125$ cubic units. The total volume is, therefore, 30(125) = 3750 units^3.

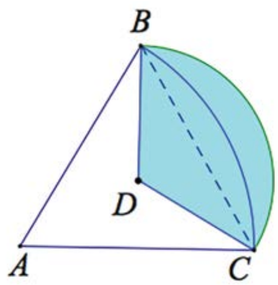
30. {4, 5, 6, 7, 8, 9}. If a, b, and c represent the smallest values possible, namely, 0, 1, and 2, then the median number is 4. If exactly two of the absent witches remember fewer than three spells, then the median number is 5. If only one of the absent witches remembers fewer than three spells, then the median number could be either 6 or 7. If one of the absent witches remem-

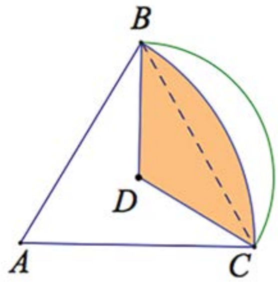
bers eight spells (and the other two remember more than eight), then 8 is the median. If all the absent witches remember at least ten spells, then 9 is the median.

31. $s^2(3\sqrt{3}-\pi)/18$. Begin by finding the area of the sector bounded by the outer arc BC and radii DB and DC (shaded blue). Radii of the equilateral triangle are 2/3 the length of an altitude of the triangle. The altitude—found by an application of the Pythagorean theorem or by using 30°-60°-90° relationships—is $s\sqrt{3}/2$, so $DB = s\sqrt{3}/3$. Now $m\angle BDC = 120°$, which means the sector is 1/3 the area of the circle with radius $s\sqrt{3}/3$. Calculate:

$$\left(\frac{\pi}{3}\right)\left(\frac{s\sqrt{3}}{3}\right)^2 = \frac{\pi s^2}{9}$$

Next we'll find the area of the region bounded by inner arc BC and segments DB and DC (shaded orange). Since $m\angle A = 60°$, the sector bounded by inner arc BC and segments AB and AC has 1/6 the area of the circle of radius AB: $\pi s^2/6$.





Subtract 2/3 the area of the triangle to obtain the area shaded orange:

$$\left(\frac{\pi s^2}{6}\right)-\left(\frac{2}{3}\right)\left(\frac{s^2\sqrt{3}}{4}\right)=\frac{\pi s^2}{6}-\frac{s^2\sqrt{3}}{6}$$
$$=\frac{s^2\left(\pi-\sqrt{3}\right)}{6}$$

One last subtraction yields the area of the lune:

$$\frac{\pi s^2}{9}-\frac{s^2\left(\pi-\sqrt{3}\right)}{6}=\frac{s^2(3\sqrt{3}-\pi)}{18}$$